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A Proof of the Unitarity of S-Matrix in a Nonlocal Quantum Field Theory

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Abstract. Using the formfactors which are entire analytic functions in a momentum space, nonlocality is introduced for a wide class of interaction Lagrangians in the quantum theory of one-component scalar field $\phi(x)$. We point out a regularization procedure which possesses the following features:

1. The regularized S^δ matrix is defined and there exists the limit

$$\lim_{\delta\to 0} S^\delta = S .$$

2. The Green positive-frequency functions which determine the operation of multiplication in $S \cdot S^+ \underset{\mathrm{Df}}{=} S \circledast S^+$ can be also regularized $\circledast^\delta$ and there exists the limit

$$\lim_{\delta\to 0} \circledast^\delta = \circledast \equiv \cdot \,.$$

3. The operator $J(\delta_1, \delta_2, \delta_3) = S^{\delta_1} \circledast^{\delta_2} S^{\delta_3 +}$ is continuous at the point $\delta_1 = \delta_2 = \delta_3 = 0$.

4. $$S^\delta \circledast^\delta S^{\delta +} \equiv 1 \quad \text{at} \quad \delta > 0 .$$

Consequently, the S-matrix is unitary, i.e.

$$S \circledast S^+ = S \cdot S^+ = 1 .$$

1. Introduction

The postulate of unitarity of S-matrix in quantum field theory is one of the principal requirements for the theory to be regarded as self-consistent and physically acceptable. Therefore, the proof of unitarity is crucial in constructing the S-matrix for various models of quantum field theory.

If the Lagrangian of interacting quantum field is given, the S-matrix is then looked for in the form of a formal expansion in powers of a function of "switching on" the interaction $g(x)$ (see, for example [1]):

$$S[g] = 1 + \sum_{n \geqq 1} \frac{1}{n!} \int dx_1 \ldots \int dx_n \, g(x_1) \ldots g(x_n) \, S_n(x_1, \ldots, x_n) \quad (1.1)$$

while defining the operator expression $S_n(x_1, \ldots, x_n)$ in (1.1) one finds out that the coefficient functions $K_{\ldots\alpha\ldots\beta\ldots}(x_1 x_n)$ in the expansion of $S_n(x_1, \ldots, x_n)$ in normal products of the quantized fields $u_\alpha(x)$

$$S_n(x_1, \ldots, x_n) = \sum_{\ldots\alpha\ldots\beta\ldots} K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n) \colon \ldots u_\alpha \ldots u_\beta \ldots \colon \tag{1.2}$$

are expressed in terms of the products of the causal functions of the field operators under consideration.

Since the causal functions have fairly strong singularities on the light cone, the products of such functions are not mathematically definite. This generates one of the main problems of quantum field theory – the so-called problem of ultraviolet divergences.

There are different ways of solving this problem. These are the subtractive procedure interpreted in varying manners in a local field theory [1, 2], the summation of asymptotic series for the Green functions and superpropagators [3], a nonlocal generalization of the theory [4], etc. An important point is that all these methods use an intermediate regularization making the S-matrix elements mathematically meaningful.

The coefficient functions in (1.2) are distributions or, in another words, generalized functions defined on some space of the test functions. They are constructed as a limit of the locally integrable functions $K^\delta_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)$ by introducting a regularization precedure given by the parameter δ, so that in the unproper sense [1], there exists a limit

$$\lim_{\delta\to 0} K^\delta_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n) = K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n) \tag{1.3}$$

or, differently,

$$S[g] = \lim_{\delta\to 0} S^\delta[g]\,. \tag{1.4}$$

It is absolutely obscure whether the S-matrix so obtained satisfies the initial axioms and especially the unitarity relation:

$$S[g]\, S^+[g] = S^+[g]\, S[g] = 1\,. \tag{1.5}$$

In the local theory [1, 2] the unitarity and causality conditions are directly used to formulate the subtractive method of regularization. This circumstance ensures the fulfilment of unitarity of the S-matrix in each order of perturbation theory, at least in the case of renormalizable interactions.

It has been proved [5] that the unitary S-matrix can be constructed in the case of local nonrenormalizable interactions of the polynomial and nonpolynomial type. Although the problem of constructing the unitary S-matrix in each order of perturbation theory has not yet been solved, some promising results have been obtained [3].

In the present paper we will prove the unitarity of S-matrix in a nonlocal quantum field theory [4].

Section 2 formulates the proof scheme; Section 3 describes the class of the interaction Lagrangians under consideration and the way of introducing nonlocality into the theory; Sections 4 and 5 give the regularization procedure employed; Section 6 investigates the algebraic implications of unitarity and presents the proof of unitarity of S-matrix in the n-th order of perturbation theory.

2. Proof Scheme of the Unitarity of S-Matrix

Suppose that S-matrix is known in the form of a functional expansion (1.1), where the operators $S_n(x_1, \ldots, x_n)$ are given by expansions of the type (1.2). If the S-matrix is finite and satisfies the axioms of quantum field theory (see [1]), then the coefficient function $K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)$ satisfy the following requirements:

1. They are translationally invariant, i.e.

$$K_{\ldots\alpha\ldots\beta\ldots}(x_1+a, \ldots, x_n+a) = K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n).$$

2. They are integrable on some space of sufficiently smoothly varying test functions A, i.e. there exists an integral

$$\int \cdots \int dx_1 \ldots dx_n K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n) < \infty$$

for any $f(x_1, \ldots, x_n) \in A$.

Below we shall describe in details the space of the test functions in reference.

If the coefficient functions $K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)$ are known, then the expansion for $S^+[g]$ is known as well.

