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On the Free Boson Gas with Spin

R. H. Critchley and J. T. Lewis

Dublin Institute for Advanced Studies, Dublin 4, Ireland

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Abstract. The generating functionals of the grand canonical and canonical thermodynamic equilibrium states of several models of free bosons with spin are calculated and the properties of the states discussed. In particular the distribution of condensate over the degenerate ground state is described, and it is shown that spinning bosons interacting with a magnetic field exhibit spontaneous magnetization at sufficiently low temperatures.

§ 1. Introduction and Summary of Results

Cannon [1], Lewis and Pulè [2] have rigorously studied the thermodynamic limit of the canonical and grand canonical equilibrium states of the free boson gas at arbitrary density and temperature. We extend their methods to three models in which the free bosons have spin ν. The aim is to prove the existence of Bose-Einstein condensation, and to describe its nature. In particular we look at the fluctuations in the density of condensate, comparing the canonical and grand canonical ensembles, and we investigate a suggestion of Buckingham [4] and Blatt [3] that a condensed gas of spinning bosons with a magnetic moment would behave like a ferromagnet.

In § 2 we discuss the basic model of free spinning bosons in both the grand canonical ensemble (g.c.e.) and the canonical ensemble (c.e.). It is similar to the case $\nu=0$. There is a critical density ϱ_c, and for densities greater than this critical value Bose-Einstein condensation occurs, the condensate being equally distributed over the $2\nu+1$ zero energy states. In both ensembles the density of condensate in each of the condensed states fluctuates (more so in the g.c.e. than in the c.e.), but only in the canonical ensemble do these fluctuations cancel out, so that the total condensate density does not fluctuate [see Eqs. (2.13)–(2.17)].

The spin plays a very subsidiary role in this model. Essentially it provides a means of achieving a degenerate ground state. We could have considered a non-interacting system of $2\nu+1$ different species of particles with identical masses (and so identical energy spectra) and arrived at the same answer; or, with only a slight modification, we could have treated the case where the different species have different masses. Qualitatively the result would be unchanged.

In the model considered in § 3 the spin plays a more significant role; the various spin states (or, equivalently, the various species) have different properties. In this model, in addition to demanding that the mean particle density is fixed at $\bar{\varrho}$, we also require the mean spin density to take a prescribed value $\bar{s}$. The condensation is more complicated than in the model of § 2 and its nature depends on the value

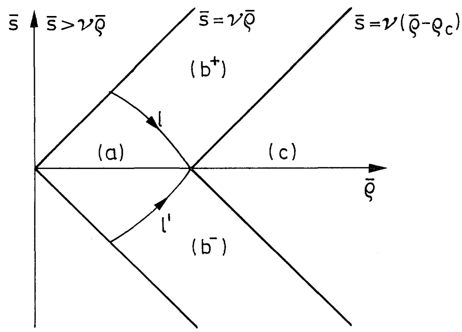


Fig. 1. The $(\bar{\varrho}, \bar{s})$ plane. The line l is given parametrically by $\bar{\varrho}=(2\pi\beta)^{-\frac{3}{2}}\{g_{\frac{3}{2}}(1)+\sum_{j=-\nu}^{\nu-1} g_{\frac{3}{2}}(y^{\nu-j})\}$, $\bar{s}=(2\pi\beta)^{-\frac{3}{2}}\{\nu g_{\frac{3}{2}}(1)+\sum_{j=-\nu}^{\nu-1} j g_{\frac{3}{2}}(y^{\nu-j})\}$; l' is its reflection

of $\bar{s}$ as well as on $\bar{\varrho}$. In fact the $(\bar{\varrho}, \bar{s})$ plane splits, as in Fig. 1, into 3 regions labelled (a), (b), and (c). For $(\bar{\varrho}, \bar{s})$ in (a), there is no condensation; for $(\bar{\varrho}, \bar{s})$ in (b) there is condensation into the state with zero energy and either spin $\nu(\bar{s}>0)$ or spin $-\nu(\bar{s}<0)$. For $(\bar{\varrho}, \bar{s})$ in (c), condensate occupies the $2\nu+1$ zero energy states and it is distributed over them in such a way that the mean spin density of the condensate is $\bar{s}$. The mean spin density of the normal component of the gas is zero. In other words the condensate is carrying all the spin (see Theorem 3).

In § 4 we consider a generalized canonical ensemble for this model in which the particle density ϱ and the spin density s are fixed. In § 5 we restrict our attention to the case $\nu=1$ in order to provide a more satisfactory comparison between the grand canonical and the canonical ensembles. In particular we can compare the distribution of condensate over the zero energy states for $(\bar{\varrho}, \bar{s})$ in the region (c) (see Fig. 2) and show that, as in the model of § 2, the density of condensate in each zero energy state fluctuates in both ensembles but the total density of condensate does not fluctuate in the canonical ensemble [see (5.4)–(5.9)].

In § 6 we study a model in which our spinning bosons interact with a constant magnetic field h. This enables us to discuss the conjecture of Buckingham and Blatt [3, 4]. Their argument was that in the condensed phase a gas of spinning bosons (with a magnetic moment associated with the spin) would exhibit spontaneous magnetization because even the slightest magnetic field would produce a splitting of the ground state levels and all the condensed particles would go into the lowest state.

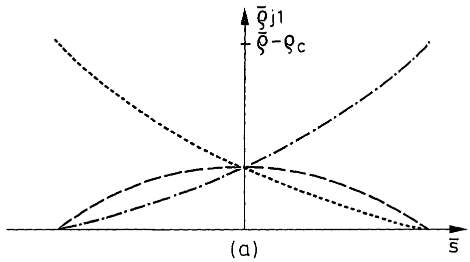
We justify this rigorously by computing the magnetization $m^*(h)$ and showing that there is a critical temperature T_c such that for $T>T_c$

$$\lim_{h\to 0^+} m^*(h)=0,$$

whereas for $T<T_c$

$$\lim_{h\to 0^+} m^*(h)\neq 0.$$

It is interesting to compare the spontaneous magnetization exhibited by this model with that shown by the usual model of a ferromagnetic system, the



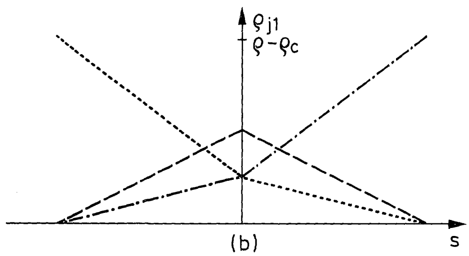


Fig. 2a and b. The distribution of condensate over the zero energy states in (a) grand canonical ensemble, (b) canonical ensemble. (— · — · — spin 1, — — — spin 0, · · · · · spin −1)

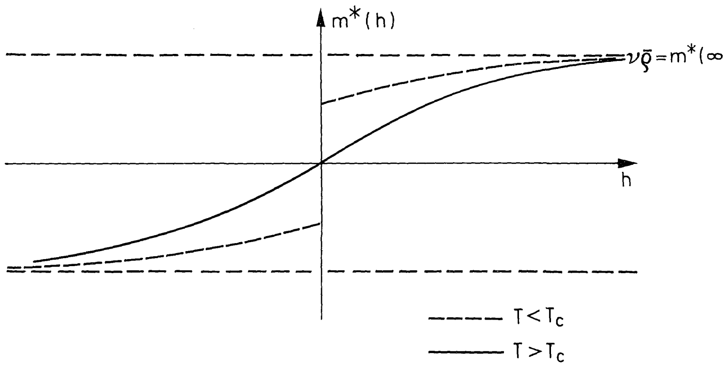


Fig. 3. Magnetization isotherms

Ising model [15]. The isotherms sketched in Fig. 3 are similar, but the cause of the spontaneous magnetization is different. In the Ising model it is a cooperative phenomenon caused by the interaction between the spins (which tends to make them align themselves) overcoming the disordering effects of the temperature. In our model this is not so, since there is no interaction between the spins. The phase transition is caused by a splitting of the degenerate ground state, as suggested by Blatt [3].

