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Volume Dependence of Schwinger Functions in the Yukawa_2 Quantum Field Theory

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Abstract. We prove upper bounds on the partition function and Schwinger functions for the Euclidean Yukawa_2 quantum field theory which depend on the interaction volume Λ only through a term of the form $(\text{const})^{|\Lambda|}$. We also prove a lower bound of the form $(\text{const})^{|\Lambda|}$ for the partition function. We work throughout in the Matthews-Salam representation with the fermions integrated out.

I. Introduction

We study the Yukawa_2 quantum field theory in a finite volume Λ as a Euclidean boson field theory with the fermions "integrated out". The possibility of integrating out the fermions in the Yukawa theory was first demonstrated, in the external boson field case, by Matthews and Salam [1, 2], and in the finite volume interacting theory, by Seiler [3] who showed that the resulting Fredholm determinants are integrable functions of the boson field. As a step towards taking the infinite volume limit of Yukawa_2 we show in this paper that these determinants approximately factor over a decomposition of the space-time volume into subvolumes. While the determinants do not factor exactly, we exhibit upper and lower bounds which factor. The existence of such an approximate factoring is related to the exponential decoupling of distant regions in the free boson and fermion two point functions—i.e., to the nonzero free boson and fermion masses μ_0, m_0.

Our principal results are bounds on the un-normalized finite volume Schwinger functions

$$(ZS)^{(\Lambda)}(f_1,\ldots,f_n;g_1,\ldots,g_m;h_1,\ldots,h_m) \equiv \left\langle \prod_{i=1}^n \phi(f_i) \prod_{j=1}^m \Psi^{(1)}(g_j) \prod_{k=1}^m \Psi^{(2)}(h_k) e^{-V(\Lambda)} \right\rangle,$$

and on the partition function $Z^{(\Lambda)} \equiv \langle e^{-V(\Lambda)}\rangle$. Here f_i, g_j, h_k are functions in the boson and fermion test-function spaces: $\mathscr{H}_{-1}^{(\mu_0)}$ and $\mathscr{H}^* = \mathscr{H}_{-\frac{1}{2}}^{(m_0)} \otimes C^2$, where $\mathscr{H}_s^{(m)} = L_2(R^2, (k^2+m^2)^s d^2k)$. We cover space-time with a lattice of unit squares Δ_α, with centers $\alpha \in Z^2$, and we suppose that the f_i are localized in unit squares.

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Denoting by n_α the number of f_i localized in Δ_α we have:

Theorem 1.1. *With constants depending only on m_0, μ_0 and the coupling constant,*

$$|(ZS)^{(\Lambda)}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)| \leqq c_1^{|\Lambda|}c_2^{n+m}\prod_{\alpha\in Z^2} n_\alpha!^{\frac{1}{2}}\prod_{i=1}^n\|f_i\|_{-1}\prod_{j=1}^m\|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}}.$$

Theorem 1.2. *If the space-time cutoff $g(x)$ is of Hamiltonian form: $g(x)=\chi_{[0,t]\times[0,l]}(x)$ then for a strictly positive constant:*

$$Z^{(\Lambda)}\geqq(\text{const})^{|\Lambda|}.$$

Theorem 1.1 is proved in Section II and Theorem 1.2 in Section III. The proof of Theorem 1.2 requires the use of the Feynman-Kac formula of Osterwalder and Schrader [4] and hence the restriction on the form of cutoff. We consider the case of scalar Yukawa_2 theory, but all results apply equally to the pseudoscalar theory with only trivial modifications.

Before publication of Seiler's paper [3], which for the first time applied fully Euclidean methods to the Yukawa_2 model, most of the rigorous results on the Yukawa_2 model were proved in the Hamiltonian formalism. Glimm [5, 6] showed the existence of a semibounded finite volume Hamiltonian $H(g)$ and Glimm and Jaffe [7, 8] showed that $H(g)$ is selfadjoint and generates dynamics with a finite propagation speed. Schrader [9] proved upper and lower bounds proportional to the volume for the vacuum energy $E(g)=\inf\operatorname{spec}H(g)$ and showed existence of an infinite volume Yukawa_2 theory; Brydges [10] has given an alternate proof of the lower bound for $E(g)$ using semi-Euclidean techniques. The proof of the Haag-Kastler axioms for the infinite volume theory was completed by McBryan and Park [11] who proved Lorentz covariance. Our upper and lower bounds on $Z^{(\Lambda)}$ are the Euclidean equivalent of Schrader's bounds on $E(g)$[1]. For notation and standard results for compact operators, we refer the reader to the books of Dunford and Schwartz [12].

We include here for convenience a short formal description of the procedure of "integrating out" the fermions in the Yukawa_2 theory. The possibility of doing so is due to the field equations being linear in the fermi fields [1, 2]. In terms of the Euclidean fermi fields $\Psi^{(i)}$, $i=1,2$, introduced by Osterwalder and Schrader [4],

$$\langle\Psi^{(i)}(x)\Psi^{(i)}(y)\rangle=0,\qquad\langle\Psi^{(1)}(x)\Psi^{(2)}(y)\rangle=S_0(x,y),$$

the un-renormalized interaction for the spacecutoff Yukawa_2 model is $V_I=\lambda\int dx g(x)\Psi^{(2)}(x)\Psi^{(1)}(x)\phi(x)$. The Schwinger functions and partition function may be expressed as:

$$\begin{aligned}&(ZS)(x_1,\dots,x_n;y_1,\dots,y_m;z_1,\dots,z_m)\\&\quad\equiv\langle\prod_{i=1}^n\phi(x_i)\prod_{j=1}^m\Psi^{(1)}(y_j)\prod_{k=1}^m\Psi^{(2)}(z_k)e^{-V_I}\rangle\\&\quad=\int d\mu_0\prod_{i=1}^n\phi(x_i)T_m(y_1,\dots,y_m;z_1,\dots,z_m),\end{aligned}$$

$$T_m(y_1,\dots,y_m;z_1,\dots,z_m)\equiv\langle\prod_{j=1}^m\Psi^{(1)}(y_j)\prod_{k=1}^m\Psi^{(2)}(z_k)e^{-V_I}\rangle_{\Omega_f},$$

$$Z=\int d\mu_0 Z_f,\qquad Z_f=\langle e^{-V_I}\rangle_{\Omega_f},$$

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[1] We note that Schrader's bounds follow immediately, see Bridges[10], from Theorems 1.1, 1.2 and the Feynman-Kac formula of Osterwalder and Schrader [4].

where $d\mu_0$ is the free boson measure and Ω_f denotes expectation in the free Euclidean fermi vacuum. Decomposing $\Psi^{(1)}(y_1)$, in T_m, into creation and annihilation operators and contracting the latter to the right, we obtain:

$$T_m(y_1,\dots,y_m;z_1,\dots,z_m)=\sum_{i=1}^m(-)^{m-i}S_0(y_1,z_i)T_{m-1}(y_2,\dots,y_m;z_1,\dots,\not{z}_i,\dots,z_m)$$
$$-\lambda\int dyg(y)\phi(y)S_0(y_1,y)T_m(y,y_2,\dots,y_m;z_1,\dots,z_m)\,. \quad (1.1)$$

In terms of the integral operator K with kernel $K(x,y)\equiv S_0(x,y)g(y)\phi(y)$, the integral equations (1.1) have the solution:

$$T_m(y_1,\dots,y_m;z_1,\dots,z_m)$$
$$=\sum_{i=1}^m(-)^{m-i}((1+\lambda K)^{-1}S_0)(y_1,z_i)T_{m-1}(y_2,\dots,y_m;z_1,\dots,\not{z}_i,\dots,z_m)\,.$$

Iterating this equation m times we arrive at:

$$T_m(y_1,\dots,y_m;z_1,\dots,z_m)=(-)^{[m/2]}\det_{jk}S'(y_j,z_k)Z_f \quad (1.2)$$

where $S'(x,y)=((1+\lambda K)^{-1}S_0)(x,y)$. To compute Z_f we differentiate with respect to λ:

$$(d/d\lambda)Z_f(\lambda)=-\int dxg(x)\phi(x)\langle\Psi^{(2)}(x)\Psi^{(1)}(x)e^{-V_I}\rangle_{\Omega_f}$$
$$=\int dxg(x)\phi(x)((1+\lambda K)^{-1}S_0)(x,x)Z_f(\lambda)$$
$$=\int dx((1+\lambda K)^{-1}K)(x,x)Z_f(\lambda)=\mathrm{Tr}\,K(1+\lambda K)^{-1}Z_f(\lambda)\,,$$

where in the second line we have used (1.2) with $m=1$. Thus

$$Z_f(\lambda)=e^{\mathrm{Tr}\ln(1+\lambda K)}=\det(1+\lambda K)\,.$$

Renormalization of V_I requires that we subtract (i) $\langle V_I\rangle_{\Omega_f}=\mathrm{Tr}\,\lambda K$ which normal orders V_I, (ii) a vacuum energy counter-term $-\frac{1}{2}\langle :V_I:^2\rangle=\frac{1}{2}\langle\mathrm{Tr}\,\lambda^2K^2\rangle$ and (iii) a boson mass counter-term $-\frac{1}{2}\lambda^2\delta m^2:\phi^2(g^2):$, $\delta m^2=-2(2\pi)^{-2}\int d^2p(p^2+m_0^2)^{-1}$. Thus we must replace Z_f above by:

$$D(\phi)=e^{-\mathrm{Tr}\,\lambda K+\frac{1}{2}\lambda^2\langle\mathrm{Tr}\,K^2\rangle+\frac{1}{2}\lambda^2\delta m^2:\phi^2(g^2):}\det(1+\lambda K)$$
$$=e^{\frac{1}{2}\lambda^2:F(\phi):}\det{}_3(1+\lambda K)\,,$$

where $F(\phi)=-\mathrm{Tr}\,K^2+\delta m^2\phi^2(g^2)$, and

$$\det{}_p(1+\lambda K)=\det(1+\lambda K)\times\exp\{\textstyle\sum_{n=1}^{p-1}(-)^n n^{-1}\mathrm{Tr}(\lambda K)^n\}\,,$$

see Dunford and Schwartz [12]. Thus the representation for the renormalized Schwinger functions with the fermions "integrated out" is:

$$(ZS)(x_1,\dots,x_n;y_1,\dots,y_m;z_1,\dots,z_m)$$
$$=\int d\mu_0\prod_{i=1}^n\phi(x_i)(-)^{[m/2]}\det_{jk}S'(y_j,z_k)D(\phi)\,. \quad (1.3)$$

The formal expressions above are all well-defined if we replace the fields $\phi,\Psi^{(i)}$ by momentum cutoff fields ϕ_κ, $\Psi_\kappa^{(i)}$. Furthermore Seiler [3], has shown that the quantity on the right of (1.3) converges as $\kappa\to\infty$. Thus we may view (1.3) as a definition of the Schwinger functions without momentum cutoffs. For technical reasons it is convenient to separate the term $F(\phi)$, in $D(\phi)$, as a sum:

$$F(\phi)=-G(\phi)-\tfrac{1}{2}\mathrm{Tr}(K^*+K)^2$$

where $-G(\phi)\equiv \mathrm{Tr}\, K^*K+\delta m^2\phi^2(g^2)$. This completes the motivation for formula (2.1), and the subsequent definitions, in Section II.

