Commun. math. Phys. 42, 1—7 (1975)

Higher Order Estimates for the Yukawa$_2$ Quantum Field Theory*

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Received March 27, 1974

Abstract. Higher order estimates of the form

$$\prod_{1}^{n} N_{\tau_i} \leqq \text{const}(H(g)+I)^n\,, \qquad \sum_{1}^{n} \tau_i < 1\,, \tau_i \geqq 0$$

are proved for the Yukawa$_2$ models with and without SU$_3$ symmetry. We also prove norm convergence of $\Pi_1^n N_{\tau_i}^{\frac{1}{2}} \cdot R_\kappa^{n/2+\delta}$ as $\kappa \to \infty$ where $R_\kappa = (H(g,\kappa)+I)^{-1}$.

Introduction and Results

Higher order estimates, bounding powers of the fractional energy operator by powers of the Hamiltonian, have proved useful in studying the $\mathscr{P}(\phi)_2$ model [1]. In this paper we obtain similar estimates for the Yukawa$_2$ model as well as for the Yukawa$_2$ model with internal SU$_3$ symmetry discussed in [2].

In the following we will use even, positive odd and negative odd values of ε to label bosons, fermions and anti-fermions respectively. Thus $b(k, \varepsilon)$ denotes the annihilation operator for free particles of momentum k and type ε. The fractional energy operator is:

$$N_\tau = \sum_{\varepsilon} N_\tau^{(\varepsilon)} = \sum_{\varepsilon} \int dk\, \mu(k,\varepsilon)^\tau\, b^*(k,\varepsilon)\, b(k,\varepsilon)\,,$$

$$\mu(k,\varepsilon) = (k^2 + m(\varepsilon)^2)^{\frac{1}{2}}\,,$$

where $m(\varepsilon) = m$ for bosons, $m(\varepsilon) = M$ for fermions. For convenience we define $E(k) = \sqrt{k^2+1}$. We will work with the dense domain $\mathscr{D}$ of vectors in Fock space with finite numbers of particles and wave functions in Schwartz space.

Formally, the finite volume Hamiltonian $H(g)$ has the form

$$\begin{aligned} H(g) &= H_0 + H_I(g) + C(g) \\ &= N_1 + \lambda \int dx g(x) : \overline{\psi}\psi\phi : - \tfrac{1}{2}\delta m^2 \int dx g^2(x) : \phi^2 : - E(g)\,, \end{aligned}$$

where $g \geqq 0 \in C_0^\infty$ and δm^2, $E(g)$ provide infinite renormalizations. To define the momentum cutoff Hamiltonian $H(g, \kappa)$ we multiply the momentum space kernels $w^c(k, p_1, p_2)$, $w(k, p_1, p_2)$ of the interaction term $H_I(g)$ by a general momentum cutoff function $\chi_\kappa(k, p_1, p_2)$ in the sense of [3]. The renormalization constants

* Supported in part by the National Research Council of Canada and by the National Science Foundation, grant NSF-GP-31239X.

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are then defined as:

$$\delta m_\kappa^2 = -\frac{\lambda^2}{2\pi}\int dk\,\omega(k)^{-1}|\chi_\kappa(0,k/2,-k/2)|^2 + \text{const} + o(1)\,,$$

$$E(g,\kappa) = -\int dk\,dp_1\,dp_2|w^c(k,p_1,p_2)\,\chi_\kappa(k,p_1,p_2)|^2\,(\mu(k)+\omega(p_1)+\omega(p_2))^{-1} + \text{const} + o(1)\,,$$

where $\mu(k) \equiv \mu(k,0)$, $\omega(k) = \mu(k,1)$. The SU_3 Hamiltonians involve slight generalizations and are defined in [2]. Glimm and Jaffe [3] have shown that the Hamiltonians, with a suitable choice of the constant in $E(g,\kappa)$, define positive self-adjoint operators converging in the sense of resolvents to a positive self-adjoint operator $H(g)$ which satisfies:

$$N_\tau \leqq \text{const}(H(g)+I)\,, \qquad \tau < 1\,.$$

Furthermore the operators $H(g,\kappa)$ are essentially self-adjoint on $\mathscr{D}$ and satisfy κ-dependent estimates of the form:

$$\prod_{i=1}^n N_{\tau_i} \leqq C_\kappa(H(g,\kappa)+I)^n\,, \qquad \sum_{i=1}^n \tau_i < 1\,.$$