The method we use is to calculate the infinite volume limit of the generating functional of the grand canonical state using the results of [2]. But Kac [5] noted that the grand canonical state is an average of the canonical states at different densities with respect to a probability distribution. Specifically, there is a probability density function $K(\bar{\varrho}, \varrho)$ such that if $\bar{a}(\bar{\varrho})$ is the value of a quantity A in the grand canonical state, and $a(\varrho)$ is its value in the canonical state, then

$$\bar{a}(\bar{\varrho}) = \int_0^\infty K(\bar{\varrho}, \varrho) a(\varrho) d\varrho . \tag{1.1}$$

We call $K(\bar{\varrho}, \varrho)$ the Kac density. It is evaluated as follows. In finite volume Λ, take $A = e^{i\xi N/|\Lambda|}$ (N is the number operator, $|\Lambda|$ is the volume of Λ) then $a(\varrho) = e^{i\xi\varrho}$ and so the finite volume Kac density $K_\Lambda(\bar{\varrho}, \varrho)$ satisfies

$$\langle e^{i\xi N/|\Lambda|} \rangle_{g.c.e.}^{\bar{\varrho}} = \int_0^\infty K_\Lambda(\bar{\varrho}, \varrho) e^{i\xi\varrho} d\varrho = \hat{K}_\Lambda(\bar{\varrho}, \xi) \tag{1.2}$$

where $\hat{}$ denotes the Fourier transform. Thus, theoretically at least, $\hat{K}_\Lambda(\bar{\varrho}, \xi)$ can be evaluated and so we take as the Kac density the function $K(\bar{\varrho}, \varrho)$ whose Fourier transform is the thermodynamic limit of $\hat{K}_\Lambda(\bar{\varrho}, \xi)$.

We then define the canonical state by means of the grand canonical state and the Kac density, as in (1.1). Cannon [1] has shown that for the case of zero spin, this is equivalent to the usual definition of the canonical state as the thermodynamic limit of canonical Gibbs states. It seems likely that his proof will generalize to the examples of non-zero spin that we consider.

Finally we note that it is not necessary to demand that v be an integer. The only requirement is that $2v$ should be integral. Thus "bosons" with $\frac{1}{2}$-integral spin are covered by our analysis.

§ 2. Bosons with Spin

In order to study the thermodynamic limit of a system of bosons of spin v we require a sequence of finite volumes $\{\Lambda_L\}_{L \geqq 1}$ in $\mathbb{R}^3$ which increase without bound. It is well known [8, 9] that it is necessary to place restrictions on the sequence; we follow [2] and take the following definition. Let Λ_1 be a bounded region in $\mathbb{R}^3$ with unit volume, which contains the origin, and whose boundary $\partial\Lambda_1$ satisfies a regularity condition [2]. For $L \geqq 1$ define

$$\Lambda_L = \{x \in \mathbb{R}^3 : L^{-1}x \in \Lambda_1\} . \tag{2.1}$$

Let Δ_L be the Laplacian on $L^2(\Lambda_L)$ together with a boundary condition of the form $\partial\phi/\partial n + a\phi/L = 0$ and let $h_L = -\frac{1}{2}\Delta_L + C_L$ where the constant C_L is chosen so that the smallest eigenvalue of h_L is zero. The eigenvalues $\{\eta_p^L\}_{p=1}^\infty$ are ordered so that

$$0 = \eta_1^L < \eta_2^L \leqq \eta_3^L \leqq \dots .$$

It is a consequence of the definition (2.1) and the choice of boundary condition, that the eigenvalues and corresponding eigenfunctions scale:

$$\eta_p^L = L^{-2}\eta_p , \qquad \phi_p(x) = L^{-\frac{3}{2}}\phi_p(L^{-1}x)$$

where $\eta_p = \eta_p^1$ and $\phi_p = \phi_p^1$.

Let $M_{jL} = L^2(\Lambda_L)$ for each j and let $M_L = \bigoplus_{-v}^{v} M_{jL}$. The Fock space $\mathscr{F}(M_L)$ can be decomposed in the following way

$$\mathscr{F}(M_L) = \bigotimes_{-v}^{v} \mathscr{F}(M_{jL}) \tag{2.2}$$

and the Fock representation W_F of the canonical commutation relation on M_L can be likewise decomposed

$$W_F(\underline{h}) = \bigotimes_{j=-\nu}^{\nu} W_{jF}(h_j) \tag{2.3}$$

where $\underline{h} = \bigoplus_{-\nu}^{\nu} h_j$ and W_{jF} is the Fock representation on $\mathscr{F}(M_{jL})$.

We take H_L as the operator on $\mathscr{F}(M_L)$ induced by $\bigoplus_{-\nu}^{\nu} h_L$ on M_L, and N_L as the number operator on Fock space:

$$N_L = \sum_{j,p} \psi_j^*(\phi_p^L)\psi(\phi_p^L) = \sum_{j,p} n_{jp}^L \tag{2.4}$$

where $n_{jp}^L = \psi_j^*(\phi_p^L)\psi_j(\phi_p^L)$ is the number of particles with spin j in the p'th level, and ψ_j and ψ_j^* are canonical annihilation and creation operators defined from $W_{jF}(h_j)$ in the usual way (the algebraic background to this work is discussed in [1, 2, 6]).

The grand canonical equilibrium state at temperature $T = \beta^{-1}$ and density $\bar{\varrho}$ of a system of free particles in Λ_L is determined by the following density matrix on $\mathscr{F}(M_L)$

$$\sigma_L = \exp\{-\beta(H_L - \gamma_L N_L)\}/\mathrm{trace}[\exp\{-\beta(H_L - \gamma_L N_L)\}] \tag{2.5}$$

where γ_L is a Lagrange multiplier (the chemical potential) determined by the constraint

$$L^{-3}\langle N_L\rangle \equiv L^{-3}\mathrm{trace}(\sigma_L N_L) = \bar{\varrho}\,. \tag{2.6}$$

Since the particles do not interact, the density matrix can be decomposed as

$$\sigma_L = \bigotimes_{j=-\nu}^{\nu} \sigma_{jL} \tag{2.7}$$

where each σ_{jL} is the grand canonical density matrix of a system of free spinless bosons with chemical potential γ_L. The state σ_L is completely determined by its generating functional [11, 12] which is defined on M_L by

$$\mu_{L,\bar{\varrho}}(\underline{h}) = \mathrm{trace}(W_F(\underline{h})\sigma_L)\,.$$

Using the decompositions (2.3) and (2.7) we can compute the generating functional as in [2, 6, 10]:

$$\begin{aligned}\mu_{L,\varrho}(\underline{h}) &= \prod_{j=-\nu}^{\nu} \mathrm{trace}_j(\sigma_{jL} W_{jF}(h_j)) \\ &= \mu_F(\underline{h})\exp\{-\tfrac{1}{2}Q_L(\underline{h},\underline{h})\}\end{aligned}$$

where $\mu_F(\underline{h}) = \exp\{-\tfrac{1}{4}\|\underline{h}\|^2\}$,

$$Q_L(\underline{h},\underline{h}) = \sum_{j=-\nu}^{\nu} \langle h_j, z_L(e^{\beta h_L} - z_L)^{-1} h_j\rangle \tag{2.8}$$

and $z_L = \exp\beta\gamma_L$. From the properties of generating functionals and from (2.8), or by explicit calculation we have

$$\langle n_{jp}^L\rangle = z_L(e^{\beta\eta_p^L} - z_L)^{-1} \tag{2.9}$$

so that the constraint Eq. (2.6) becomes

$$\bar{\varrho} = L^{-3}(2\nu+1)\sum_p z_L(e^{\beta\eta_p^L} - z_L)^{-1}\,.$$

The generating functional of the thermodynamic limiting state is defined by

$$\mu_{\bar{\varrho}}(\underline{h}) = \lim_{L\to\infty} \mu_{L,\bar{\varrho}}(\underline{h})$$

provided the RHS exists for $\underline{h}$ in some dense subset of $\bigoplus_{j=-\nu}^{\nu} L^2(\mathbb{R}^3)$. To determine $\mu_\varrho(\underline{h})$ it is necessary to consider the asymptotic behaviour of the solution z_L of the constraint equation, and the asymptotic behaviour of the quadratic form $Q_L(\underline{h}, \underline{h})$. The notation required is as follows: For $z \in [0, 1]$, the operator $F(z)$: $L^1 \cap L^2(\mathbb{R}^3) \to L^1 \cap L^2(\mathbb{R}^3)$ and the function $g_\alpha : [0, 1] \to \mathbb{R}$ are defined by

$$[F(z)f](x) = \int_{\mathbb{R}^3} F(x, y; z) f(y) dy \tag{2.10}$$

where

$$\begin{gathered} F(x, y; z) = (2\pi\beta)^{-\frac{3}{2}} \sum_{n=1}^{\infty} e^{-\|x-y\|^2/2n\beta} n^{-\frac{3}{2}} z^n \\ g_\alpha(z) = \sum_{n=1}^{\infty} n^{-\alpha} z^n, \qquad (\alpha > 1). \end{gathered} \tag{2.11}$$

Let

$$\varrho_c = (2\pi\beta)^{-\frac{3}{2}} (2\nu+1) g_{\frac{3}{2}}(1). \tag{2.12}$$

The following theorem is a straightforward extension of the case $\nu = 0$ [2].