We note that our methods give an alternate proof of Seiler's results when the perturbation expansion presented in Section II is carried out once in the interaction volume instead of repeatedly in unit volumes.

II. Upper Bounds on Schwinger Functions

The Schwinger functions for the Yukawa_2 model are expressed in terms of solutions of a Fredholm equation by [1, 2, 3]:

$$\begin{aligned}&(\mathrm{ZS})^{(\Lambda)}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)\\&\quad=\int d\mu_0 \prod_{i=1}^n \phi(f_i)(-)^{[m/2]}\det{}_{jk}S'(g_j,h_k;\phi)D(\phi)\,. \qquad (2.1)\end{aligned}$$

Here $S'(g,h;\phi)\equiv(g,(1+\lambda K)^{-1}S_0h)$, $S_0(x,y)=(2\pi)^{-2}\int d^2p e^{ip\cdot(x-y)}(\not p+m_0)/(p^2+m_0^2)$,

$$\begin{aligned}&D(\phi)\equiv e^{-\frac{1}{2}\lambda^2:(\phi,G\phi)_1:}e^{-\frac{1}{4}\lambda^2:\mathrm{Tr}(K^*+K)^2:}\det{}_3(1+\lambda K)\,,\\&G(p,q)\equiv\mu(p)^{-2}\mu(q)^{-2}\int d^2kG(k)\tilde g(p-k)\tilde g(k-q),\qquad \mu(p)^2=p^2+\mu_0^2\,,\\&\quad G(k)\equiv-2(2\pi)^{-2}\int d^2l\{1/\omega(l+k/2)\omega(l-k/2)-1/\omega(l)^2\}\,,\qquad \omega(p)^2=p^2+m_0^2\,,\\&K(p,q)\equiv(2\pi)^{-1}(\not p+m_0)\,(p^2+m_0^2)^{-1}(\phi g)^\frown(p-q)(q^2+m_0^2)^{-\frac{1}{2}}\,.\end{aligned}$$

The Fredholm operator $K(\phi)=K(\phi g)$ is the compact operator with kernel $K(p,q)$, on $\mathscr{H}=\mathscr{H}_{1/2}^{(m_0)}\otimes C^2$ while G defines a positive Hilbert-Schmidt operator on $\mathscr{H}_1^{(\mu_0)}$. We will take the space-time cutoff $g(\cdot)$ to be in $C_0^\infty(R^2)$ or else to be the characteristic function of a bounded region and we define $\Lambda=\mathrm{suppt}\,g(\cdot)$. Seiler [3] has shown that the quantities defined above are a.e. defined functions of the boson field ϕ.

Applying the Schwarz inequality to (2.1) we obtain:

$$\begin{aligned}&|(\mathrm{ZS})^{(\Lambda)}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)|\\&\quad\leqq\Big\|\prod_{i=1}^n\phi(f_i)\Big\|_2\,\|\det{}_{jk}S'(g_j,h_k;\phi)D(\phi)\|_2\,.\end{aligned}$$

By the hypercontractivity of the free boson field, and checkerboard estimates [13] it follows that:

$$\begin{aligned}\Big\|\prod_{i=1}^n\phi(f_i)\Big\|_2&\leqq\prod_{\alpha\in Z^2}\Big\|\prod_{i\in\Delta_\alpha}\phi(f_i)\Big\|_{8\beta^2},\qquad \beta=(e^{\mu_0}+1)/(e^{\mu_0}-e^{-\mu_0})\,,\\&\leqq\prod_{\alpha\in Z^2}(8\beta^2-1)^{(n_\alpha-1)/2}\Big\|\prod_{i\in\Delta_\alpha}\phi(f_i)\Big\|_2\,,\\&\leqq\prod_{\alpha\in Z^2}(8\beta^2-1)^{(n_\alpha-1)/2}\prod_{i\in\Delta_\alpha}\|f_i\|_{-1}(\pi^{-1/2}2^{n_\alpha}\Gamma(n_\alpha+\tfrac{1}{2}))^{1/2}\,,\\&\leqq\mathrm{const}\,(\mathrm{const})^n\prod_{\alpha\in Z^2}n_\alpha!^{1/2}\prod_{i=1}^n\|f_i\|_{-1}\,.\end{aligned}$$

Thus to prove Theorem 1.1 we need only show that

$$\|\det{}_{jk}S'(g_j,h_k;\phi)D(\phi)\|_2\leqq(\mathrm{const})^m(\mathrm{const})^{|\Lambda|}\prod_{j=1}^m\|g_j\|_{-1/2}\,\|h_j\|_{-1/2}\,. \qquad (2.2)$$

We again use the Schwarz inequality and the result, proved below:

Lemma 2.1. $\|e^{-1/2\,\lambda^2:(\phi,G\phi)_1:}\|_p\leqq(\mathrm{const})^{|\Lambda|}$, $p\geqq1$, *uniformly in* $|\Lambda|$.

This reduces the problem to studying

$$X(\phi) \equiv \det_{jk} S'(g_j, h_k; \phi) e^{-(\lambda^2/4):\mathrm{Tr}(K^*+K)^2:} \det_3(1+\lambda K)$$
$$= (\psi, \bigotimes^m (1+\lambda K)^{-1} \phi)_{\Lambda^m \mathscr{H}} e^{-(\lambda^2/4):\mathrm{Tr}(K^*+K)^2:} \det_3(1+\lambda K),$$

where $\Psi = D_0 g_1 \Lambda \dots \Lambda D_0 g_m$, $\Phi = S_0 g_1 \Lambda \dots \Lambda S_0 g_m$, are vectors in the m-fold anti-symmetric tensor product space $\Lambda^m \mathscr{H}$, and D_0 is multiplication by $\omega(p)^{-1}$. Both D_0 and S_0 are isometries from $\mathscr{H}^*$ to $\mathscr{H}$ and thus:

$$|X(\phi)| \leqq \prod_{j=1}^m \|g_j\|_{-1/2} \|h_j\|_{-1/2} e^{-(\lambda^2/4):\mathrm{Tr}(K^*+K)^2:}$$
$$\cdot \|\bigotimes^m (1+\lambda K)^{-1} \det_3(1+\lambda K)\|_{\Lambda^m \mathscr{H}}.$$

Hence (2.2), and Theorem 1.1, are proved once we have shown that:

Theorem 2.2. *Let* $Y_m(\phi) = e^{-(\lambda^2/4):\mathrm{Tr}(K^*+K)^2:} \|\bigotimes^m (1+\lambda K)^{-1} \det_3(1+\lambda K)\|_{\Lambda^m \mathscr{H}}$. *Then*

$$\|Y_m(\phi)\|_p \leqq (\mathrm{const})^m (\mathrm{const})^{|\Lambda|}, \qquad p \geqq 1.$$

Proof of Lemma 2.1. By explicit Gaussian integration we have:

$$\|e^{-1/2\,\lambda^2:(\phi, G\phi)_1:}\|_p^p = \det_2(1+p\lambda^2 G)^{-1/2} \leqq e^{1/2\,\|p\lambda^2 G\|_2^2}.$$

The Hilbert-Schmidt norm, $\|G\|_2^2$, is given by:

$$\|G\|_2^2 = \int d^2p\, d^2q\, \mu(p)^{-2} \mu(q)^{-2} |\int d^2k\, G(k) \tilde{g}(p-k) \tilde{g}(k-q)|^2.$$

Denote by $Z_2^{(\Lambda)}$ the smallest subset of Z^2 such that $\Lambda \subset \bigcup_{\alpha \in Z_2^{(\Lambda)}} \Delta_\alpha$, and let $\chi_\alpha(x)$ denote the characteristic function of Δ_α. We will use the notation $\alpha \in \Lambda$ to denote $\alpha \in Z_2^{(\Lambda)}$ in the sequel. Introducing the decomposition $g(x) = \sum_{\alpha \in \Lambda} \chi_\alpha(x) g(x) \equiv \sum_{\alpha \in \Lambda} g_\alpha(x)$ we obtain

$$\|G\|_2^2 = \sum_{\alpha, \beta, \alpha', \beta' \in \Lambda} I(\alpha, \beta, \alpha', \beta'),$$
$$I(\alpha, \beta, \alpha', \beta')$$
$$= \int d^2p\, d^2q\, d^2k\, d^2l\, \mu(p)^{-2} \mu(q)^{-2} G(k) G(l) \tilde{g}_\alpha(p-k) \tilde{g}_\beta(k-q) \tilde{g}_{\alpha'}(l-p) \tilde{g}_{\beta'}(q-l).$$