The proof of these estimates requires the essential self-adjointness of $H(g,\kappa,\sigma)^n$ on $\mathscr{D}$, where $H_I(g,\kappa,\sigma)$ and $C(g,\kappa,\sigma)$ have momentum space kernels in Schwartz space converging to those of $H_I(g,\kappa)$ and $C(g,\kappa)$ as $\sigma\to\infty$, which follows by techniques similar to those of Jaffe, Lanford, and Wightman [6]. For notation and general techniques we refer to Glimm and Jaffe [3], Dimock [4], and McBryan [2, 5]. Our main results are:

Theorem 1. *Provided* $\tau = \sum_{i=1}^n \tau_i < 1$, $\tau_i \geqq 0$, *then there is a constant, depending on* n, τ, g, *such that*

$$\prod_{i=1}^n N_{\tau_i} \leqq \text{const}(H(g)+I)^n\,. \tag{1}$$

The restriction $\tau < 1$ is indicated by perturbation theory and so we expect that (1) are the most general n^{th} order estimates. It is also useful to have estimates controlling the inequality (1) as the momentum cutoff κ is removed. With $R_\kappa = (H(g,\kappa)+I)^{-1}$, $R \equiv R_\infty = (H(g)+I)^{-1}$, we define $\delta R^\beta = R^\beta_{\kappa_2} - R^\beta_{\kappa_1}$ where one of κ_1, κ_2 may be infinite.

Theorem 2. *Provided* $\tau = \sum_{i=1}^n \tau_i < 1$, $\tau_i \geqq 0$, *and* $\delta > 0$, *there is a constant and an* $\varepsilon > 0$ *such that*

(i) $\prod_{i=1}^n N_{\tau_i}^{\frac{1}{2}} \cdot \delta R^{n/2}$ *converges weakly to* 0 *as* $\kappa = \min(\kappa_1,\kappa_2)$ *tends to* ∞.

(ii) $\|\prod_1^n N_{\tau_i}^{\frac{1}{2}-\delta} \cdot \delta R^{n/2}\| \leqq \text{const}\,\kappa^{-\varepsilon}$.

(iii) $\|\prod_1^n N_{\tau_i}^{\frac{1}{2}} \cdot \delta R^{n/2+\delta}\| \leqq \text{const}\,\kappa^{-\varepsilon}$.

Theorem 1 follows from two lemmas:

Lemma 3. *For* $\psi \in D(\prod_{i=1}^n N_{\tau_i}^{\frac{1}{2}})$ *and an arbitrary choice of* ε_i, τ_i:

$$\left\|\prod_{i=1}^n N_{\tau_i}^{(\varepsilon_i)^{\frac{1}{2}}}\psi\right\|^2 = \sum_{r=1}^n \int dk_1\ldots dk_r \sum_{1=i_1<\cdots i_r}^n P^{\tau,\varepsilon}_{i_1\ldots i_r}(k_1,\ldots,k_r) \cdot \|b(k_r,\varepsilon_{i_r})\ldots b(k_1,\varepsilon_{i_1})\,\psi\|^2\,, \tag{2}$$

where $P^{\tau,\varepsilon}_{i_1\dots i_r}$ are homogeneous expressions of degree $\Sigma_{i=1}^n \tau_i$ in $\mu(k_i,\varepsilon_j)$. Explicitly:

$$P^{\tau,\varepsilon}_{i_1\dots i_r}(k_1,\dots,k_r)=\prod_{t=1}^{r}\left[\mu^{\tau_{i_t}}(k_t,\varepsilon_{i_t})\sum_{s=i_t+1}^{i_{t+1}-1}\cdot\left\{\sum_{l=1}^{t}\delta_{\varepsilon_{i_l},\varepsilon_s}\mu^{\tau_s}(k_l,\varepsilon_s)\right\}\right],$$

where for $t=r$ we define $i_{r+1}-1\equiv n$ and a vacuous product is always taken to be 1, *i.e.*, $\Pi_{s=3}^2\{\ \}\equiv 1$.