Theorem 1. *The quadratic form $Q_L(\underline{h}, \underline{h})$ converges pointwise for $h \in \bigoplus_{-\nu}^{\nu} C_0^\infty$ to $Q(\underline{h}, \underline{h})$ where*

(a) $\bar{\varrho} < \varrho_c : Q(\underline{h}, \underline{h}) = \sum_{j=-\nu}^{\nu} \langle h_j, F(z_\infty) h_j \rangle$,

(b) $\bar{\varrho} \geqq \varrho_c : Q(\underline{h}, \underline{h}) = \dfrac{(\bar{\varrho} - \varrho_c)}{2\nu+1} |\phi_1(0)|^2 \sum_{j=-\nu}^{\nu} |\hat{h}_j(0)|^2 + \sum_{j=-\nu}^{\nu} \langle h_j, F(1) h_j \rangle$

where z_∞ is the unique solution in $[0, 1]$ *of*

$$\bar{\varrho} = (2\pi\beta)^{-\frac{3}{2}} (2\nu+1) g_{\frac{3}{2}}(z_\infty).$$

Furthermore

(a) $\bar{\varrho} < \varrho_c : z_L$ *converges to* z_∞,

(b) $\bar{\varrho} \geqq \varrho_c : z_L$ *converges to* 1 *with* $(2\nu+1) L^{-3} z_L (1 - z_L)^{-1} \to \bar{\varrho} - \varrho_c$.

[$\hat{h}_j(0)$ is the Fourier transform of h_j, evaluated at zero]. The pointwise convergence is enough to ensure that the corresponding states converge operationally [13]. In a sense, this is a physical convergence.

To determine the canonical ensemble we must evaluate the Kac density and then decompose the grand canonical ensemble with respect to it.

Lemma 1. *The Kac density is given by*

(a) $\bar{\varrho} \leqq \varrho_c : K(\bar{\varrho}, \varrho) = \delta(\varrho - \bar{\varrho})$

(b) $\bar{\varrho} > \varrho_c : K(\bar{\varrho}, \varrho) = 0 \qquad \varrho \leqq \varrho_c$

$$= \frac{(\varrho - \varrho_c)^{2\nu}}{2\nu} \left(\frac{2\nu+1}{\bar{\varrho} - \varrho_c} \right)^{2\nu+1} \exp\left\{ -\frac{(2\nu+1)(\varrho - \varrho_c)}{\bar{\varrho} - \varrho_c} \right\}, \qquad \varrho > \varrho_c.$$

To prove this lemma we recall (see § 1) that, by definition, the Fourier transform $\hat{K}(\bar{\varrho}, \xi)$ of the Kac density is

$$\hat{K}(\bar{\varrho}, \xi) = \lim_{L\to\infty} \hat{K}_L(\bar{\varrho}, \xi) = \lim_{L\to\infty} \langle \exp\{i\xi N_L/L^3\}\rangle_{g.c.e.}$$

The grand canonical expectation value can readily be evaluated to give

$$\hat{K}_L(\bar{\varrho}, \xi) = \left\{\prod_p \frac{1 - z_L e^{-\beta\eta_p^L}}{1 - z_L e^{-\beta\eta_p^L} e^{i\xi/L^3}}\right\}^{2v+1}$$

Comparison of this with the case $v=0$, given by Cannon [1], enables us to deduce the thermodynamic limit. It is

(a) $\bar{\varrho} \leqq \varrho_c : \hat{K}(\bar{\varrho}, \xi) = e^{i\xi\bar{\varrho}}$

(b) $\bar{\varrho} > \varrho_c : \hat{K}(\bar{\varrho}, \xi) = e^{i\xi\varrho_c}/\{1 - i\xi(2v+1)^{-1}(\bar{\varrho} - \varrho_c)\}^{2v+1}$.

The Fourier transform can be inverted to complete the proof of the lemma. We can now compute the canonical generating functional.

Theorem 2. *The generating functional of the canonical ensemble is given as follows:*

(a) $\varrho \leqq \varrho_c : \mu_\varrho(\underline{h}) = \mu_F(\underline{h}) \exp\{-\frac{1}{2}\sum_j \langle h_j, F(z_\infty) h_j\rangle\}$

(b) $\varrho > \varrho_c : \mu_\varrho(\underline{h}) = \mu_F(\underline{h}) \dfrac{C J_{2v}(\alpha(\varrho))}{\alpha(\varrho)^{2v}} \exp\{-\frac{1}{2}\sum_j \langle h_j, F(1) h_j\rangle\}$

where

$$C = (2v)!\,2^{2v}, \qquad \alpha(\varrho) = \sqrt{2}(\varrho - \varrho_c)^{\frac{1}{2}} |\phi_1(0)|^2 \sum_{-v}^{v} |\hat{h}_j(0)|^2 .$$

Proof. (a) is trivial. (b) is an application of a result in [14] (see also [6]).

From Theorem 1 we see that the mean density of particles in the lowest energy states ($p=1, j=-v, \ldots, v$) satisfies

$$L^{-3}\langle n_{j1}^L\rangle = \frac{z_L L^{-3}}{(1-z_L)} \to \begin{cases} 0 , & \bar{\varrho} \leqq \varrho_c , \\ \dfrac{(\bar{\varrho} - \varrho_c)}{2v+1} , & \bar{\varrho} > \varrho_c . \end{cases} \tag{2.13}$$

But for all other energy states (i.e. $p > 1$)

$$L^{-3}\langle n_{jp}^L\rangle = \frac{z_L L^{-3}}{(e^{\beta\eta_p L^{-2}} - z_L)} \to 0$$

for all values of the mean density $\bar{\varrho}$. Similarly there is macroscopic occupation of the degenerate ground state in the canonical ensemble.

It is interesting to compute the fluctuations in these quantities for densities larger than ϱ_c. For example the fluctuation in ϱ_{j1} (the density of particles in the ground state with spin j) in the grand canonical ensemble is

$$\Delta_{g.c.}(\varrho_{j1}) = \{\langle \varrho_{j1}^2\rangle_{g.c.} - \langle \varrho_{j1}\rangle_{g.c.}^2\}^{\frac{1}{2}} = \frac{\bar{\varrho} - \varrho_c}{2v+1} . \tag{2.14}$$

Similarly in the canonical ensemble

$$\Delta_c(\varrho_{j1}) = \left(\frac{v}{v+1}\right)^{\frac{1}{2}} \left(\frac{\bar{\varrho} - \varrho_c}{2v+1}\right). \tag{2.15}$$

[(2.15) is obtained by decomposing $\langle \varrho_{j1}^2 \rangle_{g.c.}$ and $\langle \varrho_{j1} \rangle_{g.c.}$ with respect to the Kac density.] We see from these that the density of condensate in each of the spin states fluctuates macroscopically in each ensemble. However this is not true of the total density $\varrho_1 = \sum_{-v}^{v} \varrho_{j1}$ of particles in the ground state:

$$\Delta_{g.c.}(\varrho_1) = \frac{\bar{\varrho} - \varrho_c}{(2v+1)^{\frac{1}{2}}}, \tag{2.16}$$

$$\Delta_c(\varrho_1) = 0. \tag{2.17}$$

Thus macroscopic fluctuations of this quantity occur in the grand ensemble but not in the canonical one.