For $\gamma \geqq 1$ we denote by $\mathscr{I}(\gamma)$ the set of space-time cutoffs $g(\cdot)$ which are of one of the forms: (i) $\sum_{\alpha \in \Lambda} \chi_\alpha(x)$, (ii) $\sum_{\alpha_0 \in \Lambda_0} \chi_{\alpha_0}(x_0) g_1(x_1)$, or (iii) $g(x)$, with $g_1(x_1) \in C_0^\infty(R^1)$, $g(x) \in C_0^\infty(R^2)$ and

$$\sup_{x_1} |\partial g_1 / \partial x_1| \leqq \gamma, \quad \text{or} \quad \sup_x \{|\partial g/\partial x_0|, |\partial g/\partial x_1|, |\partial^2 g/\partial x_0 \partial x_1|\} \leqq \gamma,$$

respectively. The function $G(k)$ may be computed [3]:

$$G(k) = \pi^{-1} \ln \tfrac{1}{4} \{2 + (k^2/m_0^2 + 4)^{1/2}\}.$$

By integration by parts in each of the variables p_i, q_i, k_i, l_i we find

$$|I(\alpha, \beta, \alpha', \beta')|$$
$$\leqq \mathrm{const}(1+\gamma)^4 \{\prod_{i=0,1} (1+|\alpha_i - \beta_i|)(1+|\alpha_i' - \beta_i'|)(1+|\alpha_i - \alpha_i'|)(1+|\beta_i - \beta_i'|)\}^{-1}$$

for $g \in \mathscr{I}(\gamma)$, with the constant independent of g. It follows that

$$\|G\|_2^2 \leqq \mathrm{const}(1+\gamma)^4 \sum_{\alpha, \beta, \alpha', \beta' \in \Lambda} |I(\alpha, \beta, \alpha', \beta'| \leqq \mathrm{const}(1+\gamma)^4 |\Lambda|,$$

where $|\Lambda|$ is defined to be $|Z_2^{(\Lambda)}|$. This completes the proof of Lemma 2.1, with the constant independent of g for $g \in \mathscr{I}(\gamma)$.

Proof of Theorem 2.2. In order to study $Y_m(\phi)$, we note that

$$Y_m(\phi)=e^{-\frac{1}{4}\lambda^2:\mathrm{Tr}(K^*+K)^2:}\left(\prod_{i=1}^m v_i(1+\lambda K)\right)^{-1}\det{}_3(1+\lambda K),$$

where $v_i(C)$ are the eigenvalues of the operator $|C|$ in increasing order, counted by multiplicity. It is convenient to work with the self-adjoint operator $A\equiv\lambda K+\lambda K^*+\lambda^2K^*K\geqq -1$, and then

$$Y_m(\phi)=e^{-(\lambda^2/4):\mathrm{Tr}(K^*+K)^2_:-\lambda^3\mathrm{Tr}K^*K^2-(\lambda^4/4)\mathrm{Tr}(K^*K)^2}\left(\det{}_3(1+A)/\prod_{i=1}^m v_i(1+A)\right)^{1/2}.$$

We introduce a sequence of momentum cutoffs $\kappa_r=\mu_0(e^r-1)$, $r=0,1,\ldots$, and fields $\phi_r=\chi_{\kappa_r}*\phi$ where $\chi_\kappa(x)=\kappa^2\chi(\kappa x)$, $\chi\in C_0^\infty(R^2)$, $\mathrm{suppt}\,\chi\in\bigotimes^2[-1/8,1/8]$ and $\int d^2x\chi(x)=1$. Since $Y_m(\phi_n)\to Y_m(\phi)$ in $L_p(d\mu_0)$, [3], Theorem 2.2 will follow from a bound of the form $\|Y_m(\phi_n)\|_p\leqq(\mathrm{const})^m(\mathrm{const})^{|A|}$, with constants uniform in $n,m,|A|$. We consider for convenience that the unit volumes $\Delta_\alpha\in A$ are ordered in some way, say $\alpha_1,\ldots,\alpha_{|A|}$. To every positive integer n, every sequence

$$r=\{r_\alpha|\alpha\in A,\ r_\alpha\in Z^+,\ r_\alpha\leqq n\}$$

and every double sequence

$$s=\{s^{(\alpha)}|\alpha\in A,\ s^{(\alpha)}=(s_0^{(\alpha)},\ldots,s_{n-1}^{(\alpha)}),\ s_0^{(\alpha)}=1\geqq s_1^{(\alpha)}\geqq\ldots\geqq s_{n-1}^{(\alpha)}\geqq 0\},$$

we define integral operators

$$K^{(\alpha)}(r_\alpha;s^{(\alpha)})=\sum_{i=1}^{r_\alpha}(s_{i-1}^{(\alpha)}-s_i^{(\alpha)})K_i^{(\alpha)}+s_{r_\alpha}^{(\alpha)}K_{r_\alpha+1}^{(\alpha)},$$

$$K_n^{(\alpha)}(r_\alpha;s^{(\alpha)})=\sum_{i=1}^{r_\alpha}(s_{i-1}^{(\alpha)}-s_i^{(\alpha)})K_i^{(\alpha)}+s_{r_\alpha}^{(\alpha)}K_n^{(\alpha)},$$

$$K_\alpha(r;s)=\sum_{\beta\leqq\alpha}K^{(\beta)}(r_\beta;s^{(\beta)})+\sum_{\beta>\alpha}K_n^{(\beta)},$$

$$K_{n,\alpha}(r;s)=\sum_{\beta<\alpha}K^{(\beta)}(r_\beta;s^{(\beta)})+K_n^{(\alpha)}(r_\alpha;s^{(\alpha)})+\sum_{\beta>\alpha}K_n^{(\beta)},$$

where $K_r^{(\alpha)}\equiv K(\phi_r g_\alpha)$ – i.e. has momentum cutoff κ_r and is localized in region α. Thus the $K_\alpha(r;s)$ have maximum cutoffs of $\kappa_{r_\beta+1}$ in Δ_β, $\beta\leqq\alpha$ but maximum cutoff still at κ_n in regions with $\beta>\alpha$; $K_{n,\alpha}(r;s)$ has similar cutoffs except that in region α the cutoff interpolates between κ_n and κ_{r_α}.

For any operator C we denote the eigenvalues ordered in decreasing absolute value, counting multiplicity, by $\lambda_i(C)$. For each of the operators K defined above, we introduce $A\equiv(1+\lambda K^*)(1+\lambda K)-1$, and define functions

$$H(K)\equiv-\tfrac{1}{4}\lambda^2:\mathrm{Tr}(K^*+K)^2:-\lambda^3\mathrm{Tr}K^*K^2-\tfrac{1}{4}\lambda^4\mathrm{Tr}(K^*K)^2,$$

$$D_m(A)\equiv\prod_{i\neq(m)}^\infty(1+\lambda_i(A))e^{-\lambda_i(A)+1/2\,\lambda_i(A)^2},$$

where $\prod_{i\neq(m)}^\infty$ is defined to mean leaving out of the product the lowest m eigenvalues below zero. With the definitions above, we have

$$\begin{aligned}Y_m(\phi_n)^p&=e^{pH(K(\phi_n))}\prod_{i=1}^m(v_i(1+A(\phi_n))^{-p/2}D_0(A(\phi_n))^{p/2}\\&\leqq e^{mp}e^{pH(K(\phi_n))}D_m(A(\phi_n))^{p/2}.\end{aligned}$$

The factor e^{mp} gives the required $(\mathrm{const})^m$, and we now show that $Z_m(K)\equiv e^{pH(K)}D_m(A)^{p/2}$, satisfies

$$\|Z_m(K(\phi_n))\|_1\leqq\mathrm{const}^{|A|}, \tag{2.3}$$

uniformly in $m,n,|A|$, completing the proof of Theorem 2.2.

To prove the bound (2.3) we show the existence of an increasing sequence of upper bounds for $Z_m(K(\phi_n))$ with the final bound dominated by $(\text{const})^{|A|}$. The bounds are obtained essentially by applying a finite perturbation expansion (which is exact) of order n in each unit volume and by bounding each of the terms so obtained. To generate the perturbation expansion in region α, we lower the cutoff κ_n in that region to κ_{r_α}, $r_\alpha=1,2,\ldots$ by replacing $K_{n,\alpha}(\ldots,r_{\alpha-1},0;s)$ by $K_\alpha(\ldots,r_\alpha;s)$ with a remainder, given by $K_{n,\alpha}(\ldots,r_\alpha;s)$, involving interpolation between κ_n and κ_{r_α} in Δ_α. On reaching $r_\alpha=n-1$, the expansion terminates in region α. Noting that $K_{n,\alpha}(\ldots,r_{\alpha-1},n-1;s)=K_{\alpha+1}(\ldots,r_{\alpha-1},n-1,0;s)$ and that $K_\alpha(\ldots,r_\alpha;s)=K_{n,\alpha+1}(\ldots,r_\alpha,0;s)$ are ready to perform the expansion in the next region.