Lemma 4. *For any choice of ε_i, $\Sigma_{i=1}^n\tau_i=\tau<1$ and for any θ, there is a constant independent of κ with:*

$$\int dk_1\dots dk_n\, E^{\tau_1}(k_1)\dots E^{\tau_n}(k_n)\,\|b(k_n,\varepsilon_n)\dots b(k_1,\varepsilon_1)\,R_\kappa^{n/2}\theta\|^2\leqq \text{const}\ \|\theta\|^2\,. \tag{3}$$

Proof of Theorem 1. From the form of $P^{\tau,\varepsilon}_{i_1\dots i_r}$ we have

$$|P^{\tau,\varepsilon}_{i_1\dots i_r}(k_1,\dots k_r)|\leqq \text{const}\prod_{t=1}^{r}\left[E^{\tau_{i_t}}(k_t)\prod_{s=i_t+1}^{i_{t+1}-1}\left\{\sum_{l=1}^{t}E^{\tau_s}(k_l)\right\}\right].$$

Inserting this in (2) and applying Lemma 4 with $\psi=R_\kappa^{n/2}\theta$ we obtain:

$$\left\|\prod_1^n N_{\tau_i}^{(\varepsilon_i)^{\frac12}}R_k^{n/2}\theta\right\|^2\leqq \text{const}\,\|\theta\|^2 \tag{4}$$

with a constant independent of κ. Since R_κ converges in norm to R, $\prod_1^n N_{\tau_i}^{(\varepsilon_i)^{\frac12}}R_\kappa^{n/2}$ converges weakly on $D(\prod_1^n N_{\tau_i}^{(\varepsilon_i)^{\frac12}})$ to $\prod_1^n N_{\tau_i}^{(\varepsilon_i)^{\frac12}}R^{n/2}$ and the uniform bounds (4) then apply also to the case $\kappa=\infty$. This completes the proof of Theorem 1 and of Theorem 2 (i).

Proof of Theorem 2. (ii) We use $\|A\|=\|A^*A\|^{\frac12}$. Thus

$$\begin{aligned}\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-\delta}\delta R^{n/2}\right\|&=\left\|\delta R^{n/2}\left(\prod_1^n N_{\tau_i}\right)^{1-2\delta}\delta R^{n/2}\right\|^{\frac12}\\&\leqq\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-2(1-\alpha)\delta}\delta R^{n/2}\right\|^{\frac12}\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-2\alpha\delta}\delta R^{n/2}\right\|^{\frac12}\\&\leqq\text{const}\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-2\alpha\delta}\delta R^{n/2}\right\|^{\frac12},\end{aligned}\tag{5}$$

where we have used Theorem 1 in the form

$$\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-\delta'}\delta R^{n/2}\right\|^{\frac12}\leqq\left(\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12}R_{\kappa_1}^{n/2}\right\|+\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12}R_{\kappa_2}^{n/2}\right\|\right)^{\frac12}\leqq\text{const}\,.$$

The inequality (5) allows us to reduce the exponent $\frac12-\delta$ to $\frac12-2\alpha\delta$, $0<\alpha\leqq 1$. By iterating m times and choosing m,α carefully we reduce the exponent to $\frac12-(2\alpha)^m\delta$ with $0\leqq\frac12-(2\alpha)^m\delta\leqq\dfrac{\gamma}{2n}$, $\gamma<1$. Thus

$$\begin{aligned}\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac12-\delta}\delta R^{n/2}\right\|&\leqq\text{const}\left\|\left(\prod_1^n N_{\tau_i}\right)^{\frac{\gamma}{2n}}\delta R^{n/2}\right\|^{2^{-m}}\\&\leqq\text{const}\,\|N_\tau^{\gamma/2}\delta R^{n/2}\|^{2^{-m}}\\&\leqq\text{const}\,\kappa^{-\varepsilon}\quad\text{for some }\varepsilon>0\,,\end{aligned}$$

where we have used the norm convergence of $N_\tau^{\gamma/2}\delta R^{\frac{1}{2}}$, $\gamma<1$, which we have proved in [2, Theorem 2.4.1]. There remains only the specification of m and α. A suitable choice is

$$m>\log 2\delta/\log\left(1-\frac{\gamma}{n}\right),\qquad \alpha=\frac{1}{2}\left(\frac{1}{2\delta}\right)^{\frac{1}{m+1}},\qquad \delta<\frac{1}{2}.$$