Following Bogolubov and Shirkov [1], one can show that the coefficient functions obtained by multiplying two operator functions with different independent arguments

$$S_n(x_1, \ldots, x_n) \circledast S_m^+(y_1, \ldots, y_m) \tag{2.1}$$

and having the form

$$\begin{gathered} K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)\, K_{(-)}(x-y)\, K_{\ldots\mu\ldots\nu\ldots}(y_1, \ldots, y_m) \\ K_{(-)}(x-y) = \prod_{s,t} \Delta_{\sigma\lambda}^{(-)}(x_s - y_t) \\ \tilde{\Delta}_{\sigma\lambda}^{(-)}(k) \sim 2\pi\theta(k_0)\,\delta(k^2 - m^2) \end{gathered} \tag{2.2}$$

may be defined as generalized integrable functions on the space A. The sign $\circledast$ denotes the transition to a normal product of $u_\alpha(x)$ field operators in the product (2.1), according to the Wick theorem.

So, if the S-matrix in the form of expansion (1.1) is known to us, then the product

$$J[g] = S[g]\, S^+[g] \underset{\mathrm{Df}}{=} S[g] \circledast S^+[g]$$

is given in each order of perturbation theory as a generalized operator-valued function on the space of the test functions A.

Our problem is to prove that

$$S[g] \circledast S^+[g] = 1 .$$

We shall start from the method of definition a finite S-matrix through the use of an improper limiting transition and construct out proof in the following manner.

Suppose there exists a regularization procedure which possesses the following features:

1. The regularized functions $K^{\delta}_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)$ are continuous and bounded, and

$$\lim_{\delta\to 0} K^{\delta}_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n) = K_{\ldots\alpha\ldots\beta\ldots}(x_1, \ldots, x_n)$$

i.e. the regularized $S^{\delta}[g]$ matrix is defined and there exists the improper limit

$$\lim_{\delta\to 0} S^{\delta}[g] = S[g] .$$

2. The Green positive-frequency functions which determine the multiplication in (2.1) and (2.2) can be also regularized and there exists the limit:

$$\lim_{\delta\to 0} K^{\delta}_{(-)}(x-y) = K_{(-)}(x-y)$$

or symbolically

$$\lim_{\delta\to 0} \circledast^{\delta} = \circledast .$$

3. In the relation

$$J[g] = \lim_{\delta_1\to 0}\, \lim_{\delta_2\to 0}\, \lim_{\delta_3\to 0} S^{\delta_1}[g] \circledast^{\delta_2} S^{\delta_3 +}[g]$$

the limit is independent of the order of limiting transitions to the points $\delta_1 = \delta_2 = \delta_3 = 0$, i.e. the operator $J^{\delta_1;\delta_2;\delta_3}[g] = S^{\delta_1}[g] \circledast^{\delta_2} S^{\delta_3 +}[g]$ is continuous at the point $\delta_1 = \delta_2 = \delta_3 = 0$.

4. Regularization is chosen so that

$$S^{\delta}[g] \circledast^{\delta} S^{\delta +}[g] \equiv 1 \qquad (\delta > 0) .$$

Then there holds a chain of equalities such as follows

$$J[g] = S[g] \circledast S^+[g] = \lim_{\delta_1\to 0}\, \lim_{\delta_2\to 0}\, \lim_{\delta_3\to 0} S^{\delta_1}[g] \circledast^{\delta_2} S^{\delta_3 +}[g]$$
$$= \lim_{\delta\to 0} S^{\delta}[g] \circledast^{\delta} S^{\delta +}[g] = \lim_{\delta\to 0} 1 = 1 .$$

Hence

$$S[g]\,S^+[g]=1\,.$$

Our aim is to indicate in a nonlocal quantum field theory a regularization procedure which would satisfy all the requirements listed above. This will prove the unitarity of S-matrix.

3. Interaction Lagrangians and Nonlocal S-Matrix

3.1. Interaction Lagrangians

We shall consider the theory of an one-component scalar field $\phi(x)$ with the Lagrangian

$$\begin{aligned}&\mathcal{L}=\mathcal{L}_0(x)+g\mathcal{L}_I(x)\\&\mathcal{L}_0=\tfrac{1}{2}:\{\partial_\mu\phi(x)\,\partial_\mu\phi(x)-m^2\phi(x)\}:\\&g\mathcal{L}_I=gU(\phi(x))=g\sum_{n=0}^{\infty}\frac{u_n}{n!}:\phi^n(x):\end{aligned}\tag{3.1}$$

If for a sequence of real numbers $\{u_k\}_0^\infty$, there exists in a complex ξ plane a measure $\sigma(\xi)$ such that

$$u_n=i^n\int\xi^n\,d\sigma(\xi)\tag{3.2}$$

then $\mathcal{L}_I$ may be written as

$$\mathcal{L}_I(x)=\int d\sigma(\xi):e^{i\xi\phi(x)}:\tag{3.3}$$

The representation (3.3) is convenient in that it exhausts all the forms of self-action of the scalar field now under study.

For example,

$$\mathcal{L}_I(x)=:\phi^N(x):=\frac{(-i)^N N!}{2\pi i}\oint\frac{d\xi}{\xi^{N+1}}:e^{i\xi\phi(x)}:$$

or

$$\mathcal{L}_I(x)=U(\phi(x))=\int_{-\infty}^{\infty}d\beta\,\tilde{U}(\beta):e^{i\beta\phi(x)}:$$

for any continuous function $U(u)$ absolutely integrable over $(-\infty,+\infty)$.

If $U(z)$ is analytic in some neighbourhood of the real axis $|\mathrm{Im}\,z|<d$ then

$$\int_{-\infty}^{\infty}d\beta\,|\tilde{U}(\beta)|\,e^{a|\beta|}<\infty$$

for $0<a<d$.

Since the Lagrangian is Hermitian, the measure $\sigma(\xi)$ satisfies the condition

$$i^n\int\xi^n\,d\sigma(\xi)=(-i)^n\int(\xi^*)^n[d\sigma(\xi)]^*\,.\tag{3.4}$$

Furthermore, we assume that there exists a strictly increasing non-negative function $M(u)$ such that

$$\int |d\sigma(\xi)|\, e^{NM(|\xi|)} < \infty \tag{3.5}$$

for any $N>0$.