The basic expansion step in region α is given by:

$$Z_m(K_{n,\alpha}(r;s))=Z_m(K_\alpha(r;s))+\int_0^{s_{r_\alpha}^{(\alpha)}} ds_{r_\alpha+1}^{(\alpha)}\,(\partial/\partial s_{r_\alpha+1}^{(\alpha)})Z_m(K_{n,\alpha}(\ldots,r_\alpha+1;s)),$$

$$|(\partial/\partial s_{r_\alpha}^{(\alpha)})Z_m(K_{n,\alpha}(r;s))|=e^{pH(K_{n,\alpha}(r,s))}|p(\partial/\partial s_{r_\alpha}^{(\alpha)})H(K_{n,\alpha}(r;s))D_m(A_{n,\alpha}(r;s))^{p/2}$$
$$+\tfrac{1}{2}pD_m(A_{n,\alpha}(r;s))^{p/2-1}(\partial/\partial s_{r_\alpha}^{(\alpha)})D_m(A_{n,\alpha}(r;s))|.$$

Assuming, for the moment, differentiability of $\lambda_i=\lambda_i(A_{n,\alpha}(r;s))$:

$$|(\partial/\partial s_{r_\alpha}^{(\alpha)})D_m(A_{n,\alpha}(r;s))|$$
$$=|\textstyle\sum_{i\neq(m)}(1/3)(\partial\lambda_i^3/\partial s_{r_\alpha}^{(\alpha)})e^{-\lambda_i+1/2\lambda_i^2}\prod_{j\neq i,(m)}^{\infty}(1+\lambda_j)e^{-\lambda_j+1/2\lambda_j^2}|$$
$$\leqq\textstyle\sum_{i=1}^{\infty}|\partial\lambda_i^3/\partial s_{r_\alpha}^{(\alpha)}|(e^2/3)D_{m+1}(K_{n,\alpha}(r;s)).$$

To bound the sum of derivatives of eigenvalues, we will use the following Lemma which we prove below (note: $\|A\|_1=\mathrm{Tr}|A|$):

Lemma 2.3. *Let $A(s)$ be self-adjoint compact operators, holomorphic as functions of s, and define $\lambda_i(s)=\lambda_i(A(s))$. Then for any $\varepsilon>0$ and almost all s:*

$$\textstyle\sum_{i=1;|\lambda_i|>\varepsilon}^{\infty}|\partial\lambda_i^3(s)/\partial s|\leqq\|\partial A(s)^3/\partial s\|_1.$$

Assuming that Lemma 2.3 may also be applied in the limit $\varepsilon=0$, we have therefore

$$|(\partial/\partial s_{r_\alpha}^{(\alpha)})Z_m(K_{n,\alpha}(r;s))|\leqq P_{n,\alpha}(r;s)Z_{m+1}(K_{n,\alpha}(r;s)),$$
$$P_{n,\alpha}(r;s)\equiv e^2p\{|(\partial/\partial s_{r_\alpha}^{(\alpha)})H(K_{n,\alpha}(r;s))|+\tfrac{1}{6}\|\partial A_{n,\alpha}^3(r;s)/\partial s_{r_\alpha}^{(\alpha)}\|_1\},\tag{2.4}$$

and so, noting that $Z_m(K_\alpha(r;s))=Z_m(K_{n,\alpha+1}(\ldots,r_\alpha,0;s))$, our basic expansion step takes the form:

$$Z_m(K_{n,\alpha}(\ldots,r_\alpha;s))\leqq Z_m(K_{n,\alpha+1}(\ldots,r_\alpha,0;s))$$
$$+\int_0^{s_{r_\alpha}^{(\alpha)}} ds_{r_\alpha+1}^{(\alpha)}P_{n,\alpha}(\ldots,r_\alpha+1;s)Z_{m+1}(K_{n,\alpha}(\ldots,r_\alpha+1;s)).$$

Repeating this expansion step n times in each unit volume and then moving to the next volume, we obtain the bound:

$$Z_m(K(\phi_n))\equiv Z_m(K_{n,1}(0;s))$$
$$\leqq\sum_{r_\alpha=1;\alpha\in\Lambda}^{n-1}\int d^rs\prod_\alpha Q_{n,\alpha}(r,s)\cdot Z_{m+\sum_{\alpha\in\Lambda}r_\alpha}(K_{|\Lambda|}(r;s)),\tag{2.5}$$

where

$$Q_{n,\alpha}(r;s) = \prod_{t=1}^{r_\alpha} P_{n,\alpha}(\ldots, r_{\alpha-1}, t; s),$$

and we have used the notation $\int d^r s \equiv \prod_{\alpha\in\Lambda} \int d^{r_\alpha} s^{(\alpha)}$.

As a final step we bound the remaining factors $Z_{m+\sum_{\alpha\in\Lambda} r_\alpha}(K_{|\Lambda|}(r;s))$ with a bound independent of m. Thus

$$Z_m(K) \equiv e^{pH(K)} D_m(A)^{p/2} \leqq e^{pH(K)+(p/4)\operatorname{Tr} A^2},$$

where we have applied $(1+x)e^{-x} \leqq 1$, $x > -1$, to each eigenvalue in the product D_m, and have increased the bound by addition of the missing eigenvalues λ_i^2, $i=(m)$. Now

$$\begin{aligned} H(K) + \tfrac{1}{4}\operatorname{Tr} A^2 &= -(\lambda^2/4){:}\operatorname{Tr}(K^* + K)^2{:} - \lambda^3 \operatorname{Tr} K^* K^2 \\ &\quad -(\lambda^4/4)\operatorname{Tr}(K^*K)^2 + \tfrac{1}{4}\operatorname{Tr} A^2, \\ &= (\lambda^2/4)\langle \operatorname{Tr}(K^* + K)^2\rangle. \end{aligned}$$

Therefore returning to (2.5), we obtain the m-independent bound:

$$Z_m(K(\phi_n)) \leqq \sum_{r_\alpha=1;\alpha\in\Lambda}^{n-1} \int d^r s \prod_\alpha Q_{n,\alpha}(r;s) e^{1/4\, p\lambda^2 \langle \operatorname{Tr}(K_{|\Lambda|}(r;s)^* + K_{|\Lambda|}(r;s))^2\rangle}.$$

The bound $\|Z_m(K(\phi_n))\|_1 \leqq (\text{const})^{|\Lambda|}$ now follows from:

Lemma 2.4. *There is a constant c_1, independent of $|\Lambda|, r, s$ such that*

$$\langle \operatorname{Tr}(K_{|\Lambda|}(r,s)^* + K_{|\Lambda|}(r,s))^2\rangle \leqq c_1 \sum_{\alpha\in\Lambda} \ln(1 + \kappa_{r_\alpha+1}/\mu_0).$$

Lemma 2.5. *There are constants c_2, c_3 and $\varepsilon > 0$, independent of $n, |\Lambda|, r$, such that, with $R(r) \equiv \int d^r s \prod_\alpha Q_{n,\alpha}(r,s)$,*

$$\|R(r)\|_1 \leqq \prod_{\alpha\in\Lambda} c_2^{r_\alpha} (r_\alpha!)^{c_3} \prod_{t_\alpha=1}^{r_\alpha} \kappa_{t_\alpha}^{-\varepsilon}.$$

Combining these two lemmas with the bound for $Z_m(K(\phi_n))$, we have:

$$\begin{aligned} \int d\mu_0 Z_n(K(\phi_n)) &\leqq \sum_{r_\alpha=1;\alpha\in\Lambda}^{n-1} \prod_\alpha c_2^{r_\alpha} r_\alpha!^{c_3} \prod_{t_\alpha=1}^{r_\alpha} \kappa_{t_\alpha}^{-\varepsilon} e^{c_1 \ln(1+\kappa_{r_\alpha+1}/\mu_0)} \\ &= \Big\{\sum_{r=1}^{n-1} c_2^r r!^{c_3} \prod_{t=1}^{r} \kappa_t^{-\varepsilon} e^{c_1 \ln(1+\kappa_{r+1}/\mu_0)}\Big\}^{|\Lambda|} \leqq c_4^{|\Lambda|}, \\ c_4 &\equiv \sum_{r=1}^{\infty} (c_2(2/\mu_0)^\varepsilon)^r (r!)^{c_3} e^{-\varepsilon\sum_{t=1}^r t} e^{c_1(r+1)} \\ &\leqq \sum_{r=1}^{\infty} e^{r\ln c_2(2/\mu_0)^\varepsilon + c_3 r\ln(1+r) - 1/2\,\varepsilon r(1+r) + c_1(1+r)} < \infty, \end{aligned}$$

where we have used $\kappa_r = \mu_0(e^r - 1)$.

Before giving the proofs of Lemmas 2.3, 2.4, and 2.5, we discuss the differentiability of the eigenvalues $\lambda_i(A_{n,\alpha}(r;s))$. Away from zero they are piecewise differentiable functions of s, see Kato [14], but this may no longer be true when $\lambda_i = 0$. To avoid such problems near the point of accumulation, we perform the above expansion for $Z_m^{(\varepsilon)}(K)$, defined by restricting the product over eigenvalues to $|\lambda_i(K)| > \varepsilon > 0$. The expansion above, for $Z_m^{(\varepsilon)}(K(\phi_n))$, is then certainly valid, and since the final bound is independent of ε, and since $Z_n^{(\varepsilon)}(K) \to Z_m(K)$ as $\varepsilon \to 0$, the bound applies also to $Z_m(K(\phi_n))$.

Proof of Lemma 2.3. Since the eigenvalues are continuous functions of s and are isolated for $|\lambda_i(s)| > \varepsilon$, the number of terms in the sum is piecewise constant.

Also the eigenvalues $\lambda_i(s)$, $|\lambda_i(s)| > \varepsilon$, and a corresponding set of orthonormal eigenfunctions $\phi_i(s)$ are piecewise differentiable (see Kato [14]). Differentiating the eigenvalue equation:

$$A(s)^3 \phi_i(s) \equiv \lambda_i(s)^3 \phi_i(s) ,$$

and taking the scalar product with $\phi_i(s)$ we obtain:

$$\partial \lambda_i(s)^3 / \partial s = (\phi_i(s), (\partial A(s)^3 / \partial s) \phi_i(s)) ,$$

for almost all s. Taking absolute values and summing over i with $|\lambda_i(s)| > \varepsilon$, the result follows, noting that for B trace-class:

$$\sum_{i=1}^{\infty} |\psi_i, B\psi_i)| \leqq \|B\|_1$$

for any orthonormal system ψ_i. Obviously Lemma 2.3 applies also to any differentiable function $f(\cdot)$ of the eigenvalues.