(iii) We use the inequality, valid for $\tau\geqq 0$ and $\beta\geqq 1$

$$N_\tau\leqq N_{\tau/\beta}^{\beta}.$$

Thus with $0<\delta<\frac{1}{2}$ and $\beta>1$:

$$\prod_1^n N_{\tau_i}=\prod_1^n N_{\tau_i}^{1-2\delta}\prod_1^n N_{\tau_i}^{2\delta}\leqq\prod_1^n N_{\tau_i}^{1-2\delta}\prod_1^n N_{\tau_i/\beta}^{2\beta\delta}\leqq\prod_1^n N_{\tau_i}^{1-2\delta}\cdot N_{\tau/\beta}^{2n\beta\delta},\quad \tau=\sum_{i=1}^n\tau_i.$$

We now choose β so that $\tau+\tau/\beta<1$, i.e. $\beta>\max\left(1,\dfrac{\tau}{1-\tau}\right)$, and we choose $\delta>0$ sufficiently small that $2n\beta\delta=1-2\delta$, i.e., $\delta=(2(n\beta+1))^{-1}$. Then defining $\tau_{n+1}=\tau/\beta$ we have

$$\prod_1^n N_{\tau_i}\leqq\prod_1^{n+1}N_{\tau_i}^{1-2\delta}\quad\text{and}\quad\sum_1^{n+1}\tau_i<1$$

and with $R_\kappa(\zeta)=(H(g,\kappa)-\zeta)^{-1}$, $\zeta\leqq -1$, we obtain by (ii):

$$\left\|\prod_1^n N_{\tau_i}^{\frac{1}{2}}\delta R(\zeta)^{(n+1)/2}\right\|\leqq\left\|\prod_1^{n+1}N_{\tau_i}^{\frac{1}{2}-\delta}\delta R(\zeta)^{(n+1)/2}\right\|\leqq\text{const}\,\kappa^{-\varepsilon'}.\tag{6}$$

However, by Theorem 1 we have

$$\left\|\prod_1^n N_{\tau_i}^{\frac{1}{2}}\delta R(\zeta)^{(n+1)/2}\right\|\leqq\text{const}\,|\zeta|^{-\frac{1}{2}}.\tag{7}$$

Combining (6) and (7) using $\|A\|=\|A\|^\nu\,\|A\|^{1-\nu}$, $0\leqq\nu\leqq 1$, we get

$$\left\|\prod_1^n N_{\tau_i}^{\frac{1}{2}}\delta R(\zeta)^{(n+1)/2}\right\|\leqq\text{const}\,|\zeta|^{-\frac{1}{2}+\delta/2}\kappa^{-\varepsilon},\qquad\varepsilon=\varepsilon'\delta>0,$$

where $\delta>0$ has no relation to the δ used previously. Finally, using the identity, valid for $0<\delta<\frac{1}{2}$:

$$R_\kappa(\zeta)^{n/2+\delta}=c(\delta)\int_0^\infty d\lambda\,\lambda^{-\frac{1}{2}-\delta}R_\kappa(\zeta-\lambda)^{(n+1)/2},$$

where

$$c(\delta)=\left[\int_0^\infty d\lambda\,\lambda^{-\frac{1}{2}-\delta}(1+\lambda)^{-(n+1)/2}\right]^{-1}$$

we obtain for $\delta>0$:

$$\begin{aligned}\left\|\prod_1^n N_{\tau_i}^{\frac{1}{2}}\delta R(\zeta)^{n/2+\delta}\right\|&\leqq c(\delta)\int_0^\infty d\lambda\,\lambda^{-\frac{1}{2}-\delta}\left\|\prod_1^n N_{\tau_i}^{\frac{1}{2}}\delta R(\zeta-\lambda)^{(n+1)/2}\right\|\\&\leqq c(\delta)\int_0^\infty d\lambda\,\lambda^{-\frac{1}{2}-\delta}\text{const}\,|\zeta-\lambda|^{-\frac{1}{2}+\delta/2}\kappa^{-\varepsilon}\\&\leqq\text{const}\,\kappa^{-\varepsilon}\end{aligned}$$

which completes the proof of Theorem 2.