§ 3. Bosons with Fixed Mean Spin

As noted in the introduction, the spin played a secondary role in the model of § 2. In this section a means of distinguishing between the spin states by their properties is introduced. Let

$$J_L = \sum_{j,p} j a_{jp}^* a_{jp}. \tag{3.1}$$

J_L is interpreted as the total z-component of spin. Its average in the state determined by (2.5) is zero since $\langle n_{jp}^L \rangle$ is independent of j (2.9). We consider now the grand canonical equilibrium state in which, in addition to $L^{-3}\langle N_L \rangle$ being fixed, we also demand that $L^{-3}\langle J_L \rangle$ takes a prescribed value $\bar{s}$. This state has density operator σ_L' on $\mathscr{F}(M_L)$ given by

$$\sigma_L' = \exp\{-\beta(H_L - \gamma_L N_L - \alpha_L J_L)\} / \mathrm{trace}[\exp\{-\beta(H_L - \gamma_L N_L - \alpha_L J_L)\}]. \tag{3.2}$$

The Lagrange multipliers γ_L and α_L are determined by the constraints

$$L^{-3}\langle N_L \rangle = \bar{\varrho}, \quad L^{-3}\langle J_L \rangle = \bar{s}. \tag{3.3}$$

As in § 2, standard Fock space techniques give the generating functional:

$$\mu_{L,\bar{\varrho},\bar{s}}(\underline{h}) = \mathrm{trace}\,\sigma_L' W_F(\underline{h}) = \mu_F(\underline{h}) \exp\{-\tfrac{1}{2}\mathscr{A}_L(\underline{h}, \underline{h})\} \tag{3.4}$$

where

$$\mathscr{A}_L(\underline{h}, \underline{h}) = \sum_{-v}^{v} \langle h_j, e^{\beta(\gamma_L + j\alpha_L)} (e^{\beta h_L} - e^{\beta(\gamma_L + j\alpha_L)})^{-1} h_j \rangle.$$

From this we may deduce that

$$\langle n_{jp}^L \rangle = e^{\beta(\gamma_L + j\alpha_L)} / (e^{\beta \eta_p^L} - e^{\beta(\gamma_L + j\alpha_L)}) \tag{3.5}$$

and the constraint Eqs. (3.3) become

$$L^{-3} \sum_{jp} \langle n_{jp}^L \rangle = \bar{\varrho}, \quad L^{-3} \sum_{jp} j \langle n_{jp}^L \rangle = \bar{s}. \tag{3.6}$$

Since $0 \leqq \langle n_{jp}^L \rangle < \infty$ we must have $-\gamma_L - \nu|\alpha_L| > 0$. This is only possible if we insist on the (physically very reasonable) condition that

$$-\nu\bar{\varrho} < \bar{s} < \nu\bar{\varrho}\,.$$

The constraint Eqs. (3.6) determine the multipliers γ_L, α_L. To show that there is a solution (i.e. a pair γ_L, α_L) and that it is unique we shall appeal to Theorem 1 of [7]. Let $\mathscr{R} = \{(\gamma, \alpha) \in \mathbb{R}^2 : -\gamma - \nu|\alpha| > 0\}$ and for $(\gamma, \alpha) \in \mathscr{R}$ define the free energy density function:

$$\begin{aligned} F_L(\gamma, \alpha) &= L^{-3} \log \operatorname{trace}[\exp\{-\beta(H_L - \gamma N_L - \alpha J_L)\}] - \beta\gamma\bar{\varrho} - \beta\alpha\bar{s} \\ &= L^{-3} \textstyle\sum_{j=-\nu}^{\nu} \sum_p \log(1 - e^{\beta(\gamma + j\alpha)} e^{-\beta\eta_p^L}) - \beta\gamma\bar{\varrho} - \beta\alpha\bar{s}\,. \end{aligned} \tag{3.7}$$

Then as in the case of the free boson gas [2, 7] it is easy to check that F_L is twice differentiable. Hence Theorem 1 of [7] is applicable, and proves that (γ_L, α_L) exists and is unique.

To determine the thermodynamic limiting state we must investigate the behaviour of the sequence $\{(\gamma_L, \alpha_L)\}$. We do this using Theorem 2 of [7]. Let

$$S'_L(s) = L^{-3} \textstyle\sum_{p=2}^{\infty} e^{-s\eta_p^L} \tag{3.8}$$

and let $T : \mathscr{R} \to \mathscr{D} = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\}$ be defined by

$$T(\gamma, \alpha) = (e^{\beta(\gamma + \nu\alpha)/2\nu}, e^{\beta(\gamma - \nu\alpha)/2\nu})\,.$$

For $(x, y) \in \mathscr{D}$ define

$$d_L(x, y) = \textstyle\sum_{j=-\nu}^{\nu} \sum_{n=1}^{\infty} n^{-1} S'_L(n\beta) x^{\nu+j} y^{\nu-j} - \nu\bar{\varrho} \log xy - \bar{s} \log xy^{-1}\,, \tag{3.9}$$

$$w_L(x, y) = -L^{-3} \textstyle\sum_{-\nu}^{\nu} \log(1 - x^{\nu+j} y^{\nu-j})\,. \tag{3.10}$$

Then $F_L(\gamma, \alpha) = [(d_L + w_L) \circ T](\gamma, \alpha)$. The functions d_L and w_L, and the region $\mathscr{D}$ have the following properties:

(i) w_L converges uniformly to zero on compact subsets of $\mathscr{D}$.

(ii) d_L converges uniformly in $\mathscr{D}$ to

$$d_\infty(x, y) = (2\pi\beta)^{-\frac{3}{2}} \textstyle\sum_{-\nu}^{\nu} g_{\frac{5}{2}}(x^{\nu+j} y^{\nu-j}) - (\nu\bar{\varrho} + \bar{s}) \log x - (\nu\bar{\varrho} - \bar{s}) \log y\,. \tag{3.11}$$

(iii) the boundary $\partial\mathscr{D} = \bar{\mathscr{D}} \setminus \mathscr{D}$ can be decomposed into the disjoint sets

$$\partial\mathscr{D}_1 = \{(1, y) : 0 < y \leqq 1\} \cup \{(x, 1) : 0 < x < 1\}$$

$$\partial\mathscr{D}_2 = \{(0, y) : 0 \leqq y \leqq 1\} \cup \{(x, 0) : 0 < x \leqq 1\}$$

and d_∞ can be extended continuously to $\partial\mathscr{D}_1$, and it diverges to ∞ as (x, y) approaches $\partial\mathscr{D}_2$.

(iv) d_∞ achieves its unique global minimum at (x_∞, y_∞) in $\mathscr{D} \cup \partial\mathscr{D}_1$.

(i), (ii), and (iii) can be deduced from the definitions and from the properties of the function as used by [2] and [7] (see also [6]). (iv) is a consequence of the convexity of the function $D_\infty(\gamma, \alpha) = (d_\infty \circ T)(\gamma, \alpha)$ defined on $\mathscr{R}$.

We can now invoke Theorem 2 of [7] and deduce that $\{(x_L, y_L)\} = \{T(\gamma_L, \alpha_L)\}$ converges to (x_∞, y_∞).

It remains to identify the location of (x_∞, y_∞), to consider the condensation phenomena expected when $(x_\infty, y_\infty) \in \partial\mathcal{D}_1$, and to determine the thermodynamic limit of the generating functional (3.4).

The following lemma provides the solution to the first of these problems. Its proof, which requires a close analysis of the dependence of d_∞ on $\bar{\varrho}$ and $\bar{s}$, is omitted (see [6]).

Lemma 2. *The point* (x_∞, y_∞) *depends on* $\bar{\varrho}$ *and* $\bar{s}$ *in the following way:*

(a) *If* $\bar{\varrho}$ *and* $\bar{s}$ *are such that there is a solution* (x, y) *in* $\mathcal{D}$ *of*

$$\bar{\varrho}=(2\pi\beta)^{-\frac{3}{2}}\sum_{-v}^{v} g_{\frac{3}{2}}(x^{v+j}y^{v-j}), \qquad \bar{s}=(2\pi\beta)^{-\frac{3}{2}}\sum_{-v}^{v} j g_{\frac{3}{2}}(x^{v+j}y^{v-j}) \tag{3.12}$$

then (x_∞, y_∞) *is this solution.*

(b) *If there is no solution to* (3.12) *but there is a solution to*

$$v\bar{\varrho}-|\bar{s}|=(2\pi\beta)^{-\frac{3}{2}}\sum_{-v}^{v}(v-j)g_{\frac{3}{2}}(z^{v-j}) \tag{3.13}$$

then for $\bar{s}>0$, $(x_\infty, y_\infty)=(1, z)$ *(case* b^+*) and for* $\bar{s}<0$, $(x_\infty, y_\infty)=(z, 1)$ *(case* b^-*)*

(c) *If* $v(\bar{\varrho}-\varrho_c)>|\bar{s}|$ *(or, equivalently, if no solution to either* (3.12) *or* (3.13) *exists), then* $(x_\infty, y_\infty)=(1, 1)$.

$$[\varrho_c=(2\pi\beta)^{-\frac{3}{2}}(2v+1)g_{\frac{3}{2}}(1) \text{ as in (2.12)}].$$

The three cases (a), (b), and (c) are interpreted diagrammatically in Fig. 1.