3.2. Perturbation Series for S-Matrix

The S-matrix for the theory with a Lagrangian of the form (3.3) is formally representable as

$$S[g] = \sum_{n \geq 0} \frac{i^n}{n!} \int dx_1 \ldots \int dx_n g(x_1) \ldots g(x_n) \int d\sigma(\xi_1) \ldots \int d\sigma(\xi_n) \tag{3.6}$$
$$\cdot :\exp\{i(\xi_1 \phi(x_1) + \cdots + \xi_n \phi(x_n))\}: \prod_{1 \leq i < j \leq n} \exp\{-\xi_i \xi_j \Delta_c(x_i - x_j)\}$$

where $\Delta_c(x)$ is the causal function of the scalar field:

$$\Delta_c(x) = \frac{1}{(2\pi)^4 i} \int \frac{d^4k\, e^{-ikx}}{m^2 - k^2 - i\varepsilon},$$

$g(x)$ belongs to the space of the test functions.

Introducing the notation

$$w_{ij} = w(\xi_i \xi_j, x_i - x_j) = e^{-\xi_i \xi_j \Delta_c(x_i - x_j)} - 1$$
$$= \sum_{n=1}^{\infty} \frac{(-\xi_i \xi_j)^n}{n!} [\Delta_c(x_i - x_j)]^n, \tag{3.7}$$

$$d\mu_j = dx_j d\sigma(\xi_j)\, g(x_j)\, e^{i\xi_j \phi(x_j)}, \tag{3.8}$$

the series (3.6) may be rewritten as

$$S[g] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \cdots \int : d\mu_1 \ldots d\mu_n : \prod_{1 \leq i < j \leq n} (1 + w_{ij}). \tag{3.9}$$

Here $w_{ij} = w(\xi_i \xi_j; x_i - x_j)$ is a two-point function – the so-called "super-propagator" – describing a sum of Feinman graphs such as

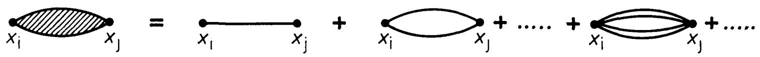

Fig. 1

In the S-matrix representation (3.9), firstly, the superpropagator $w(\xi_i; \xi_j; x_i - x_j)$ and, secondly, the superpropagator products

$$\prod_{i,j} w(\xi_i \xi_j; x_i - x_j) \tag{3.10}$$

are mathematically indefinite, because both the constructions are expressed in terms of the product of generalized singular functions.

One of the most important problems in modern quantum field theory is to specify the multiplication of generalized functions which would conform to the principial axioms of quantum field theory. There is a number of methods of definition this operation (see [5]). A detailed analysis of these methods is beyond the scope of this paper. We shall define $w(\xi_i \xi_j; x_i - x_j)$ as a nonlocal generalized function defined on a space Z_a and make a series of assumptions on its analytic properties which will enable us to define (3.10).

3.3. The Space of Test Functions Z_a

The space $Z_a(a \geqq 1)$ consists of all those entire functions $f(z_1 \dots z_n)$ of n complex variables $z_j = x_j + iy_j$ $(j = 1, \dots, n)$, which satisfy the following requirements:

1) For any $f(z_1, \dots, z_n) \in Z_a$, there exist positive

$$C > 0 \quad \text{and} \quad A_j > 0 \qquad (j = 1, \dots, n),$$

such that

$$|f(z_1, \dots, z_n)| < C \exp\left\{ \sum_{j=1}^{n} A_j |z_j|^a \right\}.$$

2) For any $y_1, \dots, y_n$

$$\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_n |f(x_1 + iy_1, \dots, x_n + iy_n)| < \infty .$$

The number a is chosen depending on the interaction Lagrangian under consideration and the way of introducing nonlocality into theory.

The space $\tilde{Z}_a$, which is a space of the Fourier transforms of the functions f of Z_a, is made up of the differentiable functions $\tilde{f}(p_1, \dots, p_n)$ which satisfy the condition:

3) There always exist positive C and B_j, such that

$$|\tilde{f}(p_1, \dots, p_n)| < C \exp\left\{ - \sum_{j=1}^{n} B_j |p_j|^\gamma \right\},$$

where

$$\gamma = \frac{a}{a-1} > 1 .$$

On the space $Z_a(a \geqq 1)$ one defines all nonlocal generalized functions $K(x)$ (for details see [6]), for which the Fourier transform $\tilde{K}(p^2)$ is an entire analytic function in the p^2-plane the order of which is

$$\varrho < \frac{1}{2}\gamma = \frac{a}{2(a-1)} .$$

3.4. Nonlocal Superpropagator

We assume that nonlocality is introduced into the theory so that the Fourier transform of the superpropagator $w(\xi_i \xi_j, x_i - x_j)$ may be written as

$$w(\xi, x) = \frac{1}{(2\pi)^4 i} \int d^4 k \, e^{-ikx} \tilde{w}(\xi, k^2), \tag{3.11}$$

$$\tilde{w}(\xi, k^2) = \int_{m^2}^{\infty} \frac{d\kappa^2 \varrho(\xi, \kappa^2, k^2)}{\kappa^2 - k^2 - i\varepsilon}. \tag{3.12}$$

Here

$$\xi = \xi_i \xi_j, \qquad x = x_i - x_j.$$

The Fourier transformation is interpreted in the sense of generalized functions.

The function $\varrho(\xi, \kappa^2, k^2)$ satisfies the conditions (A_1):

1) It is an entire analytic function in the complex k^2 plane, the order of which is

$$\frac{1}{2} \leqq \varrho < \frac{1}{2} \gamma = \frac{a}{2(a-1)}.$$

2) For $k^2 \to \pm \infty$

$$\varrho(\xi, \kappa^2, k^2) = \begin{cases} O(\exp\{A |k^2|^\varrho\}), & k^2 \to +\infty \\ O\left(\frac{1}{(-k^2)^{1+\lambda}}\right), & 0 < \lambda < 1, \; k^2 \to -\infty \end{cases}$$

for any ξ, uniformly for all $\kappa^2 \in [m^2, +\infty)$.