Proof of Lemmas 3, 4. Lemma 3 follows easily by induction from

$$b(k,\varepsilon_1)\,N_\tau^{(\varepsilon_2)^{\frac{1}{2}}} = (N_\tau^{(\varepsilon_2)} + \delta_{\varepsilon_1,\varepsilon_2}\mu(k,\varepsilon_1)^\tau)^{\frac{1}{2}}\,b(k,\varepsilon_1)$$

and

$$\|(N_\tau^{(\varepsilon)}+a)^{\frac{1}{2}}\,\psi\|^2 = \|N_\tau^{(\varepsilon)^{\frac{1}{2}}}\,\psi\|^2 + a\,\|\psi\|^2\,.$$

The proof of Lemma 4 depends on the renormalized pull-through formula [5]. This expansion takes the form:

$$\begin{aligned} b(k_n,\varepsilon_n)\dots b(k_1,\varepsilon_1)\,R_\kappa^\alpha(\zeta) = &\sum_{r=0}^{n}\ \sum_{i_r>\dots>i_1\geqq 1}^{n} (-)^{\delta_{i_r\dots i_1}^{r,\varepsilon}} \\ &\cdot U_{r,\kappa}^{(\alpha)}(\zeta-\mu(k_n,\varepsilon_n)\dots-\cancel{\mu(k_{i_r},\varepsilon_{i_r})}\dots-\cancel{\mu(k_{i_1},\varepsilon_{i_1})}\dots-\mu(k_1,\varepsilon_1);\,k_{i_r}\varepsilon_{i_r},\dots,k_{i_1}\varepsilon_{i_1}) \\ &\cdot b(k_n,\varepsilon_n)\dots \cancel{b(k_{i_r},\varepsilon_{i_r})}\dots\cancel{b(k_{i_1},\varepsilon_{i_1})}\dots b(k_1,\varepsilon_1)\,, \end{aligned} \tag{8}$$

where $0<\alpha\leqq 1$ and a slash denotes absence of a term. The indices $\delta_{i_r\dots i_1}^{r,\varepsilon}$ are given by:

$$\delta_{i_r\dots i_1}^{r,\varepsilon} = \sum_{s=1}^{r}\left(\sum_{j=i_s+1}^{i_{s+1}-1}\varepsilon_j\right)\left(\sum_{l=1}^{s}\varepsilon_{i_l}\right),$$

while the $U_{n,\kappa}^{(\alpha)}$ are defined inductively by:

$$U_{0,\kappa}^{(\alpha)}(\zeta) = R_\kappa(\zeta)^\alpha\,,$$

$$\begin{aligned} U_{n,\kappa}^{(\alpha)}(\zeta;k_n\varepsilon_n,\dots,k_1\varepsilon_1) &= b(k_n,\varepsilon_n)\,U_{n-1,\kappa}^{(\alpha)}(\zeta;k_{n-1}\varepsilon_{n-1},\dots,k_1\varepsilon_1) \\ &\quad -(-)^{\varepsilon_n\sum_1^n\varepsilon_i}\,U_{n-1,\kappa}(\zeta-\mu(k_n,\varepsilon_n);k_{n-1}\varepsilon_{n-1},\dots,k_1\varepsilon_1)\,b(k_n,\varepsilon_n)\,. \end{aligned}$$

For a fuller treatment of these defining relations and of (8) see [5]. In [5] we have proved that:

$$\int dk_1\dots dk_n\,E^{\tau_n}(k_n)\dots E^{\tau_1}(k_1)\,\|U_{n,\kappa}^{(\frac{1}{2})}(\zeta;k_n\varepsilon_n,\dots,k_1\varepsilon_1)\|^2 \leqq \text{const}\,, \tag{9}$$

uniformly in κ provided $\Sigma_{i=1}^n\tau_i<1$.