To investigate condensation phenomena we must use [7]:

$$\frac{\partial d_\infty}{\partial x}(x_\infty, y_\infty)=-\lim_{L\to\infty}\frac{\partial w_L}{\partial x}(x_L, y_L), \qquad \frac{\partial d_\infty}{\partial y}(x_\infty, y_\infty)=-\lim_{L\to\infty}\frac{\partial w_L}{\partial y}(x_L, y_L). \tag{3.14}$$

From the definition

$$\frac{\partial}{\partial x}w_L(x, y)=\frac{1}{L^3 x}\sum_{-v}^{v}\frac{(v+j)x^{v+j}y^{v-j}}{1-x^{v+j}y^{v-j}},$$

$$\frac{\partial}{\partial y}w_L(x, y)=\frac{1}{L^3 y}\sum_{-v}^{v}\frac{(v-j)x^{v+j}y^{v-j}}{1-x^{v+j}y^{v-j}}$$

We then have

Theorem 3. *For* $(\bar{\varrho}, \bar{s})$ *in each of the cases* (a), (b), (c) *of Lemma* 2 *we have*

$$\text{(a)} \quad \lim_{L\to\infty}\frac{L^{-3}x_L^{v+j}y_L^{v-j}}{1-x_L^{v+j}y_L^{v-j}}=0 \quad \text{for each } j, \tag{3.15}$$

$$(\mathrm{b}^+) \quad \lim_{L\to\infty}\frac{L^{-3}x_L^{v+j}y_L^{v-j}}{1-x_L^{v+j}y_L^{v-j}}=\delta_{jv}(\bar{\varrho}-\varrho_1(y_\infty)), \quad \bar{s}>0, \tag{3.16}$$

where

$$\varrho_1(y)=(2\pi\beta)^{-\frac{3}{2}}\sum_{-v}^{v} g_{\frac{3}{2}}(y^{v-j}). \tag{3.17}$$

$$\text{(c)} \quad \lim_{L\to\infty}\frac{L^{-3}x_L^{v+j}y_L^{v-j}}{1-x_L^{v+j}y_L^{v-j}}=c_j=\frac{2abv}{b(v+j)+a(v-j)}$$

where a and b depend on $\bar{\varrho}$ and $\bar{s}$ and are determined by

$$\bar{\varrho}-\varrho_c=\sum_{-\nu}^{\nu} c_j\,, \quad \bar{s}=\sum_{-\nu}^{\nu} jc_j\,. \tag{3.18}$$

The proofs of parts (a) and (b) are straightforward. To prove part (c) we note that (3.14) is equivalent to

$$\bar{\varrho}-\varrho_c=\lim_{L\to\infty}\sum_{-\nu}^{\nu} x_L^{\nu+j}y_L^{\nu-j}(1-x_L^{\nu+j}y_L^{\nu-j})^{-1}$$

$$\bar{s}=\lim_{L\to\infty}\sum_{-\nu}^{\nu} jx_L^{\nu+j}y_L^{\nu-j}(1-x_L^{\nu+j}y_L^{\nu-j})^{-1}$$

and that if $L^3(1-x_L^{2\nu})\to a^{-1}$ and $L^3(1-y_L^{2\nu})\to b^{-1}$ then

$$\frac{x_L^{\nu+j}y_L^{\nu-j}}{L^3(1-x_L^{\nu+j}y_L^{\nu-j})}\to\frac{2ab\nu}{b(\nu+j)+a(\nu-j)}=c_j\,.$$

Theorem 3 has the following interpretation. We fix $\bar{s}>0$ and assume it to be small enough so that when $\bar{\varrho}\sim\bar{s}/\nu$, the point $(\bar{\varrho},\bar{s})$ is in region (a). The bosons then behave as a normal fluid. If $\bar{\varrho}$ is increased, it reaches a first critical density ϱ_1 which depends on $\bar{s}$ and is equal to $\varrho_1(y_0)$ [Eq. (3.17)] with y_0 determined by

$$\bar{s}=(2\pi\beta)^{-\frac{3}{2}}\sum_{-\nu}^{\nu} jg_{\frac{3}{2}}(y_0^{\nu-j})\,.$$

This corresponds to the transition from region (a) to region (b), and for $\bar{\varrho}>\varrho_1$ condensation into the single state with zero energy and spin ν occurs. As $\bar{\varrho}$ is increased further so the density of condensate increases. But not all the extra particles go into the condensing state as is the case in the free boson gas. For a fixed s, the point y_∞ appearing in theorem 3(b) is an increasing function of $\bar{\varrho}$ [see (3.13)]; the density of condensate is $\bar{\varrho}-\varrho_1(y_\infty)$ and not $\bar{\varrho}-\varrho_1(y_0)$. The remaining particles are added to the other spin states until they too become critically occupied. This occurs at a density $\varrho_2=\varrho_c+\bar{s}/\nu$. The zero energy states, non-degenerate in finite volume, can be thought of as asymptotically degenerate for densities greater than ϱ_2. For $\bar{\varrho}>\varrho_2$, condensate occupies all the zero energy states, its distribution over these states being such as to maintain the fixed value of the spin density. Indeed, as can be seen from Theorem 3(c), the mean spin density of the condensate is $\bar{s}$, whereas the mean spin density of the normal component of the fluid is zero. Thus if $\bar{\varrho}$ is increased, the extra particles go into the condensed states and are distributed over them so as to maintain the mean spin density of the condensate.

Using the results of [2] on the case $\nu=0$ we can write down the thermodynamic limit of the grand canonical ensemble generating functional.

Theorem 4. *The generating functional $\mu_{\bar{\varrho},\bar{s}}(\underline{h})$ is given by*

$$\mu_{\bar{\varrho},\bar{s}}(\underline{h})=\mu_F(\underline{h})\exp\{-\tfrac{1}{2}\mathscr{A}(\underline{h},\underline{h})\}$$

where

(a) $\mathscr{A}(\underline{h},\underline{h})=\sum_{-\nu}^{\nu}\langle h_j, F(x_\infty^{\nu+j}y_\infty^{\nu-j})h_j\rangle\,,$

(b$^+$) $\mathscr{A}(\underline{h},\underline{h})=|\hat{h}_\nu(0)|^2|\phi_1(0)|^2(\bar{\varrho}-\varrho_1(y_\infty))+\sum_{j=-\nu}^{\nu}\langle h_j, F(y_\infty^{\nu-j})h_j\rangle\,, \quad \bar{s}>0\,,$

(c) $\mathscr{A}(\underline{h},\underline{h})=|\phi_1(0)|^2\sum_{-\nu}^{\nu} c_j|\hat{h}_j(0)|^2+\sum_{-\nu}^{\nu}\langle h_j, F(1)h_j\rangle\,.$

§ 4. Bosons with Fixed Spin

In the grand canonical ensemble of the previous chapter it was the mean densities $\bar{\varrho}$ and $\bar{s}$ of particles and spin that were fixed. In the generalized canonical ensemble that we discuss now it is the densities ϱ and s that take prescribed values.