3) The function $\varrho(\xi, \kappa^2, k^2)$ is defined for all $\kappa^2 \in [m^2, +\infty)$, and decreases as $\kappa^2 \to \infty$, so that the function

$$v(\xi, k^2) = \int_{m^2}^{\infty} d\kappa^2 \varrho(\xi, \kappa^2, k^2)$$

is entire in ξ and k^2 planes.

Here the following estimation is true

$$|v(\xi, k^2)| \leqq C \exp\{A |k^2|^\varrho + M_1(|\xi|)\},$$

where the function $M_1(u)$ is such that for $u > 0$

$$M_1(u_1 u_2) \leqq M(u_1) + M(u_2)$$

(see (3.5)). As $k^2 \to -\infty$

$$v(\xi; k^2) = O\left(\frac{1}{(-k^2)^{1+\lambda}}\right).$$

4) The function $\varrho(\xi, \kappa^2, k^2)$ is normalized by the condition

$$\varrho(\xi, \kappa^2, k^2)|_{\kappa^2=k^2} = \varrho_0(\xi, k^2)$$

$$= -\xi\delta(\kappa^2 - m^2) + \sum_{n=2}^{\left[\frac{\kappa}{m}\right]} \frac{(-\xi)^n}{n!} \Omega_n(\kappa^2);$$

$$\Omega_n(\kappa^2) = \frac{1}{(2\pi)^{3(n-1)}} \int \frac{d\boldsymbol{k}_1}{2\omega_1} \cdots \int \frac{d\boldsymbol{k}_n}{2\omega_n} \delta^{(4)}(p - k_1 - \cdots - k_n)$$

$$\omega = \sqrt{m^2 + \boldsymbol{k}^2}, \quad p^2 = p_0^2 - \boldsymbol{p}^2 = \kappa^2 .$$

5) $$[\varrho(\xi, \kappa^2, k^2)]^* = \varrho(\xi^*, \kappa^2, (k^2)^*) .$$

The methods available in a nonlocal quantum field theory (see [4, 5]) lead to the superpropagators $w(\xi, k^2)$ having all the properties listed above.

Consider one of the possible versions. We assume that nonlocality is introduced into the theory so that the causal function of the scalar field changes according to

$$\frac{1}{m^2 - k^2 - i\varepsilon} \to \frac{V(l^2 k^2)}{m^2 - k^2 - i\varepsilon}, \tag{3.13}$$

where the function $V(z)$ satisfies the conditions (A_2):

1) It is an entire analytic function in the z-plane of some finite order $\frac{1}{2} \leqq \varrho < \infty$,

2) $$[V(z)]^* = V(z^*),$$

3) $$V(l^2 m^2) = 1,$$

4) $$V(z) = O\left(\frac{1}{|z|^{1+\lambda}}\right), \quad 0 < \lambda < 1, \quad \mathrm{Re}\, z \to -\infty .$$

For example,

$$V_1(l^2 k^2) = e^{-l^2(m^2 - k^2)}, \tag{3.14a}$$

$$V_2(l^2 k^2) = \left[\frac{\mathrm{Sin}\, l\sqrt{m^2 - k^2}}{l\sqrt{m^2 - k^2}}\right]^4, \tag{3.14b}$$

$$V_3(l^2 k^2) = ml\left[I_0(lk^2)\, K_1(ml) + \frac{I_1(l\sqrt{k^2})}{l\sqrt{k^2}}\, ml\, K_0(ml)\right] \tag{3.14c}$$

where $I_\nu(z)$ and $K_\nu(z)$ are the Bessel functions.

The change the propagator (3.13) in the Euclidean x-space means that the two propagators

$$\Delta(m, x_E^2) = \frac{1}{(2\pi)^4} \int \frac{d^4 k_E\, e^{ik_E x_E}}{m^2 + k_E^2} \tag{3.15}$$

and

$$D(x_E^2)=\frac{1}{(2\pi)^4}\int\frac{d^4k_E\,V(-l^2k_E^2)}{m^2+k_E^2}\,e^{ik_Ex_E} \tag{3.16}$$

are related by

$$D(x_E^2)=\Delta(m,x_E^2)\,\vartheta(x_E^2)\,.$$

It is easily to show that the function $\vartheta(x_E^2)$ is real and satisfies the conditions

$$\vartheta(u)=\begin{cases}O(u^\sigma), & \sigma=1+\lambda,\quad u\to 0\\ 1+O\left(\exp\left\{-au^{\frac{\varrho}{2\varrho-1}}\right\}\right), & u\to+\infty\end{cases} \tag{3.17}$$

where a is a constant.

Conversely, if $D(x_E^2)=\Delta(m,x_E^2)\,\vartheta(x_E^2)$, where $\vartheta(x_E^2)$ satisfies the conditions (3.17), then in the momentum representation

$$\tilde{D}(k^2)=\frac{V(l^2k^2)}{m^2-k^2-i\varepsilon},$$

where $V(z)$ satisfies the conditions (A_2).