Returning to the proof of Lemma 4, we pull all of the $b(k_i,\varepsilon_i)$ through the first $R_\kappa^{\frac{1}{2}}$ in (3) obtaining by (8):

$$\begin{aligned} &\int dk_1\dots dk_n\,E^{\tau_n}(k_n)\dots E^{\tau_1}(k_1)\,\|b(k_n,\varepsilon_n)\dots b(k_1,\varepsilon_1)\,R^{n/2}\,\theta\|^2 \\ &\quad\leqq 2^n\int dk_1\dots dk_n\,E^{\tau_n}(k_n)\dots E^{\tau_1}(k_1)\,\|R^{\frac{1}{2}}(1-\mu(k_n,\varepsilon_n)\dots-\mu(k_1,\varepsilon_1)) \\ &\quad\cdot b(k_n,\varepsilon_n)\dots b(k_1,\varepsilon_1)\,R^{(n-1)/2}\,\theta\|^2 \\ &\quad+2^n\int dk_1\dots dk_n\,E^{\tau_n}(k_n)\dots E^{\tau_1}(k_1)\sum_{r=1}^{n}\ \sum_{i_r>\dots>i_1\geqq 1}^{n}\,, \\ &\|U_{r,\kappa}^{(\frac{1}{2})}(1-\mu(k_n,\varepsilon_n)\dots-\cancel{\mu(k_{i_r},\varepsilon_{i_r})}\dots-\cancel{\mu(k_{i_1},\varepsilon_{i_1})}\dots-\mu(k_1,\varepsilon_1);\,k_{i_r}\varepsilon_{i_r},\dots,k_{i_1}\varepsilon_{i_1})\|^2 \\ &\|b(k_n,\varepsilon_n)\dots\cancel{b(k_{i_r},\varepsilon_{i_r})}\dots\cancel{b(k_{i_1},\varepsilon_{i_1})}\dots b(k_1,\varepsilon_1)\,R^{(n-1)/2}\,\theta\|^2\,, \end{aligned} \tag{10}$$

where we have supressed κ for convenience and used

$$\left\|\sum_{i=1}^{m}a_i\right\|^2 \leqq m\sum_{i=1}^{m}\|a_i\|^2\,.$$

The proof of Lemma 4 now follows by induction on n. We assume the result (3) for all possible choices of τ_i, $i = 1, \ldots, m$ with $\Sigma_{i=1}^{m} \tau_i \leqq \tau$ and for all m, $1 \leqq m \leqq (n-1)$. The first term in (6) is

$$\begin{aligned}
&2^n \int dk_n \ldots dk_1 \, E^{\tau_n}(k_n) \ldots E^{\tau_1}(k_1) \\
&\quad \cdot \| R^{\frac{1}{2}}(\zeta - \mu(k_n, \varepsilon_n) \cdots - \mu(k_1, \varepsilon_1)) \, b(k_n, \varepsilon_n) \ldots b(k_1, \varepsilon_1) \, R^{(n-1)/2} \theta \|^2 \\
&= 2^n \int dk_{n-1} \ldots dk_1 \, E^{\tau_{n-1}}(k_{n-1}) \ldots E^{\tau_1}(k_1) \{ \int dk_n E^{\tau_n}(k_n) \, \| R^{\frac{1}{2}}(\zeta - \mu(k_n, \varepsilon_n) \ldots \mu(k_1, \varepsilon_1)) \\
&\quad \cdot (N_{\tau_n}^{(\varepsilon_n)} + \mu^{\tau_n}(k_n, \varepsilon_n))^{\frac{1}{2}} \, b(k_n, \varepsilon_n) \, N_{\tau_n}^{(\varepsilon_n) - \frac{1}{2}} b(k_{n-1}, \varepsilon_{n-1}) \ldots b(k_1, \varepsilon_1) \, R^{(n-1)/2} \theta \|^2 \} \\
&\leqq \text{const} \int dk_{n-1} \ldots dk_1 \, E^{\tau_{n-1}}(k_{n-1}) \ldots E^{\tau_1}(k_1) \, \| b(k_{n-1}, \varepsilon_{n-1}) \ldots b(k_1, \varepsilon_1) \, R^{(n-1)/2} \theta \|^2 \\
&\leqq \text{const} \, \|\theta\|^2
\end{aligned}$$

by the induction hypothesis.