The canonical state $\mu_{\varrho,s}(\underline{h})$ is derived from the grand canonical state $\mu_{\bar{\varrho},\bar{s}}(\underline{h})$ by means of a generalized Kac density $K(\bar{\varrho},\bar{s};\varrho,s)$. $K(\cdot,\cdot;\cdot,\cdot)$ is such that if $a(\varrho,s)$ is the value of an operator in the canonical state, and $\bar{a}(\bar{\varrho},\bar{s})$ its value in the grand canonical state then

$$\bar{a}(\bar{\varrho},\bar{s})=\int K(\bar{\varrho},\bar{s};\varrho,s)a(\varrho,s)d\varrho ds\,.$$

As in § 1 we take the view that the canonical state is defined through the medium of the Kac density, and that the Kac density is defined by the following limiting process

$$\begin{aligned}\hat{K}(\bar{\varrho},\bar{s};\xi,\eta)&=\lim_{L\to\infty}\hat{K}_L(\bar{\varrho},\bar{s};\xi,\eta)\\&=\lim_{L\to\infty}\langle e^{i\xi N_L/L^3}e^{i\eta J_L/L^3}\rangle_{g.c.}\,.\end{aligned}$$

$\hat{K}_L(\bar{\varrho},\bar{s};\xi,\eta)$ can be evaluated by the usual methods of Fock space:

$$\hat{K}_L(\bar{\varrho},\bar{s};\xi,\eta)=\prod_{jp}\left\{\frac{1-x_L^{\nu+j}y_L^{\nu-j}e^{-\beta\eta_p^L}}{1-x_L^{\nu+j}y_L^{\nu-j}e^{-\beta\eta_p^L}e^{i\xi/L^3}e^{ij\eta/L^3}}\right\}.$$

As in [6], the thermodynamic limit of this can be deduced from the case $\nu=0$ [1].

Lemma 3. *The Fourier transform $\hat{K}(\bar{\varrho},\bar{s};\xi,\eta)$ of the Kac density is*

(a) $\hat{K}(\bar{\varrho},\bar{s};\xi,\eta)=\exp\{i(\xi\bar{\varrho}+\eta\bar{s})\}$,

$$(\mathrm{b}^+)\quad \hat{K}(\bar{\varrho},\bar{s};\xi,\eta)=\exp\{i(\xi\varrho_1(y_\infty)+\eta s_1(y_\infty))\}/\{1-i(\bar{\varrho}-\varrho_c)(\xi+\nu\eta)\} \tag{4.1}$$

where

$s_1(y)=(2\pi\beta)^{-\frac{3}{2}}\sum_{-\nu}^{\nu}jg_{\frac{3}{2}}(y^{\nu-j})$,

(c) $\hat{K}(\bar{\varrho},\bar{s};\xi,\eta)=\exp\{i\xi\varrho_c\}/\prod_{j=-\nu}^{\nu}[1-i(\xi+j\eta)c_j]$.

[The three cases (a), (b^+), (c) correspond to $(\bar{\varrho},\bar{s})$ lying in the regions (a), (b^+), (c) of Fig. 1. Case (b^-) is similar to case (b^+).]

Inverting the Fourier transform gives

Lemma 4. *The Kac density is*

(a) $K(\bar{\varrho},\bar{s};\varrho,s)=\delta(\bar{\varrho}-\varrho)\delta(\bar{s}-s)$,

$$\begin{aligned}(\mathrm{b})\quad K(\bar{\varrho},\bar{s};\varrho,s)&=\frac{e^{-(\varrho-\varrho_1(y_\infty))/(\bar{\varrho}-\varrho_1(y_\infty))}}{\bar{\varrho}-\varrho_1(y_\infty)}\delta(\nu(\varrho-\varrho_1(y_\infty))-(s-s_1(y_\infty)))\,,\quad \varrho\geqq\varrho_1(y_\infty)\\&=0\,,\quad \varrho<\varrho_1(y_\infty)\,,\end{aligned}$$

$$\begin{aligned}(\mathrm{c})\quad K(\bar{\varrho},\bar{s};\varrho,s)&=0\,,\quad \varrho<\varrho_c\\&=\frac{e^{-\{(\varrho-\varrho_c)(a+b)-\frac{s}{\nu}(a-b)\}/2ab}}{(\prod_j c_j)(2\nu)!(2\nu-1)!}\sum_{j=[s/\varrho-\varrho_c]}^{\nu}\binom{2^\nu}{\nu+j}(-1)^{\nu+j}\\&\quad\cdot(j(\varrho-\varrho_c)-s)^{2\nu-1}\,,\quad \varrho>\varrho_c\,,\end{aligned}$$

where $[x]$ is the smallest integer $\geqq x$ and a and b are functions of $\bar{\varrho}$ and $\bar{s}$.

We note that the (ϱ, s)-plane can be decomposed as was the $(\bar{\varrho}, \bar{s})$-plane (Fig. 1) and that, according to Lemma 4, $K(\bar{\varrho}, \bar{s}; \varrho, s)$ is non-zero only if $(\bar{\varrho}, \bar{s})$ and (ϱ, s) lie in the same region of their respective planes. This considerably simplifies evaluation of the canonical ensemble as it means that each region can be treated independently. An essential step in the proof of this remark, and indeed in any application of Lemma 4, is to understand the significance of the delta function in case (b). $\varrho_1(y_\infty)$ and $s_1(y_\infty)$ depend on $\bar{\varrho}$ and $\bar{s}$ since y_∞ is the solution of (3.13). For (ϱ, s) in region (b^+), let y'_∞ be the solution of

$$\nu\varrho - s = (2\pi\beta)^{-\frac{3}{2}} \sum_{j=-\nu}^{\nu} (\nu - j) g_{\frac{3}{2}}(y^{\nu-j})$$

then the delta function restricts the integration to the line $y_\infty = y'_\infty$ in region (b^+).

We can now prove

Theorem 5. *The generating functional of the canonical ensemble is given by*

(a) $\mu(\underline{h}) = \mu_F(\underline{h}) \exp\{-\frac{1}{2}\sum_j \langle h_j, F(x_\infty^{\nu+j} y_\infty^{\nu-j}) h_j\rangle\}$,

(b^+) $\mu(\underline{h}) = \mu_F(\underline{h}) J_0(|\hat{\underline{h}}(0)| |\phi_1(0)| 2^{\frac{1}{2}} (\varrho - \varrho_1(y_\infty))^{\frac{1}{2}}) \exp\{-\frac{1}{2}\sum_j \langle h_j, F(y_\infty^{\nu-j}) h_j\rangle\}$,

(c) $\mu(\underline{h}) = \mu_F(\underline{h}) g(\underline{h}; \varrho - \varrho_c, s) \exp\{-\frac{1}{2}\sum_j \langle h_j, F(1) h_j\rangle\}$

where the Fourier transform of $g(\underline{h}; t, s)$ *satisfies*

$$\int_{\mathbb{R}^2} \frac{\hat{g}(\underline{h}; \xi, \eta)}{\prod_j (1 - i(\xi + j\eta) c_j)} \frac{d\xi d\eta}{(2\pi)^2} = \exp\{-\tfrac{1}{2}\textstyle\sum_j |\hat{h}_j(0)|^2 |\phi_1(0)|^2 c_j\}.$$

Proof. The remarks before the statement of the theorem reduce the proofs of part (a) and (b) to an elementary calculation. For part (c) we note that the canonical generating functional can clearly be written in this form and that the function $g(\underline{h}; t, s)$ must satisfy

$$\int_{\mathbb{R}^2} g(\underline{h}; t, s) K(\bar{\varrho}, \bar{s}; \varrho_c + t, s) dt ds = \exp\{-\tfrac{1}{2}\textstyle\sum_j |\hat{h}_j(0)|^2 |\phi_1(0)|^2 c_j\}.$$

Part (c) is just a restatement of this in terms of Fourier transforms. We have expressed it in this way as it may be possible to solve the equation directly for $g(\underline{h}; \xi, \eta)$.

The Kac density in Lemma 4(c) is, essentially, a product of two Laplace transformation kernels: $\exp\left\{-\frac{1}{2a\nu}[\nu(\varrho - \varrho_c) + s]\right\}$ and $\exp\left\{-\frac{1}{2b\nu}[\nu(\varrho - \varrho_c) - s]\right\}$. Thus, since the Laplace transform of $J_0(x^{\frac{1}{2}})$ is $\varrho^{-1} e^{-1/4\varrho}$, the term $g(\underline{h}; \varrho - \varrho_c, s)$ in Theorem 5 (c) is, again essentially, a convolution of $2\nu + 1$ zero order Bessel functions. Its precise nature is too complicated to be worth recording here (but see [6]); for the case $\nu = 1$, however, we give it explicitly in the next section. We also postpone a quantitative discussion of condensation until then, when the differences between the canonical and grand canonical states can be more clearly displayed.