Proof of Lemma 2.4. As in [3], formula (A.7), we have:

$$E_A(r, s) \equiv \tfrac{1}{2} \mathrm{Tr}(K_{|A|}(r; s)^* + K_{|A|}(r; s))^2$$
$$= \int d^2k (F_{\mathrm{reg}}(k) + G_{\mathrm{reg}}(k)) |\phi_A(r, s)^\sim(k)|^2 ,$$

$$\phi_A(r, s)(x) \equiv \sum_{\alpha \in A} g_\alpha(x) \phi(r_\alpha, s^{(\alpha)})(x) ,$$

$$\phi(r_\alpha, s^{(\alpha)})(x) \equiv \sum_{i=1}^{r_\alpha} (s_{i-1}^{(\alpha)} - s_i^{(\alpha)}) \phi_i + s_{r_\alpha}^{(\alpha)} \phi_{r_\alpha + 1} .$$

and where $F_{\mathrm{reg}}(k) + G_{\mathrm{reg}}(k) \leqq$ const. Thus

$$E_A(r, s) \leqq \int d^2x \phi_A(r, s)^2 (x) = \sum_{\alpha \in A} \int d^2x g_\alpha^2(x) \phi(r_\alpha, s^{(\alpha)})^2 (x) .$$

Since $\langle \phi_i^2(x) \rangle$ is an increasing function of i and since

$$s_{i-1}^{(\alpha)} - s_i^{(\alpha)} \geqq 0, \quad \sum_{i=1}^{r_\alpha} (s_{i-1}^{(\alpha)} - s_i^{(\alpha)}) + s_{r_\alpha}^{(\alpha)} = 1 ,$$

it follows that

$$\langle \phi(r_\alpha, s^{(\alpha)})^2 (x) \rangle \leqq \langle \phi_{r_\alpha + 1}^2 (x) \rangle \leqq \text{const} \ln(1 + \kappa_{r_\alpha + 1} / \mu_0) .$$

This proves Lemma 2.4 because $\int d^2x g_\alpha^2(x) \leqq 1$.

Proof of Lemma 2.5. The proof of this lemma involves four types of basic estimates to control number divergences, boson localization, fermion localization and convergence in momentum cutoffs. The estimates used are combinatorial arguments, checkerboard estimates, exponential decay of fermi two-point functions and estimates on simple Feynman diagrams respectively. We begin by discussing the first two types of estimate, assuming results proved later for the last two estimation problems.

By definition, $R(r) = \int d^r s \prod_{\alpha \in A} \prod_{t_\alpha = 1}^{r_\alpha} P_{n,\alpha}(\ldots, r_{\alpha-1}, t_\alpha; s)$ with $P_{n,\alpha}(r; s)$ given by (2.4). We show below that for each n, α, r, s and each $d \geqq 0$ there is an upper bound of the form:

$$P_{n,\alpha}(r; s) \leqq \delta P_{n,\alpha}(r; s) S_{n,\alpha}(r; s) ,$$
$$S_{n,\alpha}(r; s) = \sum_{\beta_1, \ldots, \beta_5 \in A} \prod_{i=1}^{5} S_{n,\alpha}^{(i,\beta_i)}(r; s) (1 + |\beta_i - \alpha|)^{-d} . \tag{2.6}$$

Here $\delta P_{n,\alpha}(r;s)^6$ and $S_{n,\alpha}^{(i,\beta)}(r;s)^6$ are polynomials of degree at most $=12$ in the boson field, with $\delta P_{n,\alpha}(r;s)$ localized in Δ_α and $S_{n,\alpha}^{(i,\beta)}(r;s)$ localized in Δ_β. The quantities $\delta P_{n,\alpha}(r;s)$ are functions of $(\partial/\partial s_{r_\alpha}^{(\alpha)})K_{n,\alpha}(r;s)$ and are thus expected to be small for large r_α, while the $S_{n,\alpha}^{(i,\beta)}(r;s)$, contributions due (for $\beta \neq \alpha$) to the non-locality of the determinants, should be bounded uniformly in $n, \alpha, i, \beta, r, s$. The associated factors $(1+|\beta_i-\alpha|)^{-d}$ reflect the fact that the coupling between distant regions caused by the fermi two-point function falls off rapidly with distance. To estimate $\|R(r)\|_1 = \int d\mu_0 R(r)$, we use a Holder inequality to separate the contributions $\delta P_{n,\alpha}$ and $S_{n,\alpha}$ and then checkerboard estimates [13], on each of the two resulting terms to seperate contributions from different regions. Thus

$$\begin{aligned}
&\|R(r)\|_1 \\
&\quad \leqq \Big\| \int d^r s \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s) S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_1 \\
&\quad \leqq \int d^r s \Big\| \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_2 \Big\| \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_2 \\
&\quad \leqq \prod_\alpha (r_\alpha!)^{-1} \sup_s \prod_{\alpha\in\Lambda} \Big\| \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_{8\beta^2} \\
&\qquad \cdot \Big\| \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_2 ,
\end{aligned}$$

where $\beta = (e^{3\mu_0/4}+1)/(e^{3\mu_0/4} - e^{-5\mu_0/4})$. By hypercontractivity we have

$$\Big\| \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_{8\beta^2} \leqq ((4/3)\beta^2-1)^{6r_\alpha} \Big\| \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_{12} .$$

To estimate the norm on the right, we decompose each $\delta P_{n,\alpha}^6$ into normal ordered components $\delta P_{n,\alpha;j}(r,s)$, $0 \leqq j \leqq 12$ with momentum space kernels $\delta P_{n,\alpha;j}(r;s)[\cdot]$. We show below that there is an $\varepsilon > 0$ and a constant c_5 with

$$\|\delta P_{n,\alpha;j}(r,s)[\cdot]\|_{L_2(R^j)} \leqq c_5 \kappa_{r_\alpha}^{-\varepsilon}, \quad \text{uniformly in} \quad |\Lambda|, n, \alpha, r, s, j , \tag{2.7}$$

from which it follows that for a constant c_6:

$$\Big\| \prod_{t_\alpha=1}^{r_\alpha} \delta P_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_{12} \leqq c_6^{r_\alpha} (r_\alpha!)^6 \prod_{t_\alpha=1}^{r_\alpha} \kappa_{t_\alpha}^{-\varepsilon} . \tag{2.8}$$

Returning to (2.6), we see that to complete the proof of Lemma 2.5 we need only show the existence of constants c_7, c_8 uniform in $|\Lambda|, n, \alpha, r, s$ such that

$$\Big\| \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_2 \leqq c_7^{\sum_\alpha r_\alpha} \prod_\alpha (r_\alpha!)^{c_8} .$$

Introducing the decomposition (2.5) into local parts for each $S_{n,\alpha}$:

$$\begin{aligned}
&\Big\| \prod_{\alpha\in\Lambda} \prod_{t_\alpha=1}^{r_\alpha} S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\Big\|_2 \\
&\quad \leqq \sum_{\beta(\cdot)} \Big\| \prod_{v\in M} S_{n,v}^{\beta(\cdot)}(r;s)(1+|\beta(v)-\alpha(v)|)^{-d}\Big\|_2 ,
\end{aligned}$$

where $M = M(r) \equiv \{v = (\alpha, t, i) | \alpha\in\Lambda, 1 \leqq t \leqq r_\alpha, 1 \leqq i \leqq 5\}$, the sum running over all functions $\beta(\cdot): M \to \Lambda$, and where for $v = (\alpha(v), t(v), i(v)) \in M$ we define $S_{n,v}^{\beta(\cdot)}(r;s) \equiv S_{n,\alpha(v)}^{(i(v),\beta(v))}(\dots, r_{\alpha(v)-1}, t(v); s)$. For each function $\beta(\cdot)$ let $N_\gamma(\beta) = |\{v\in M | \beta(v) = \gamma\}|$. Then by a checkerboard estimate and hypercontractivity,

$$\begin{aligned}
&\Big\| \prod_{v\in M} S_{n,v}^{\beta(\cdot)}(r;s)(1+|\beta(v)-\alpha(v)|)^{-d}\Big\|_2 \\
&\quad \leqq \prod_{\gamma\in\Lambda} ((4/3)\beta^2-1)^{6N_\gamma(\beta)} \Big\| \prod_{v:\beta(v)=\gamma} S_{n,v}^{\beta(\cdot)}(r;s)(1+|\gamma-\alpha(v)|)^{-d}\Big\|_{12} .
\end{aligned}$$

As with $\delta P_{n,\alpha}(r;s)$, we decompose each $S^{(i,\beta)}_{n,\alpha}(r;s)^6$ into normal ordered components $S^{(i,\beta)}_{n,\alpha;j}(r;s)$, $0 \leqq j \leqq 12$, with momentum space kernels $S^{(i,\beta)}_{n,\alpha;j}(r;s)[\cdot]$, and we show below that for a constant c_9:

$$\|S^{(i,\beta)}_{n,\alpha;j}(r;s)[\cdot]\|_{L^2(R^J)} \leqq c_9, \quad \text{uniformly in} \quad |\Lambda|, n, \alpha, i, \beta, j, r, s. \tag{2.9}$$

It follows that there is a constant c_{10} with

$$\|\prod_{v:\beta(v)=\gamma} S^{\beta(\cdot)}_{n,v}(r;s)\|_{12} \leqq c_{10}^{N_\gamma(\beta)} N_\gamma(\beta)!^6,$$

and thus for a constant c_{11}:

$$\begin{aligned}&\|\prod_{\alpha\in\Lambda}\prod_{t_\alpha=1}^{r_\alpha} S_{n,\alpha}(\dots, r_{\alpha-1}, t_\alpha; s)\|_2 \\ &\leqq c_{11}^{\Sigma_\alpha r_\alpha} \sum_{\beta(\cdot)} \prod_{\gamma\in\Lambda} (N_\gamma(\beta)!)^6 \prod_{v:\beta(v)=\gamma} (1+|\gamma-\alpha(v)|)^{-d}.\end{aligned}$$

The required bound (2.8), completing the proof of Lemma 2.5, is now the result of a purely combinatorial estimate, the proof of which we relegate to the appendix (Lemma A.1):

$$\sum_{\beta(\cdot)\in\Lambda^{M(r)}} \prod_{\gamma\in\Lambda} \{N_\gamma(\beta)!^6 \prod_{\substack{v\in M(r):\\ \beta(v)=\gamma}} (1+|\gamma-\alpha(v)|)^{-d}\} \leqq (c^{5\Sigma_\alpha r_\alpha} \prod_\alpha (5r_\alpha)!^2)^6, \tag{2.10}$$

with c uniform in $|\Lambda|$ and $r = \{r_\alpha, \alpha\in\Lambda\}$, provided $d > 24$.