Consider now the function $w(\xi,x)$ in the Euclidean x-space. We perform identity transformations such as

$$\begin{aligned} w(\xi,x_E)&=\exp\{-\xi D(x_E^2)\}-1\\ &=\sum_{n=1}^{\infty}\frac{(-\xi)^n}{n!}[D(x_E^2)]^n=\sum_{n=1}^{\infty}\frac{(-\xi)^n}{n!}[\Delta(m,x_E^2)]^n\,\vartheta^n(x_E^2)\\ &=\sum_{n=1}^{\infty}\frac{(-\xi)^n}{n!}\int_{(mn)^2}^{\infty}d\kappa^2\,\Omega_n(\kappa^2)\,\Delta(\kappa,x_E^2)\,\vartheta^n(x_E^2)\\ &=\int_{m^2}^{\infty}d\kappa^2\sum_{n=1}^{\infty}\frac{(-\xi)^n}{n!}\,\Omega_n(\kappa^2)\,\Delta(\kappa,x_E^2)\,\vartheta^n(x_E^2)\,. \end{aligned} \tag{3.18}$$

Since $\vartheta^n(u)$ satisfies (3.17), where $\sigma=n(1+\lambda)$, we get

$$\tilde{w}(\xi,k_E^2)=\int_{m^2}^{\infty}\frac{d\kappa^2\,\varrho(\xi,\kappa^2,k_E^2)}{\kappa^2+k_E^2}, \tag{3.19}$$

$$\varrho(\xi,\kappa^2,k_E^2)=\sum_{n=1}^{\left[\frac{\kappa}{m}\right]}\frac{(-\xi)^n}{n!}\,\Omega_n(\kappa^2)\,V_n(\kappa^2,k_E^2)\,. \tag{3.20}$$

Here the function

$$V_n(\kappa^2,k_E^2)=(\kappa^2+k_E^2)\int d^4x_E\,\Delta(\kappa,x_E^2)\,[\vartheta(x_E^2)]^n\,e^{ik_Ex_E}$$

is entire in the k^2 plane and satisfies all the conditions (A_2). Moreover, for some $C>0$ and $a>0$

$$|V_n(\kappa^2, z)| \leqq C \frac{e^{a|z|^\varrho}}{(\kappa^2)^{n(1+\lambda)}} . \tag{3.21}$$

For example, for the formfactor $V_3(l^2k^2)$ in (3.14c) the corresponding function is $\vartheta(u) = \theta(u-l^2)$ and $V_n(\kappa^2, l^2k^2) = V_3(l^2k^2)$ at $m=\kappa$.

Going in (3.19) and (3.20) to the Minkowski space, we obtain the representation (3.12), where the function $\varrho(\xi, \kappa^2, k^2)$ satisfies all the conditions (A_1).

For the superpropagators (3.12) we can uniquely define a product integrable on the space Z_a through the use of the intermediate regularization $R^\delta(\frac{1}{2} \leqq \varrho < 1)$ or the postulate of "Euclideanness" $(\varrho \geqq 1)$. This ensures the finiteness of S-matrix in each order of perturbation theory.

In the present paper we shall consider another regularization which enables one to uniquely define the product of superpropagators for any finite order ϱ.

4. Regularization Procedure

4.1. Mellin Representation for the Function $\varrho(\xi, \kappa^2, k^2)$ and the Introduction of Regularization

The Mellin representation

$$\varrho(\xi, \kappa^2, k^2) = \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\zeta\, R(\xi, \kappa^2, \zeta)}{\sin \pi\zeta} (\kappa^2 - k^2 - i\varepsilon)^\zeta \tag{4.1}$$

with $1<\beta<2$ holds for the function $\varrho(\xi, \kappa^2, k^2)$ (which was discussed in 3.4) in the region $k^2<\kappa^2$.

Proceeding from the properties of the function $\varrho(\xi, \kappa^2, k^2)$, one can obtain:

1) The function $R(\xi, \kappa^2, \zeta)$ is regular in the half-plane $\mathrm{Re}\,\zeta > -2$, and in this region

$$|R(\xi, \kappa^2, \zeta)| \leqq \frac{C(|\xi|, \kappa^2)\, e^{\pi|y|}}{(1+|y|)^N \Gamma\left(1+\frac{|x|}{\varrho}\right)} \tag{4.2}$$

for $\forall N>0$, $\zeta = x+iy$.

At the point $\zeta = -1$ the function has zero of the first order at least.

2) $$R(\xi, \kappa^2, 0) = \varrho_0(\xi, \kappa^2) .$$

3) $$[R(\xi, \kappa^2, \zeta)]^* = R(\xi^*, \kappa^2, \zeta^*) .$$

4) The function

$$r(\xi,\zeta)=\int\limits_{m^2}^{\infty} d\kappa^2 R(\xi,\kappa^2,\zeta)$$

is entire in the ξ plane and regular in the half-plane $\mathrm{Re}\,\zeta>-2$.

The following estimation holds for this function:

$$|r(\xi,\zeta)|\leqq C\frac{e^{\pi|y|+M_1(|\xi|)}}{(1+|y|)^N\,\Gamma\left(1+\frac{|x|}{\varrho}\right)}$$

for any $N>0$. At $\zeta=-1$ the function has zero of the first order.

We emphasise once more that the representation (4.1) holds for $k^2<\kappa^2$. For the passage to the region $k^2>\kappa^2$, one has to go over from the integration in the ζ-plane over the contour

$$L=L_0=\{\zeta : x=-\beta, \forall\, y\}$$

to that over the contour

$$L_\theta=\left\{\zeta:\mathrm{Re}\,\zeta>-2,\ \arg\zeta=\pm\left(\frac{\pi}{2}-\theta\right),\ \text{for}\ |\zeta|\to\infty\right\},$$

as shown in Fig. 2.

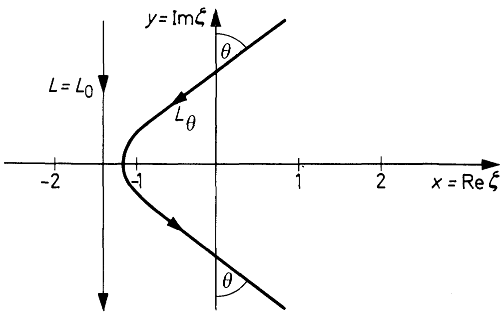

Fig. 2

The formula (4.1) with the integral over the contour $L_\theta\left(0<\theta\leqq\frac{\pi}{2}\right)$ gives the representation of $\varrho(\xi,\kappa^2,k^2)$ for all $k^2\in(-\infty,+\infty)$.