§ 5. Bosons with Spin 1

As an example of the convoluted arguments of the previous section we consider the case $\nu = 1$. The case $\nu = \frac{1}{2}$ is too simple to illustrate adequately the ideas. We shall only discuss region (c) as regions (a) and (b) are similar to the free boson gas.

Let $t=\varrho-\varrho_c$, then from Lemma 4 the Kac density is

$$K(\bar{\varrho},\bar{s};t+\varrho_c,s)=\left(\frac{a+b}{4a^2b^2}\right)e^{-\frac{1}{2a}(s+t)}e^{-\frac{1}{2b}(t-s)}\sum_{j=[s/t]}^{1}(-1)^j\binom{2}{j+1}(s-t_j)\,.$$

Let σ and τ be defined by $\sigma=2^{-1}(s+t)$, $\tau=2^{-1}(t-s)$ and let the Jacobian of this transformation be included in the transformed Kac density $k(\bar{\varrho},\bar{s};\sigma,\tau)$:

$$k(\bar{\varrho},\bar{s};\sigma,\tau)=\frac{a+b}{a^2b^2}e^{-\sigma/a}e^{-\tau/b}\tau\,,\qquad \sigma>\tau\,,$$

$$=\frac{a+b}{a^2b^2}e^{-\sigma/a}e^{-\tau/b}\sigma\,,\qquad \sigma<\tau\,.$$

To determine the canonical generating functional we must determine a function $f(\sigma,\tau)$ such that for all a, b

$$\int_0^\infty\int_0^\infty f(\sigma,\tau)e^{-\sigma/a}e^{-\tau/b}d\sigma d\tau=\frac{a^2b^2}{a+b}e^{-\frac{1}{2}a\alpha_1^2}e^{-\frac{1}{2}\frac{2ab}{a+b}\alpha_0^2}e^{-\frac{1}{2}b\alpha_{-1}^2} \tag{5.1}$$

where $\alpha_j^2=|\phi_1(0)|^2|\hat{h}_j(0)|^2$. Then the function $g(\underline{h};t,s)$ appearing in Theorem 5(c) will be given by

$$g(\underline{h};t,s)=\frac{f\left(\dfrac{s+t}{2},\dfrac{t-s}{2}\right)}{\min\left[\dfrac{s+t}{2},\dfrac{t-s}{2}\right]} \tag{5.2}$$

Theorem 6. *For* (ϱ,s) *in region* (c) *and* $\nu=1$, *the canonical generating functional is*

$$s>0:\ \mu_c(\underline{h})=\mu_F(\underline{h})\left\{\frac{1}{\tau}\int_0^\tau J_0(2^{\frac{1}{2}}\alpha_1(\sigma-\theta)^{\frac{1}{2}})J_0(2\alpha_0\theta^{\frac{1}{2}})J_0(2^{\frac{1}{2}}\alpha_{-1}(\tau-\theta)^{\frac{1}{2}})d\theta\right\}\mu_N(\underline{h})$$

$$s<0:\ \mu_c(\underline{h})=\mu_F(\underline{h})\left\{\frac{1}{\sigma}\int_0^\sigma J_0(2^{\frac{1}{2}}\alpha_1(\sigma-\theta)^{\frac{1}{2}})J_0(2\alpha_0\theta^{\frac{1}{2}})J_0(2^{\frac{1}{2}}\alpha_{-1}(\tau-\theta)^{\frac{1}{2}})d\theta\right\}\mu_N(\underline{h})$$

where $\sigma=(t+s)/2$, $\tau=(t-s)/2$ *and*

$$\mu_N(\underline{h})=\exp\{-\tfrac{1}{2}\textstyle\sum_j\langle h_j,F(1)h_j\rangle\}\,.$$

Proof. We use the fact that the Laplace transform of $J_0(x^{\frac{1}{2}})$ is $\varrho^{-1}e^{-\frac{1}{4}\varrho}$. In (5.1) let $x=1/a$, $y=1/b$, then the right hand side can be written as follows:

$$\frac{e^{-\frac{1}{2}\alpha_1^2/x}}{x}\cdot\frac{e^{-\frac{1}{2}\alpha_0^2/(x+y)}}{x+y}\cdot\frac{e^{-\frac{1}{2}\alpha_{-1}^2/y}}{y}$$

$$=\int_0^\infty e^{-\sigma x}d\sigma\int_0^\sigma J_0(2^{\frac{1}{2}}\alpha_1(\sigma-\theta)^{\frac{1}{2}})J_0(2\alpha_0\theta^{\frac{1}{2}})e^{-\theta y}d\theta\frac{e^{-\frac{1}{2}\alpha_{-1}^2/y}}{y}$$

$$=\int_0^\infty e^{-\sigma x}d\sigma\int_0^\infty e^{-\tau y}d\tau\int_0^\sigma J_0(2^{\frac{1}{2}}\alpha_1(\sigma-\theta)^{\frac{1}{2}})J_0(2\alpha_0\theta^{\frac{1}{2}})J_0(2^{\frac{1}{2}}\alpha_{-1}(\tau-\theta)^{\frac{1}{2}})\chi_{[0,\tau]}(\theta)d\theta$$

where $\chi_{[0,\tau]}$ is the characteristic function of $[0,\tau]$. Together with (5.1) and (5.2) this gives Theorem 6.

The proof given here can be extended to the more general case (i.e. other than $\nu=1$), and so we see how the convolution of Bessel functions mentioned at the end of § 4 arises.

The series expansion for $J_0(x^{\frac{1}{2}})$ is uniformly convergent so can be integrated term by term to give

$$\int_0^\tau J_0(x2^{\frac{1}{2}}(\sigma-\theta)^{\frac{1}{2}})d\theta=\frac{2^{\frac{1}{2}}\sigma^{\frac{1}{2}}}{\alpha}J_1(2^{\frac{1}{2}}\alpha\sigma^{\frac{1}{2}})-\frac{2^{\frac{1}{2}}(\sigma-\tau)^{\frac{1}{2}}}{\alpha}J_1(2^{\frac{1}{2}}\alpha(\sigma-\tau)^{\frac{1}{2}}). \tag{5.3}$$

We use this to evaluate single spin correlations. Let K^{-1} denote the operation of taking the inverse with respect to the Kac density, i.e. evaluating a canonical average by decomposing the corresponding grand canonical one. Then (5.3) shows that

$$s>0: K^{-1}(e^{-\alpha^2 a/2})=\frac{(2\sigma)^{\frac{1}{2}}}{\alpha\tau}J_1((2\sigma)^{\frac{1}{2}}\alpha)-\frac{2^{\frac{1}{2}}(\sigma-\tau)^{\frac{1}{2}}}{\alpha\tau}J_1(2^{\frac{1}{2}}(\sigma-\tau)^{\frac{1}{2}}\alpha),$$

$$s<0: K^{-1}(e^{-\alpha^2 a/2})=\frac{2^{\frac{1}{2}}}{\alpha\sigma^{\frac{1}{2}}}J_1((2\sigma)^{\frac{1}{2}}\alpha).$$

Together with the analogous expressions for $K^{-1}(e^{-\alpha_0^2 C_0/2})$ and $K^{-1}(e^{-\alpha_{-1}^2 b/2})$ these give:

$$\begin{aligned}
\langle\varrho_{11}\rangle_{c.e.}&=K^{-1}(a) &&=\begin{cases}(3s+t)/4, & s>0,\\ (s+t)/4, & s<0,\end{cases}\\
\langle\varrho_{01}\rangle_{c.e.}&=K^{-1}\left(\frac{2ab}{a+b}\right)&&=\begin{cases}(t-s)/2, & s>0,\\ (t+s)/2, & s<0,\end{cases}\\
\langle\varrho_{-11}\rangle_{c.e.}&=K^{-1}(b) &&=\begin{cases}(t-s)/4, & s>0,\\ (t-3s)/4, & s<0.\end{cases}
\end{aligned} \tag{5.4}$$

[All are, of course, zero for $|s|>t$.] It is interesting to compare these with their values in the grand canonical ensemble. Solving the equations for a and b in Theorem 3(c) gives

$$\begin{aligned}
\langle\varrho_{11}\rangle_{g.c.e.}&=a &&=(\bar{t}+3\bar{s}+(\bar{t}^2+3\bar{s}^2)^{\frac{1}{2}})/6,\\
\langle g_{01}\rangle_{g.c.e.}&=\frac{2ab}{a+b}&&=(2\bar{t}-(\bar{t}^2+3\bar{s}^2)^{\frac{1}{2}})/3,\\
\langle\varrho_{-11}\rangle_{g.c.e.}&=b &&=(\bar{t}-3\bar{s}+(\bar{t}^2+3\bar{s}^2)^{\frac{1}{2}})/6.
\end{aligned} \tag{5.5}$$

where $\bar{t}=\bar{\varrho}-\varrho_c$. The comparison is best appreciated diagrammatically (see Fig. 2).