Finally we must demonstrate the validity of the decomposition (2.5) of $P_{n,\alpha}(r;s)$ into local parts and prove the uniform bounds (2.7) and (2.9). These bounds for $P_{n,\alpha}(r;s)$ will follow if we prove similar bounds for each term $P_{m;n,\alpha}(r;s)$ in an upper bound for $P_{n,\alpha}(r;s)$ of the form:

$$P_{n,\alpha}(r;s) \leqq \sum_{m=1}^{M} P_{m;n,\alpha}(r;s), \qquad P_{m;n,\alpha}(r;s) \geqq 0, \tag{2.11}$$

since $\delta P_{n,\alpha}(r;s) \equiv (\sum_m \delta P_{m;n,\alpha}(r;s)^6)^{1/6}$ and $S^{(i,\beta)}_{n,\alpha}(r;s) \equiv (\sum_m S^{(i,\beta)}_{m;n,\alpha}(r;s)^6)^{1/6}$ then certainly satisfy (2.5), (2.7), (2.9). We now demonstrate a bound of the form (2.11) with $M = 118$.

From (2.4) and $|\mathrm{Tr}\, B| \leqq \|B\|_1$, we have

$$\begin{aligned}&P_{n,\alpha}(r,s) \\ &\quad\leqq e^2 p\{(\lambda^2/2)|:(\partial/\partial s^{(\alpha)}_{r_\alpha})\mathrm{Tr}(K_{n,\alpha}(r;s)^* + K_{n,\alpha}(r;s))^2:| \\ &\qquad + \lambda^3 \|(\partial/\partial s^{(\alpha)}_{r_\alpha}) K^*_{n,\alpha}(r;s) K_{n,\alpha}(r;s))^2\|_1 \\ &\qquad + \lambda^4 \|(\partial/\partial s^{(\alpha)}_{r_\alpha})(K^*_{n,\alpha}(r;s) K_{n,\alpha}(r;s))^2\|_1 + \|(\partial/\partial s^{(\alpha)}_{r_\alpha}) A_{n,\alpha}(r,s)^3\|_1\} \\ &\quad\equiv e^2 p\{A_1 + A_2 + A_3 + A_4\}.\end{aligned} \tag{2.12}$$

By calculating the indicated trace we find

$$A_1 = \lambda^2 |:\int d^2k E(k) |\phi_{n,\alpha}(r;s)\tilde{}(k)|^2:|,$$

where $\phi_{n,\alpha}(r,s)$ is defined in terms of $\phi^{(\beta)}_r \equiv \phi_r g_\beta$ by the same interpolation formulas used to define $K_{n,\alpha}(r;s)$ in terms of $K^{(\beta)}_r$, and

$$\begin{aligned}E(k) \equiv &\int d^2p\, \omega(p+k/2)^{-2} \omega(p-k/2)^{-2} \\ &\cdot \{\omega(p+k/2)\omega(p-k/2) - (p+k/2)\cdot(p-k/2) + m_0^2\}.\end{aligned} \tag{2.13}$$

Seiler [3] has studied $E(k)$ and shows that asymptotically: $E(k) = 4\pi \ln 2 + 0(m_0/|k|)$.

Thus defining $e(k)=E(k)-4\pi\ln 2$, and introducing the derivative $\delta\phi^{(\alpha)}_{n,r_\alpha}\equiv(\partial/\partial s^{(\alpha)}_{r_\alpha})\phi_{n,\alpha}(r;s)$, we have

$$\begin{aligned}A_1 &\leqq 8\pi\lambda^2\ln 2|:\int d^2x\delta\phi^{(\alpha)}_{n,r_\alpha}(x)\phi_{n,\alpha}(r;s)(x):| \\ &\quad +2\lambda^2|\int d^2ke(k)\delta\phi^{(\alpha)\sim}_{n,r_\alpha}(-k)\phi_{n,\alpha}(r,s)^\sim(k)| \\ &\quad +2\lambda^2|\int d^2ke(k)\langle\delta\phi^{(\alpha)\sim}_{n,r_\alpha}(-k)\phi_{n,\alpha}(r,s)^\sim(k)\rangle| \\ &\equiv\sum_{m=1}^3 P_{m;n,\alpha}(r;s).\end{aligned}$$

We define $\delta P_{1;n,\alpha}(r;s)=P_{1;n,\alpha}(r;s)$ and $S^{(i,\beta)}_{1;n,\alpha}(r;s)=\delta_{\alpha\beta}$. To treat $P_{2;n,\alpha}$ we introduce $\eta\in C_0^\infty(R^2)$, $0\leqq\eta\leqq 1$, $\eta=1$ on Δ_0, $\operatorname{suppt}\eta\in\bigotimes^2[-5/8,5/8]$, and we define $\eta_\alpha(\cdot)=\eta(\cdot-\alpha)$. Expanding $\phi_{n,\alpha}(r,s)$ into local contributions $\phi^{(\beta)}_{n,\alpha}(r,s)\equiv(\phi_{n,\alpha}(r,s)g_\beta)$ we obtain:

$$\begin{aligned}P_{2;n,\alpha}(r;s)&\leqq 2\lambda^2\sum_\beta|\int d^2l_1d^2l_2F_{\alpha,\beta}(l_1,l_2)\delta\phi^{(\alpha)\sim}_{n,r_\alpha}(l_1)\phi^{(\beta)}_{n,\alpha}(r;s)^\sim(l_2)|,\\ F_{\alpha,\beta}(l_1,l_2)&\equiv(2\pi)^{-2}\int d^2ke(k)\tilde\eta_\alpha(-k-l_1)\tilde\eta_\beta(k-l_2).\end{aligned}\tag{2.14}$$

From the representation (2.13) for $E(k)$, we see that $e(k)$ and its derivatives to arbitrary order are bounded by $\operatorname{const}\omega(k)^{-1}$ and so, integrating by parts,

$$|F_{\alpha,\beta}(l_1,l_2)|\leqq c(d)(1+|\alpha-\beta|)^{-2d}\omega(l_1)^{-1/4}\omega(l_2)^{-1/4}\omega(l_1+l_2)^{-3/2},\quad\text{and}\quad d\geqq 0.$$

Using a Schwarz inequality in (2.14) it follows that a bound for $P_{2;n,\alpha}(r;s)$ of the form (2.5) is given by choosing $\delta P_{2;n,\alpha}(r;s)\equiv(\int d^2l\omega(l)^{-1/2}|\delta\phi^{(\alpha)\sim}_{n,r_\alpha}(l)|^2)^{1/2}$, $S^{(1,\beta)}_{2;n,\alpha}(r;s)\equiv\operatorname{const}(\int d^2l\omega(l)^{-1/2}|\phi^{(\beta)}_{n,\alpha}(r,s)^\sim(l)|^2)^{1/2}$ and $S^{(i,\beta)}_{2;n,\alpha}(r;s)=\delta_{\alpha\beta}$, $i>1$. Similarly for $P_{3;n,\alpha}(r;s)$ we take

$$\begin{aligned}\delta P_{3;n,\alpha}(r;s)&=(\int d^2l\omega(l)^{-1/2}\langle|\delta\phi^{(\alpha)\sim}_{n,r_\alpha}(l)|^2\rangle)^{1/2},\\ S^{(1,\beta)}_{3;n,\alpha}(r;s)&=\operatorname{const}(\int dl\omega(l)^{-1/2}\langle|\phi^{(\beta)}_{n,\alpha}(r;s)^\sim(l)|^2\rangle)^{1/2}\end{aligned}$$

and $S^{(i,\beta)}_{3;n,}(r;s)=\delta_{\alpha\beta}$, $i>1$.

To study the remaining terms A_2,A_3,A_4 in (2.12) we define $\delta K_\alpha=(\partial/\partial s^{(\alpha)}_{r_\alpha})K_{n,\alpha}(r;s)$ and we introduce the expansion $K_{n,\alpha}(r;s)=\sum_\beta K_\beta$. where $K_\beta=K^{(\beta)}_{n,\alpha}(r;s)\equiv K(\phi^{(\beta)}_{n,\alpha}(r;s))$. We have suppressed r,s, and the subscripts n,α in our notation. Differentiating within the trace norms and using the triangle inequality we obtain

$$\sum_{i=2}^4 A_i\leqq\sum_{l=2}^5\lambda^{l+1}\sum_{j_l=1}^{M_l}\sum_{\beta_1,\dots,\beta_l\in\Lambda}T^{m(l,j_l)}_{\alpha;\beta_1,\dots,\beta_l},\qquad m(l,j_l)=j_l+3+\sum_{i=1}^{l-1}M_i,$$

where $M_2=27$, $M_3=52$, $M_4=30$, and $M_5=6$. A typical term $T^m_{\alpha;\beta_1,\dots,\beta_l}$ has the form

$$T^m_{\alpha;\beta_1,\dots,\beta_l}=\|K^\#_{\beta_1}\dots K^\#_{\beta_{r-1}}\delta K^\#_\alpha K^\#_{\beta_r}\dots K^\#_{\beta_l}\|_1,\qquad 1\leqq r<l,$$

where $K^\#_\beta$ denotes either K_β or K^*_β, and is thus a function of boson fields localized in regions $\beta_1,\dots\beta_l$. Unless all of the β_i are close to α, we expect $T^m_{\alpha;\beta_1,\dots,\beta_l}$ to be small. To bound $T^m_{\alpha;\beta_1,\dots,\beta_l}$ we note that $K_\beta=Q_\beta L_\beta P_\beta$, where Q_β,P_β,L_β, are the bounded operators on $\mathscr{H}$ with kernels:

$$\begin{aligned}Q_\beta(p,q)&=(2\pi)^{-1}(\not p+m_0)(p^2+m_0^2)^{-1}\tilde\eta_\beta(p-q),\\ P_\beta(p,q)&=(2\pi)^{-1}\tilde\eta_\beta(p-q)(q^2+m_0^2)^{-1/2},\\ L_\beta(p,q)&=(2\pi)^{-1}(p^2+m_0^2)^{-1/2}(\phi^{(\beta)}_{n,\alpha}(r;s))^\sim(p-q)(q^2+m_0^2)^{-1/2}.\end{aligned}\tag{2.15}$$