Let us introduce in our discussion a regularized function

$$\varrho^\delta(\xi,\kappa^2,k^2)=\frac{1}{2i}\int\limits_{L_\theta}\frac{d\zeta\, R(\xi,\kappa^2,\zeta)\, e^{\delta\zeta^2}}{\sin\pi\zeta}(\kappa^2-k^2-i\varepsilon)^\zeta \tag{4.3}$$

where

$$0 \leqq \theta \leqq \frac{\pi}{4}.$$

One can readily recognize that with $\delta > 0$ the function $\varrho^\delta(\xi, \kappa^2, k^2)$ is

a) regular in all the complex k^2-plane, except for the branch point with $k^2 = \kappa^2 - i\varepsilon$,

b) $$\varrho^\delta(\xi, \kappa^2, k^2) = O\left(\frac{1}{|k^2|^{1+\lambda}}\right), \quad |k^2| \to \infty. \tag{4.4}$$

c) $$\lim_{\delta \to 0} \varrho^\delta(\xi, \kappa^2, k^2) = \varrho(\xi, \kappa^2, k^2). \tag{4.5}$$

4.2. Regularization of the Superpropagator w_{ij}

Using (4.3), we obtain a regularized expression

$$\tilde{w}^\delta(\xi, k^2) = \int_{m^2}^{\infty} \frac{d\kappa^2\, \varrho^\delta(\xi, \kappa^2, k^2)}{\kappa^2 - k^2 - i\varepsilon} \tag{4.6}$$

for the superpropagator $\tilde{w}(\xi, k^2)$.

The function $\tilde{w}^\delta(\xi, k^2)$ satisfies the conditions (A_3):

1) It is analytic in the whole k^2 plane, except for the branch line along the beam $[\kappa^2, +\infty)$.

2) $$\tilde{w}^\delta(\xi, k^2) = O\left(\frac{1}{|k^2|^{2+\lambda}}\right), \quad |k^2| \to \infty, \quad 0 < \lambda < 1.$$

3) $$\lim_{\delta \to 0} \tilde{w}^\delta(\xi, k^2) = \tilde{w}(\xi, k^2)$$

for any k^2.

4) There exists the Fourier transform

$$\begin{aligned} w^\delta(\xi, x) &= \frac{1}{(2\pi)^4 i} \int d^4k\, \tilde{w}^\delta(\xi, k^2)\, e^{-ikx} \\ &= \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\zeta\, e^{\delta\zeta^2}}{\sin\pi\zeta} \int_{m^2}^{\infty} d\kappa^2 R(\xi, k^2, \zeta)\, D(\zeta, \kappa^2, x) \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} D(\zeta, \kappa^2, x) &= \frac{1}{(2\pi)^4 i} \int \frac{d^4k\, e^{-ikx}}{(\kappa^2 - k^2 - i\varepsilon)^{1-\zeta}} \\ &= i \frac{2^\zeta \kappa^{1+\zeta} e^{-i\pi\zeta}}{8\pi\Gamma(1-\zeta)} \frac{H^{(2)}_{1+\zeta}(\kappa\sqrt{x^2 - i\varepsilon})}{(\sqrt{x^2 - i\varepsilon})^{1+\zeta}}. \end{aligned} \tag{4.8}$$

$H^{(2)}_\nu(u)$ is the Hankel function.

The function $w^\delta(\xi, x)$ is bounded at zero, i.e., $w^\delta(\xi, 0) < \infty$ since

$$D(\zeta, \kappa^2, 0) = \frac{(\kappa^2)^{1+\zeta}}{4\pi^2} \frac{1}{\zeta(1+\zeta)}.$$

In this sense of generalized functions

$$\lim_{\delta\to 0} w^\delta(\xi, x) = w(\xi, x),$$

i.e.

$$\lim_{\delta\to 0} \int d^4 x w^\delta(\xi, x)\, f(x) = \int d^4 x w(\xi, x)\, f(x)$$

for any

$$f(x) \in Z_a.$$

Thus, with $\delta > 0$ the superpropagator $w^\delta(\xi, x)$ is a continuous bounded function, so that the products of such functions are well defined.

The regularized S-matrix may be written as

$$S^\delta[g] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \cdots \int : d\mu_1 \ldots d\mu_n : \prod_{1\leqq i<j\leqq n} (1 + w_{ij}^\delta). \tag{4.9}$$

Using the condition (3.4), we obtain for the Hermitian conjugate S^δ-matrix

$$S^{\delta +}[g] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int : d\mu_1 \ldots d\mu_n : \prod_{1\leqq i<j\leqq n} (1 + w_{ij}^{\delta +}) \tag{4.10}$$

where

$$w_{ij}^{\delta +} = [w^\delta(\xi_i^* \xi_j^*, x_i - x_j)]^*. \tag{4.11}$$

4.3. Removal of Regularization in the Matrix Elements of S Matrix

We now consider how to go over to the limit $\delta \to 0$ in the integrals defining some arbitrary matrix element of S-matrix. We shall follow the method given in [4].

