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On the Continuity of the Boosts for Each Orbit

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Abstract. The possibility of a continuous choice of the boost for each orbit is studied by making use of some mathematical theorems on fibre bundles and it is shown that this choice is possible only for massive particles.

1. Introduction

From the Wigner pioneer work, [1] it is well known the characterization of the elementary particles by the unitary irreducible representations of $\tilde{\mathscr{P}}_0$ (universal covering group of $\mathscr{P}_0$). A realization is obtained by the Wigner-Mackey technical procedure of the induced representations (for a modern exposition, see e.g. Simms [2]). For such a realization it is necessary the choice of a point $\hat{p}$ on a certain orbit, and a transformation $L(p)$ of $SL(2, C)$ mapping $\hat{p}$ in a generic point p of this orbit. This mapping must be a Borel mapping, and we can consider this boost as a Borel section of the fibre bundle

$$SL(2, C) \to SL(2, C)/G_{\hat{p}}$$

where G_p denotes the little group of the point p.

There are four different kinds of orbits (strata), whose orbits we call $\Omega_m^{\pm}$, $V_0^{\pm}$, Ω_{im}, 0, and whose little groups respectively are $SU(2)$, Δ (euclidean plane group), $SU(1, 1)$ and $SL(2, C)$ [2].

For the first kind of orbits corresponding to the massive particles the most natural choice of the point $\hat{p}$ is $\hat{p} = (0, 0, 0, m)$ and for $L(p)$, a pure Lorentz transformation because of the polar decomposition of any matrix in $SL(2, C)$. (By using that the unitary matrices lie in the little group of the point $\hat{p}$.) This choice is not only a Borel map but a continuous one, that is to say, a cross section of the bundle.

We show here, by analyzing some topological properties of fibre bundles, that it is not possible to find a cross section in the other cases.

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2. The More Usual Sections

Let us now examine the sections more frequently given in the literature, and a method for building-up these sections.

On the light cone, the point $\hat{p}$ usually choosed is $\hat{p}=(0,0,1,1)$ and there are two sections more frequently used. The first one, is a triangular matrix [4, p. 41], [5, p. 189], and it is not defined for $p_0=-p_3$ and this discontinuity is unavoidable. The second one, is the Jacob-Wick boost, more appropriate for the helicity formalism [2, p. 66] and it is the product of the pure Lorentz transformation mapping $\hat{p}$ in $(0,0,p_0,p_0)$ (which depends continuously of p), by the spatial rotation mapping this former point on the generic point $(p_1\, p_2\, p_3\, p_0)$ and whose axis is orthogonal to the trivectors $(0,0,p_0)$ and (p_1,p_2,p_3). This rotation is only well defined if $p_3 \neq \pm p_0$. But if $p_3=p_0$ it is natural to take the identity rotation. This election is continuous except for $p_3=-p_0$, and cannot be extended in such a way that conserves continuity.

An analogous situation, slightly more complicated, occurs in the case of spacelike orbits [4, p. 41].

These peculiarities appear also in a method of construction of boost that generalizes directly the polar decomposition of a matrix $A \in SL(2,C)$.

$$A=H\cdot U$$

where H is positive definite hermitian matrix and $U \in SU(2)\cdot$ Being $SU(2)$ the little group of timelike points, we can prove a similar decomposition:

$$A=G\cdot V$$

where V is a element of the little group of another orbit and G is a matrix to determine. If A map $\hat{p}$ in p, all matrices mapping $\hat{p}$ in p have the same "part" G, and we can expect that $p\mapsto G(p)$ provides a natural section. But it occurs that, for the little groups Δ and $SU(1,1)$, such decomposition is not defined for all element of $SL(2,C)$ and this fact suggest that it is not possible to find a such decomposition continuously defined for all points. (Some attempts are dealt with in [6]; the trouble is with some triangular matrices.) This property is studied in the next section.

3. There Exists ∂ Cross-Section of the Bundle $SL(2,C)\to SL(2,C)/G_{\hat{p}}$?

This point is a classical problem of the fibre bundle theory. The fibre bundle we are considering is principal, and the so-called cross-section theorem [3, p. 36] applies: A principal bundle with group G is equivalent in G to the product bundle if and only if it admits a cross-section. The equivalence theorem [3, p. 17] shows, as a particular result, that a bundle which group G is equivalent in G to the product bundle if and only if both bundles are equivalent as coordinate bundles [3, p. 11], that is, if there exists a map h of the given bundle into the product bundle which induces the identity map of the common base space. It is known that, in this case, the map h is necessarily an homeomorphism, and, then, the bundle space is, topologically, a product of the base by the fibre. We can say:

If a principal bundle admits a cross section, the bundle space is the topological product of the base by the fibre spaces. (The converse is also true.) The homotopy groups of the bundle space, are, then, direct sum of those of the fibre and the base space [3, p. 93], and this result provides a strong necessary condition for the existence of a cross section.

To apply it in this case, we first find the homotopy groups of the bundle space $SL(2, C)$ and those of orbits and little groups, which are the base and the fibre of the bundle. It is easy to see that, topologically

a) $\Omega_m^{\pm} \sim \mathbb{R}^3$ $\qquad SU(2) \sim S^3$

b) $V_0^{\pm} \sim S^2 \times \mathbb{R}$ $\qquad \Delta \sim S^1 \times \mathbb{R}^2$

c) $\Omega_{im} \sim S^2 \times \mathbb{R}$ $\qquad SU(1,1) \sim S^1 \times \mathbb{R}^2$

$$SL(2, C) \sim S^3 \times \mathbb{R}^3 .$$

Because all the homotopy groups of $\mathbb{R}^n$ vanish the problem is reduced to knowing the homotopy groups of the spheres S^1, S^2, S^3. These groups are [3] (Z denoting the additive group of integers)

$$\begin{aligned} &\pi_i(S^n) = Z && i = n \\ &\pi_i(S^n) = 0 && i < n \\ &\pi_i(S^1) = 0 && i > 1 \\ &\pi_i(S^2) \approx \pi_i(S^3) && i \geqq 3 . \end{aligned}$$

In the first case, a), corresponding to massive particles, it is clear that all the homotopy groups of the bundle are direct sum of those of the orbit and little group (these of the orbit are of course, all 0), and this result would have inferred from the existence of a cross section. But in the other cases, b), c), the first (and the second) homotopy group of S^3 is not a direct sum of those of S^1 and S^2, and the corresponding bundles do not admit a cross section.

The same result can be reached by simple application of the direct sum theorem [3, p. 92] which applies to any bundle, even to a no principal one: If a bundle admits a cross-section, the $n-$th homotopy group $(n \geqq 2)$ of the bundle is a direct sum of the corresponding groups of fibre and base space. It is clear that for $n=2$ this condition (for cases b and c) is not verified, and we are lead to the negative answer.

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