Denoting by $R_\beta^\#$ one of the operators $P_\beta, P_\beta^*, Q_\beta, Q_\beta^*$, we have

$$\begin{aligned} T^m_{\alpha;\beta_1,\dots,\beta_l} &= \|L_{\beta_1}R^\#_{\beta_1}R^\#_{\beta_2}\dots L_{\beta_{r-1}}R^\#_{\beta_{r-1}}R^\#_\alpha \delta L_\alpha R^\#_\alpha R^\#_{\beta_r}L_{\beta_r}\dots R^\#_{\beta_{l-1}}R^\#_{\beta_l}L_{\beta_l}\|_1 \\ &\leqq \|\delta L_\alpha\|_3 \prod_{i=1}^l \|L_{\beta_i}\|_3 \|R^\#_{\beta_1}R^\#_{\beta_2}\| \dots \|R^\#_{\beta_{r-1}}R^\#_\alpha\| \, \|R^\#_\alpha R^\#_{\beta_r}\| \dots \|R^\#_{\beta_{l-1}}R^\#_{\beta_l}\| \,, \end{aligned} \tag{2.16}$$

where we have used the Holder inequality for the trace norm, the monotone decrease of $\| \ \|_p$ in p and $\|AB\|_p \leqq \|A\| \, \|B\|_p$. By an elementary interpolation estimate [15, 16],

$$\|L_\beta\|_3 \leqq \|L_\beta^{(11/12)}\|_4^{2/3} \|L_\beta^{(7/6)}\|_2^{1/3} \,,$$

where $L_\beta^{(a)}$ is the operator obtained by replacing $(p^2+m_0^2)^{-1/2}$ with $(p^2+m_0^2)^{-a/2}$ in (2.15). We now define, for $4 \leqq m \leqq 118$,

$$\begin{aligned} \delta P_{m;n,\alpha}(r;s) &= \|\delta L_\alpha^{(11/12)}\|_4^{2/3} \|\delta L_\alpha^{(7/6)}\|_2^{1/3} \,, \\ S^{(i\ \beta)}_{m;n,\alpha}(r;s) &= \text{const}\, \|L_\beta^{(11/12)}\|_4^{2/3} \|L_\beta^{(7/6)}\|_2^{1/3} \,. \end{aligned} \tag{2.17}$$

The bounds (2.5) for $P_{m;n,\alpha}(r;s)$ follow from (2.16), (2.17), an estimate on the norms $\|R^\#_\beta R^\#_\gamma\|$ occuring in (2.16):

$$\|R^\#_\beta R^\#_\gamma\| \leqq c_1(d)(1+|\beta-\gamma|)^{-d}, \quad \text{any} \quad d \geqq 0 \,, \tag{2.18}$$

and the triangle inequality in the form

$$\prod_{i=1}^r (1+|\beta_i-\beta_{i-1}|)^{-rd} \leqq c_2(r) \prod_{i=1}^r (1+|\beta_i-\alpha|)^{-d}, \quad \text{if} \quad \beta_0 \equiv \alpha \,.$$

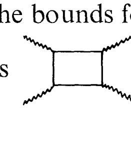
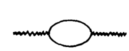
To prove (2.18), note that we may reduce the sixteen possible forms for $R^\#_\beta R^\#_\gamma$ to eight using $\|B\| = \|B^*\|$. Furthermore, since $Q_\beta = UP^*_\beta$ where U is the unitary operator on $\mathscr{H}$ with kernel $U(p,q) = (\not p + m_0)(p^2+m_0^2)^{-1}\delta(p-q)$, forms beginning with Q or ending with Q^* need not be considered, and in fact since $K = QLP$ and $K^* = P^*LQ^*$, we need only consider forms beginning with P or Q^* and ending with Q or P^*. This leaves only $P_\beta P^*_\gamma$, $Q^*_\beta Q_\gamma$, and $P_\beta Q_\gamma$, and the first two are equal since $Q^*_\beta Q_\gamma = P_\beta U^* U P^*_\gamma = P_\beta P^*_\gamma$. The corresponding kernels are:

$$\begin{aligned} P_\beta P^*_\gamma(p,q) &= (2\pi)^{-2} \int d^2l (l^2+m_0^2)^{-1/2} \tilde\eta_\beta(p-l)\tilde\eta_\gamma(l-q) \,, \\ P_\beta Q_\gamma(p,q) &= (2\pi)^{-2} \int d^2l (\not l + m_0)(l^2+m_0^2)^{-1} \tilde\eta_\beta(p-l)\tilde\eta_\gamma(l-q) \,. \end{aligned}$$

The bounds (2.18) now follow immediately by using an $L_1 - L_\infty$ norm estimate on the kernels and the fact that $(l^2+m_0^2)^{-1/2}$ and $(\not l + m_0)/(l^2+m_0^2)^{-1}$ have Fourier transforms decaying exponentially.

Finally we prove the bounds (2.7), (2.9) for the momentum space kernels $\delta P_{m;n,\alpha;j}(r;s)[\cdot]$, $S^{(i,\beta)}_{m;n,\alpha;j}(r;s)[\cdot]$ in the Wick expansion for $\delta P_{m;n,\alpha}(r;s)^6$ and $S^{(i,\beta)}_{m;n,\alpha}(r;s)^6$. By convexity in $s^{(\beta)}$ where $s^{(\beta)}$ satisfy $s^{(\beta)}_{i-1} - s^{(\beta)}_i \geqq 0$, $\sum_{i=1}^{r_\beta} (s^{(\beta)}_{i-1} - s^{(\beta)}_i) + s^{(\beta)}_{r_\beta} = 1$, it suffices to prove the bounds for a given r_α or r_β, without interpolations. For $P_{1;n,\alpha}$ these bounds follow from the convergence in $L_2(d\mu_0)$ of $:\int d^2x g_\beta(x)\phi_\kappa^2(x):$ as $\kappa \to \infty$ while for $P_{2;n,\alpha}$, $P_{3;n,\alpha}$ they follow from the convergence of

$$\int d^2l \omega(l)^{-1/2} |(\phi_\kappa g_\beta)\tilde{}(l)|^2$$

in $L_2(d\mu_0)$ as $\kappa \to \infty$. For $P_{m;n,\alpha}$, $4 \leqq m \leqq 118$, the bounds follow, by (2.17), from the convergence in $L_2(d\mu_0)$ of the Feynman graphs  and , where

〰〰 denotes a momentum cutoff boson field, —— denotes a propagator $(p^2+m_0^2)^{-11/12}$ in the first graph or $(p^2+m_0^2)^{-7/6}$ in the second graph, and the vertices are functions $g_\beta(\cdot)$ localized in a unit volume Δ_β. The bounds are uniform in the spacecutoff g provided g belongs to one of the classes $G(\gamma)$ defined in Lemma 2.1. This completes the proof of Lemma 2.5.

III. Lower Bounds for $Z^{(A)}$

In this section we assume that the space-time cutoff is of the Hamiltonian form $g(x)=\chi_{[0,t]}(x_0)\chi_{[0,l]}(x_1)$. Then the Feynman-Kac formula of Osterwalder and Schrader [4] implies that

$$Z_{t,l}\equiv Z^{(A)}=\langle e^{-V(A)}\rangle = e^{W(l)+T(t,l)}\langle \Omega_0, e^{-tH_l}\Omega_0\rangle\,, \tag{3.1}$$

where Ω_0 is the vacuum for the relativistic Fock space and $H_l=H(\chi_{[0,l]})$. The constants W, the wavefunction renormalization in second order, and T appear because the Euclidean counter-terms differ slightly from the usual Hamiltonian counter-terms; specifically

$$\begin{aligned} W(l)&=\langle H_{I,l}H_0^{-2}H_{I,l}\rangle_{\Omega_0}\,,\\ T(t,l)&=-\langle H_{I,l}H_0^{-2}e^{-tH_0}H_{I,l}\rangle_{\Omega_0}\,,\end{aligned} \tag{3.2}$$

where $H_{I,l}=\lambda\int_0^l dx{:}\bar\psi\psi\phi(x){:}$ is the unrenormalized interaction Hamiltonian. To prove (3.1), we introduce momentum cutoffs for the boson and fermi fields, in the space direction only. By the momentum cutoff Feynman-Kac formula [4], (3.1) is then valid. Glimm and Jaffe [7] have shown that the cutoff Hamiltonians converge to H_l, in the sense of resolvents, as the cutoffs are removed. Furthermore by results of Seiler [3] and McBryan [15], $Z^{(A)}$ is also the limit of the corresponding cutoff quantities. The identity (3.1) follows immediately.