In the x-space the integral for an arbitrary connected Feynman diagram G in the n-th approximation of perturbation theory of the regularized S^δ-matrix is given by

$$F^\delta(x_1, \ldots, x_n) = \prod_{i,j\in G} w^\delta(\xi_i \xi_j, x_i - x_j) \tag{4.12}$$

where $i, j = 1, \ldots, n$, according to a given connected diagram G. If in (4.12) we go over to a momentum representation, then we find

$$\begin{aligned} \tilde{F}^\delta(p_1, \ldots, p_n) &= \int dx_1 \ldots \int dx_n e^{i(p_1 x_1 + \cdots + p_n x_n)} F^\delta(x_1, \ldots, x_n) \\ &= (2\pi)^4 \delta^4(p_1 + \cdots + p_n)\, T^\delta(p_1, \ldots, p_n) \end{aligned} \tag{4.13}$$

$$T^\delta(p_1, \ldots, p_n) = \int \cdots \int \prod_{i=1}^{L} d^4 l_i \prod_{j=1}^{N} w^\delta(\xi_j, k_j^2)$$

where N is the number of internal lines in a diagram; L is the number of independent integrations. Here $\xi_j = \xi_{j1}\xi_{j2}$ and k_j is the four-momentum corresponding to a given line in the diagram which connects the points x_{j1} and x_{j2}; l_i are the four-momenta over which the integration is extended. The momentum k_j is a sum such as

$$k_j = \sum_{v=1}^{L} \theta_{jv} l_v + \sum_{\mu=1}^{n} \vartheta_{j\mu} p_\mu ,$$

where the numbers θ_{jv} and $\vartheta_{j\mu}$ can take only one of the three values $-1, 0, +1$ depending on any given line in the diagram.

The integrals in (4.13) are well convergent for $\delta > 0$, since the functions $\tilde{w}^\delta(\xi, k^2)$ decrease as $(k^2)^{-(2+\lambda)}$, according to the conditions (A_3). We now take the following step. We substitute the representation (4.6) into (4.13), and make use of the Feynman parametrization:

$$\begin{aligned} T^\delta(p_1, \ldots, p_n) &= \int_{m^2}^{\infty} d\kappa_1^2 \ldots \int_{m^2}^{\infty} d\kappa_N^2 \int \cdots \int \prod_i d^4 l_i \prod_j \frac{\varrho^\delta(\xi_j, \kappa_j^2, k_j^2)}{\kappa_j^2 - k_j^2 - i\varepsilon} \\ &= (N-1)! \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_N \, \delta\left(1 - \sum_{i=1}^{N} \alpha_i\right) \int_{m^2}^{\infty} d\kappa_1^2 \ldots \int_{m^2}^{\infty} d\kappa_N^2 \\ &\quad \cdot \int \cdots \int \prod_i d^4 l_i \frac{\prod_j \varrho^\delta(\xi_j, \kappa_j^2, k_j^2)}{\left[\sum_{j=1}^{N} \alpha_j(\kappa_j^2 - k_j^2) - i\varepsilon\right]^N} . \end{aligned} \tag{4.14}$$

From the expression standing in the denominator one can always remove the terms linear in l_j by transforming the variables of integrations like $l_j = l'_j + \sum_i A_{ji} p_i$. Then we obtain

$$\sum_{j=1}^{N} \alpha_j(\kappa_j^2 - k_j^2) = \phi(\alpha, \kappa^2, p_i p_j) + K(\alpha, l'_i l'_j) .$$

Here $K(\alpha, l'_i l'_j)$ is a uniform quadratic form of the new variables of the integration l'_j with coefficients dependent only on the parameters α_i; $\phi(\alpha, \kappa^2, p_i p_j)$ is a nonuniform quadratic form of the vectors $(p_i p_j)$ characterizing the external momenta of any given diagram.

So we have

$$\begin{aligned} T^\delta(p_1, \ldots, p_n) &= (N-1)! \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_N \int_{m^2}^{\infty} d\kappa_1^2 \ldots \int_{m^2}^{\infty} d\kappa_N^2 \\ &\quad \cdot \delta(1 - \Sigma \alpha_j) \cdot \int \cdots \int \prod_i d^4 l'_i \frac{\prod_j \varrho^\delta(\xi_j, \kappa_j^2, k_j'^2)}{[\phi(\alpha, \kappa^2, p_i p_j) + K(\alpha, l'_i l'_j) - i\varepsilon]^N} \end{aligned} \tag{4.15}$$

where $k'_j = k'_j(l', p, \alpha)$ is the momentum corresponding to the j line. After the variables are replaced, it is dependent on l'_j, p_j, α_k.

Now in the expression (4.15) we can go to the Euclidean metric, rotating the contours of integration over l_{j0} by the angle $\frac{\pi}{2}$, i.e. $l_{i0}\to i l_{i4}$. Then we get

$$T^{\delta}(p_1,\dots,p_n)=(N-1)\int_0^1 d\alpha_1\dots\int_0^1 d\alpha_N\int_{m^2}^{\infty} d\kappa_1^2\dots\int_{m^2}^{\infty} d\kappa_N^2\,\delta(1-\Sigma\alpha_j) \tag{4.16}$$

$$\cdot i^L\int\cdots\int\prod_i d^4 l_{iE}\frac{\prod_j \varrho^{\delta}(\xi_j,\kappa_j^2,[k'_j(l'_E,p,\alpha)]^2)}{[\phi(\alpha,\kappa^2,p_i p_j)+K(\alpha,l'_{iE}l'_{jE})-i\varepsilon]^N}$$

In the integral (4.16) we can go to the limit $\delta=0$, since the functions $\varrho(\xi_j,\kappa_j^2,[k'_j(l'_E,p,\alpha)]^2)$ decrease when $k_j'^2\to-\infty$, and the integral (4.16) converges in the limit $\delta=0$.

Note that while rotating the contours of integration over l_{j0}, it is necessary to take into account the cuts of the functions $\varrho^{\delta}(\xi,\kappa^2,k^2)$ in the plane l_{j0}. This results in the fact that in addition to the Euclidean integrals, there appear contributions arising from the integration over the edges of the cuts of the functions $\varrho^{\delta}(\xi,\kappa^2,k^2)$ in the planes l_{j0}. But with $\delta=0$ these contributions vanish, since the functions $\varrho^{\delta}(\xi,\kappa^2,k^2)$ at $\delta=0$ become entire analytic functions. Therefore, while rotating the countours it is important to correctly circumvent the singularities of Feynman denominators. This is ensured by the representation (4.16).