The variation is these quantities can also be evaluated. It is found that in both ensembles they fluctuate macroscopically. For example

$$\Delta_{g.c.e.}(\varrho_{11})=(\bar{t}+3\bar{s}+(\bar{t}^2+3\bar{s}^2)^{\frac{1}{2}}), \tag{5.6}$$

$$\Delta_{c.e.}(\varrho_{11})=(t-|s|)/4\sqrt{3}. \tag{5.7}$$

However, when the total condensate is considered [this involves calculating terms such as $K^{-1}\left(a\dfrac{ab}{a+b}\right)$] it is found that

$$\Delta_{g.c.e.}(\varrho_{11}+\varrho_{01}+\varrho_{-11})=(2\bar{t}^2-\bar{t}(\bar{t}^2+3\bar{s}^2)^{\frac{1}{2}}+3\bar{s}^2)^{\frac{1}{2}}/3^{\frac{1}{2}}>0\,, \tag{5.8}$$

$$\Delta_{c.e.}(\varrho_{11}+\varrho_{01}+\varrho_{-11})=0\,. \tag{5.9}$$

So, as mentioned in the introduction, the total density of condensate fluctuates macroscopically in the grand canonical ensemble but not in the canonical ensemble.

§ 6. Spinning Bosons in a Magnetic Field

We now consider a model in which our spinning bosons interact with a constant magnetic field. We assume that the field has strength h, that it interacts with the z-component of the spin, and that the coupling constant is unity. The Hamiltonian of the system in Λ_L is

$$H_L-hJ_L=\sum_{jp}(\eta_p^L-jh)a_{jp}^*a_{jp} \tag{6.1}$$

so that the finite volume density matrix is

$$\sigma_L''=\exp\{-\beta(H_L-\gamma_LN_L-hJ_L)\}/\mathrm{trace}[\exp\{-\beta(H_L-\gamma_LN_L-hJ_L)\}]\,. \tag{6.2}$$

The Lagrange multiplier γ_L is determined from the usual constraint

$$\bar{\varrho}=L^{-3}\mathrm{trace}\,\sigma_L''N_L\,.$$

This model is clearly very similar mathematically to those discussed in the earlier sections and so we just quote the main results. Let $z_L=e^{\beta\gamma_L}$.

Theorem 7. *For fixed magnetic field $h>0$ there is a critical density $\varrho_s(h)=(2\pi\beta)^{-\frac{3}{2}}\sum_j g_{\frac{3}{2}}(e^{-(\nu-j)\beta h})$ and the thermodynamic properties of the equilibrium states are*

(i) *Fugacity:* (a) $\bar{\varrho}<\varrho_s(h):z_L$ *converges to* z_∞, *the unique solution in* $[0,e^{-\nu\beta h}]$ *of* $\bar{\varrho}=(2\pi\beta)^{-\frac{3}{2}}\sum_j g_{\frac{3}{2}}(z_\infty e^{j\beta h})$.

(b) $\bar{\varrho}>\varrho_s(h):z_L$ *converges to* $e^{-\nu\beta h}$ *and* $L^{-3}z_Le^{\nu\beta h}(1-z_Le^{\nu\beta h})^{-1}$ *converges to* $\bar{\varrho}-\varrho_s(h)$.

(ii) *Grand canonical generating functional*

(a) $\mu_{\bar{\varrho}}(\underline{h})=\mu_F(\underline{h})\exp\{-\frac{1}{2}\sum_j\langle h_j,F(z_\infty e^{j\beta h})h_j\rangle\}$,

(b) $\mu_{\bar{\varrho}}(\underline{h})=\mu_F(\underline{h})\exp\{-\frac{1}{2}|\phi_1(0)|^2|\hat{h}_\nu(0)|^2(\bar{\varrho}-\varrho_s(h))\}$

$$\exp\{-\tfrac{1}{2}\textstyle\sum_j\langle h_j,F(e^{-\beta(\nu-j)h})h_j\rangle\}\,.$$

(iii) *Kac density:* (a) $K(\bar{\varrho};\varrho)=\delta(\bar{\varrho}-\varrho)$

$$\text{(b)}\quad K(\bar{\varrho};\varrho)=\frac{\exp\{-(\varrho-\varrho_s)/(\bar{\varrho}-\varrho_s)\}}{\bar{\varrho}-\varrho_s}\,,\qquad \varrho>\varrho_s\,,$$

$$=0\,,\qquad \varrho\leqq\varrho_s\,.$$

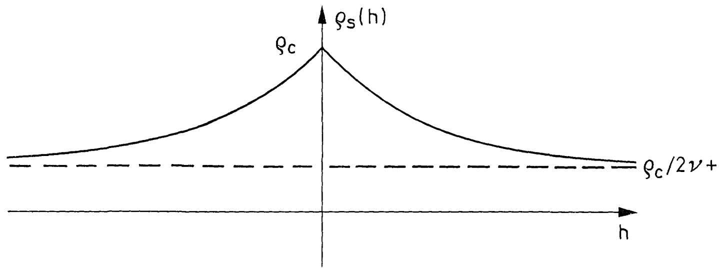


Fig. 4. The critical density as a function of the magnetic field

(iv) *Canonical generating functional*

(a) $\mu_\varrho(\underline{h}) = \mu_F(\underline{h})\exp\{-\frac{1}{2}\sum_j \langle h_j, F(z_\infty e^{j\beta h})h_j\rangle\}$,

(b) $\mu_\varrho(\underline{h}) = \mu_F(\underline{h}) J_0(2^{\frac{1}{2}}|\hat{h}_\nu(0)||\phi_1(0)|(\varrho - \varrho_s(h))^{\frac{1}{2}})\exp\{-\frac{1}{2}\sum_j \langle h_j, F(\lambda_j)h_j\rangle\}$,

where $\lambda_j = \exp\{-(\nu - j)\beta h\}$.

We define magnetization of the system by

$$m^*(h) = \lim_{L\to\infty} L^{-3}\langle J_L\rangle . \tag{6.3}$$

Lemma 6. *For $h > 0$ the magnetization is given by*

(a) $\bar{\varrho} < \varrho_s(h)$: $m^*(h) = (2\pi\beta)^{-\frac{3}{2}} \sum_j j g_{\frac{3}{2}}(z_\infty e^{j\beta h})$,

(b) $\bar{\varrho} > \varrho_s(h)$: $m^*(h) = (\bar{\varrho} - \varrho_s(h))\nu + (2\pi\beta)^{-\frac{3}{2}} \sum_j j g_{\frac{3}{2}}(e^{-(\nu - j)\beta h})$.

We say that spontaneous magnetization exists if

$$\lim_{h\to 0} m^*(h) \neq 0 .$$

Since $m^*(-h) = -m^*(h)$ this is clearly equivalent to

$$\lim_{h\to 0^+} m^*(h) \neq \lim_{h\to 0^-} m^*(h) .$$

Lemma 7. *Spontaneous magnetization exists for $\bar{\varrho} > \varrho_c$ but not for $\bar{\varrho} < \varrho_c$. Specifically, for $\bar{\varrho} > \varrho_c$, $m^*(0^+) = (\bar{\varrho} - \varrho_c)\nu$.*

Proof. Note that, as in Fig. 4, as $h \to 0$ so $\varrho_s(h)$ increases to ϱ_c. Thus for $\bar{\varrho} < \varrho_c$, we can always choose h small enough so that $\varrho < \varrho_s(h)$. The result then follows by letting h tend to zero in Lemma 6.

We can reinterpret this in terms of a critical temperature $T_c = \beta_c^{-1}$ defined at density ϱ by

$$\bar{\varrho} = (2\nu + 1)(2\pi\beta_c)^{-\frac{3}{2}} g_{\frac{3}{2}}(1) .$$

Then for $T < T_c$ there is spontaneous magnetization but for $T > T_c$ there is not. A few magnetization isotherms are sketched in Fig. 3.

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