On the right of (3.1) we apply the bound:

$$\langle\Omega_0, e^{-tH_l}\Omega_0\rangle \geqq \langle\Omega_0, e^{-H_l}\Omega_0\rangle^t\,,$$

and, using the Feynman-Kac formula again in reverse, obtain:

$$\begin{aligned} Z_{t,l}&\geqq e^{(1-t)W(l)\,+\,T(t,l)-tT(1,l)}Z_{1,l}^t\,,\\ &\geqq e^{(1-t)W(l)}Z_{l,1}^t\,,\end{aligned} \tag{3.3}$$

where in the second step we have used the Euclidean invariance of $Z_{t,l}$ and that $T(t,l)$ is negative and monotone increasing in t [see (3.2)]. We may apply (3.3) again to the right side of (3.3), obtaining:

$$Z_{t,l}\geqq e^{(1-t)W(l)+(1-l)W(1)}Z_{1,1}^{tl}\,.$$

By an elementary calculation we see that $W(l)\leqq \text{const}\, l$ and thus

$$Z_{t,l}\geqq e^{-\text{const}\,tl}Z_{1,1}^{tl}=(\text{const})^{tl}$$

which completes the proof of Theorem 1.2.

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Appendix

We now prove the bound (2.10) which controls the number divergence and fermion localization in Lemma 2.5. Let Λ be any finite subset of Z^2, let $r=\{r_\alpha:\alpha\in\Lambda, r_\alpha\in Z^+, r_\alpha\geqq 1\}$, let $l\geqq 1\in Z^+$, and define $M_\Lambda[r;l]\equiv\{v=(\alpha,t,i):\alpha\in\Lambda, 1\leqq t\leqq r_\alpha, 1\leqq i\leqq l\}$. For $v=(\alpha,t,i)\in M_\Lambda[r;l]$, define $\alpha(v)=\alpha$. Let $\Lambda^{M_\Lambda[r;l]}$ denote the set of functions $\beta(\cdot):M_\Lambda[r;l]\to\Lambda$, and for $\beta(\cdot)\in\Lambda^{M_\Lambda[r;l]}$, $\gamma\in\Lambda$, we define $N_\gamma(\beta)=|\{v\in M_\Lambda[r;l]:\beta(v)=\gamma\}|$.

Lemma A.1. *There is a constant C independent of $|\Lambda|, r, l$ such that for $a\geqq 1$ and $d>4a$:*

$$\sum_{\beta(\cdot)}\prod_\gamma\{N_\gamma(\beta)!^a\prod_{v:\beta(v)=\gamma}(1+|\gamma-\alpha(v)|)^{-d}\}\leqq(C^{l\Sigma_\alpha r_\alpha}\prod_\alpha(lr_\alpha)!^2)^a$$

where the sum is over all $\beta(\cdot)\in\Lambda^{M_\Lambda[r;l]}$, v ranges over $M_\Lambda[r;l]$ and α,γ range over Λ.

Proof. It is sufficient to consider the case $a=1$ since for $a>1$:

$$\begin{aligned}&\sum_{\beta(\cdot)}\prod_\gamma\{N_\gamma(\beta)!^a\prod_{v:\beta(v)=\gamma}(1+|\gamma-\alpha(v)|)^{-ad}\}\\ &\quad\leqq[\sum_{\beta(\cdot)}\prod_\gamma\{N_\gamma(\beta)!\prod_{v:\beta(v)=\gamma}(1+|\gamma-\alpha(v)|)^{-d}\}]^a.\end{aligned}$$

We generalize an argument of Eckmann *et al.* [17, Appendix]. Denote the sum on the left in the statement of the lemma, for $a=1$, by $S(r;l)$ and note that $S(r;l)=S(lr;1)$ where $lr\equiv\{lr_\alpha\}$. Thus it suffices to prove the result for $l=1$. and so we define $M[r]\equiv M_\Lambda[r;1]$, $S(r)\equiv S(r;1)$. For any function $\beta(\cdot)\in\Lambda^{M[r]}$ define numbers $q_{\alpha\gamma}(\beta)=|\{v\in M[r]:\alpha(v)=\alpha, \beta(v)=\gamma\}|$. The terms in the sum $S(r)$ are uniquely determined by the $q_{\alpha\gamma}(\beta)$ and thus we may replace the sum over $\beta(\cdot)$ by a sum over functions $q(\cdot,\cdot)\in Q[r]\equiv\{q(\cdot,\cdot)\in Z^{+\Lambda\times\Lambda}:\sum_\gamma q(\alpha,\gamma)=r_\alpha, \alpha\in\Lambda\}$. For each $q(\cdot,\cdot)\in Q[r]$ we define $N_\gamma(q)=\sum_\alpha q(\alpha,\gamma)$, $\gamma\in\Lambda$ and thus $\sum_\gamma N_\gamma(q)=\sum_\alpha r_\alpha$. The sum $S(r)$ may now be expressed as

$$S(r)=\sum_{N(\cdot)\in N[r]}\prod_\gamma N(\gamma)!\sum_{\substack{q(\cdot,\cdot)\in Q[r]:\\ N_\gamma(q)=N(\gamma)}}\prod_{\alpha,\gamma}(1+|\alpha-\gamma|)^{-dq(\alpha,\gamma)},$$

where $N[r]=\{N(\cdot)\in Z^{+\Lambda}:\sum_\gamma N(\gamma)=N\equiv\sum_\alpha r_\alpha\}$. Using the inequality $\prod_{\alpha,\gamma}q(\alpha,\gamma)!=\prod_\alpha\{\prod_\gamma q(\alpha,\gamma)!\}\leqq\prod_\alpha r_\alpha!$, we have

$$\begin{aligned}S(r)&\leqq\prod_\alpha r_\alpha!^2\sum_{N(\cdot)}\prod_\gamma N(\gamma)!\sum_{\substack{q(\cdot,\cdot)\in Q[r]:\\ N_\gamma(q)=N_\gamma}}\prod_{\alpha,\gamma}q(\alpha,\gamma)!^{-2}(1+|\alpha-\gamma|)^{-dq(\alpha,\gamma)},\\ &\leqq\prod_\alpha r_\alpha!^2\sum_{N(\cdot)}\prod_\gamma\{N(\gamma)!\sum_{q(\cdot)\in P[N(\gamma)]}\prod_\alpha q_\alpha!^{-2}(1+|\alpha-\gamma|)^{-dq_\alpha}\},\end{aligned}\quad\text{(A.1)}$$

where $P[n]\equiv\{q(\cdot)|q(\cdot):Z^+\to Z^+$ and $\sum_i q(i)=n\}$, and by q_α we mean $q_{m(\alpha)}$ where $m:\Lambda\to Z^+$ is some ordering of the points of Λ. Denote the expression in brackets in (A.1) by $S_\gamma(r,N(\gamma))$. For fixed γ there is an arrangement of the α for which $|\alpha-\gamma|\geqq\frac{1}{2}(\alpha^{1/2}-3)$ and thus

$$\begin{aligned}S_\gamma(r,n)&\leqq\sum_{q(\cdot)\in P[n]}n!\prod_\alpha q_\alpha!^{-2}((1+\alpha^{1/2})/4)^{-dq_\alpha}\\ &\leqq n!^{-1}(\sum_{q(\cdot)\in P[n]}n!\prod_\alpha q_\alpha!^{-1}((1+\alpha^{1/2})/4)^{-dq_\alpha/2})^2\\ &\leqq n!^{-1}(\sum_{i\in Z^+}((1+i^{1/2})/4)^{-d/2})^{2n}=n!^{-1}C_1(d)^n,\end{aligned}$$

with $C_1(d)$ finite for $d>4$. Returning to (A.1) we thus obtain:

$$\begin{aligned} S(r) &\leqq \prod_\alpha r_\alpha!^2 \sum_{N(\cdot)\in N[r]} \prod_\gamma N(\gamma)!^{-1} C_1^{N(\gamma)} \\ &= C_1^{\sum_\alpha r_\alpha} \prod_\alpha r_\alpha!^2 N!^{-1} \sum_{N(\cdot)\in N[r]} N! \prod_\gamma N(\gamma)!^{-1} \\ &= C_1^{\sum_\alpha r_\alpha} \prod_\alpha r_\alpha!^2 N!^{-1} |\Lambda|^N, \qquad N=\sum_\alpha r_\alpha \geqq |\Lambda|\,. \end{aligned}$$

But $N!\geqq N^N e^{-N}$, and thus $S(r)\leqq C^{\Sigma r_\alpha} \prod_\alpha r_\alpha!^2$, $C=eC_1$.

References

1. Matthews, P. T., Salam, A.: Nuovo Cimento **12**, 563—565 (1954)
2. Matthews, P. T., Salem, A.: Nuovo Cimento **2**, 120—134 (1955)
3. Seiler, E.: Schwinger functions for the Yukawa model in two dimensions with space-time cutoff, I.A.S. preprint
4. Osterwalder, K., Schrader, R.: Helv. Phys. Acta **46**, 277—302 (1973)
5. Glimm, J.: Commun. math. Phys. **5**, 343—386 (1967)
6. Glimm, J.: Commun. math. Phys. **6**, 120—127 (1967)
7. Glimm, J., Jaffe, A.: Ann. Phys. **60**, 321—383 (1970)
8. Glimm, J., Jaffe, A.: J. Funct. Anal. **7**, 323—357 (1971)
9. Schrader, R.: Ann. Phys. **70**, 412—457 (1972)
10. Brydges, D.: Boundedness below for fermion model theories, Part II. The linear lower bound. University of Michigan preprint
11. McBryan, O., Park, Y. M.: J. Math. Phys. **16**, 104—110 (1975)
12. Dunford, N., Schwartz, J.: Linear Operators, Parts I, II. New York: Interscience Publishers
13. Guerra, F., Rosen, L., Simon, B.: The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics. Princeton University preprint
14. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966
15. McBryan, O.: Finite mass renormalizations in the Euclidean Yukawa$_2$ field theory. Commun. math. Phys., to appear
16. Seiler, E., Simon, B.: On finite mass renormalizations in the two dimensional Yukawa model, manuscript
17. Eckmann, J.-P., Magnen, J., Seneor, R.: Commun. math. Phys. **39**, 251—271 (1975)

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