We have used the first order estimate in the form

$$\begin{aligned}
&\int dk_n E^{\tau_n}(k_n) \, \| R^{\frac{1}{2}}(\zeta - \mu(k_n, \varepsilon_n)) \, (N_{\tau_n}^{(\varepsilon_n)} + \mu^{\tau_n}(k_n, \varepsilon_n))^{\frac{1}{2}} \, b(k_n, \varepsilon_n) \, N_{\tau_n}^{(\varepsilon_n) - \frac{1}{2}} \chi \|^2 \\
&\quad \leqq \text{const} \int dk_n E^{\tau_n}(k_n) \, \| b(k_n, \varepsilon_n) \, N_{\tau_n}^{(\varepsilon_n) - \frac{1}{2}} \chi \|^2 \\
&\quad \leqq \text{const} \, \| N_{\tau_n}^{(\varepsilon_n)\frac{1}{2}} N_{\tau_n}^{(\varepsilon_n) - \frac{1}{2}} \chi \|^2 = \text{const} \, \|\chi\|^2 .
\end{aligned}$$

For the remaining terms ($r \neq 0$) in (10) we use the estimate (9) for $U_{r,\kappa}^{(\frac{1}{2})}$. Thus

$$\begin{aligned}
&2^n \int dk_n \ldots dk_1 \, E(k_n)^{\tau_n} \ldots E^{\tau_1}(k_1) \sum_{r=1}^{n} \sum_{i_r > \cdots > i_1 \geqq 1}^{n} , \\
&\quad \| U_{r,\kappa}^{(\frac{1}{2})}(1 - \mu(k_n, \varepsilon_n) \cdots - \mu\cancel{(k_{i_r} \varepsilon_{i_r})} \cdots - \mu\cancel{(k_{i_1} \varepsilon_{i_1})} \cdots - \mu(k_1, \varepsilon_1); k_{i_r} \varepsilon_{i_r}, \ldots, k_{i_1} \varepsilon_{i_1}) \|^2 , \\
&\quad \| b(k_n, \varepsilon_n) \ldots b\cancel{(k_{i_r}, \varepsilon_{i_r})} \ldots b\cancel{(k_{i_1}, \varepsilon_{i_1})} \ldots b(k_1, \varepsilon_1) \, R^{(n-1)/2} \theta \|^2 \\
&\quad \leqq \text{const} \int dk_n \ldots d\cancel{k}_{i_r} \ldots d\cancel{k}_{i_1} \ldots dk_1 \, E^{\tau_n}(k_n) \ldots E^{\tau_{i_r}} \cancel{(k_{i_r})} \ldots E^{\tau_{i_1}} \cancel{(k_{i_1})} \ldots E^{\tau_1}(k_1) \\
&\quad \cdot \| b(k_n, \varepsilon_n) \ldots b\cancel{(k_{i_r} \varepsilon_{i_r})} \ldots b\cancel{(k_{i_1} \varepsilon_{i_1})} \ldots b(k_1, \varepsilon_1) \, R^{(n-r)/2} \theta \|^2 \\
&\quad \leqq \text{const} \, \|\theta\|^2
\end{aligned}$$

by the induction hypothesis.

Since the induction hypothesis is certainly valid for $n = 1$:

$$\begin{aligned}
\int dk \, E^{\tau}(k) \, \| b(k, \varepsilon) \, R_{\kappa}^{\frac{1}{2}} \theta \|^2 &\leqq \int dk \, \text{const} \, \mu(k, \varepsilon)^{\tau} \, \| b(k, \varepsilon) \, R_{\kappa}^{\frac{1}{2}} \theta \|^2 \\
&\leqq \text{const} \, \| N_{\tau}^{(\varepsilon)\frac{1}{2}} R_{\kappa}^{\frac{1}{2}} \theta \|^2 \\
&\leqq \text{const} \, \|\theta\|^2
\end{aligned}$$

by the first order estimate, Lemma 4 follows by induction.

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