5. The Functions $w^{(\pm)}(\xi,x)$

5.1. Regularization of the Functions $w^{(\pm)}(\xi,x)$

Consider the functions

$$w_{ij}^{(\pm)}=w^{(\pm)}(\xi_i\xi_j,x_i-x_j)=e^{-\xi_i\xi_j\Delta^{(\pm)}(x_i-x_j)}-1$$
$$=\sum_{n=1}^{\infty}\frac{(-\xi_i\xi_j)^n}{n!}[\Delta^{(\pm)}(x_1-x_j)]^n. \tag{5.1}$$

Using the representation

$$[\Delta^{(\pm)}(x)]^n=\int_{(nm)^2}^{\infty} d\kappa^2\,\Omega_n(\kappa^2)\,\Delta^{(\pm)}(\kappa,x) \tag{5.2}$$

which holds for $x\neq 0$, one can easily obtain

$$w^{(\pm)}(\xi,x)=\int_{m^2}^{\infty} d\kappa^2\,\varrho_0(\xi,\kappa^2)\,\Delta^{(\pm)}(\kappa,x). \tag{5.3}$$

In general, this integral exists in improper sense all $x\neq 0$ (see, for example, [7]).

The Fourier transform of the function is equal

$$\tilde{w}^{(\pm)}(\xi, k^2) = \int_{m^2}^{\infty} d\kappa^2 \varrho_0(\xi, \kappa^2)\, \tilde{\Delta}^{(\pm)}(\kappa, k) \\ = 2\pi\theta(\mp k_0)\, \theta(k^2 - m^2)\, \varrho_0(\xi, k^2)\,. \tag{5.4}$$

Making use of the relation

$$\tilde{\Delta}^{(\pm)}(\kappa, k) = 2\pi\theta(\mp k_0)\, \delta(k^2 - \kappa^2) \\ = \frac{1}{i}\left[\frac{1}{\kappa^2 - k^2 - i\varepsilon} - \frac{1}{\kappa^2 - k^2 \mp i\varepsilon k_0}\right], \tag{5.5}$$

we introduce in our discussion the regularized functions

$$\tilde{w}^{\delta}_{(\pm)}(\xi, k) = \int_{m^2}^{\infty} d\kappa^2 \varrho^{\delta}(\xi, \kappa^2, k^2)\, \tilde{\Delta}^{(\pm)}(\kappa, k) \\ = \frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\zeta\, e^{\delta\zeta^2}}{\sin\pi\zeta} \int_{m^2}^{\infty} d\kappa^2 R(\xi, \kappa^2, \zeta)\, \tilde{D}_{(\pm)}(\zeta, k) \tag{5.6}$$

where

$$\tilde{D}_{(\pm)}(\zeta, k) = \frac{1}{i}\left[(\kappa^2 - k^2 - i\varepsilon)^{\zeta-1} - (\kappa^2 - k^2 - i\varepsilon k_0)^{\zeta-1}\right] \\ = 2\pi\theta(\mp k_0)\, \theta(k^2 - \kappa^2)\, \frac{\sin\pi\zeta}{\pi}(k^2 - \kappa^2)^{\zeta-1}\,. \tag{5.7}$$

The representation (6.7) implies that in the complex k_0-plane the contour of integration should be such as shown in Fig. 3.

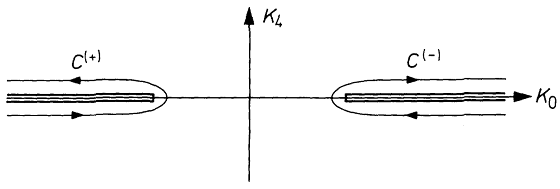

Fig. 3

In the x-representation one can get

$$w^{\delta}_{(\pm)}(\xi, x) = \frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\zeta\, e^{\delta\zeta^2}}{\sin\pi\zeta} \int_{m^2}^{\infty} d\kappa^2 R(\xi, \kappa^2, \zeta)\, D_{(\pm)}(\zeta, x)\,, \tag{5.8}$$

$$D_{(\pm)}(\zeta, x) = \frac{1}{(2\pi)^4 i}\int d^4k\, e^{-ikx}\, \tilde{D}_{(\pm)}(\zeta, k) \tag{5.9}$$

$$= i\frac{2^{\zeta}\kappa^{1+\zeta}}{8\pi\Gamma(1-\zeta)}\left\{\frac{e^{-i\pi\zeta}H^{(2)}_{1+\zeta}(\kappa\sqrt{x^2 - i\varepsilon})}{(\sqrt{x^2 - i\varepsilon})^{1+\zeta}} + 2\theta(\pm x_0)\,\theta(x^2)\frac{J_{-1-\zeta}(\kappa\sqrt{x^2})}{(\sqrt{x^2})^{1+\zeta}}\right\}.$$

It is easily seen that

$$\tilde{D}_{(\pm)}(0,k)=\tilde{\Delta}^{(\pm)}(\kappa,k)$$
$$D_{(\pm)}(0,x)=\Delta^{(\pm)}(\kappa,x)\,. \tag{5.10}$$

The functions $D_{(\pm)}(\zeta,x)$ and $\tilde{D}_{(\pm)}(\zeta,k)$, as generalized functions defined on the space Z_a, are entire analytic functions and increase as $\Gamma\left(\frac{2\zeta}{\gamma}\right)$ when $\operatorname{Re}\zeta\to+\infty$.

Since the function $R(\xi,\kappa^2,\zeta)$ decreases as $\left[\Gamma\left(\frac{\zeta}{\varrho}\right)\right]^{-1}$, and $\frac{1}{\varrho}>\frac{2}{\gamma}$, the integral in (5.8) is convergent, provided that one goes to the contour of integration $L_\theta\left(0<\theta\leqq\frac{\pi}{4}\right)$ in the plane ζ. Than we can go to the limit $\delta\to0$, obtaining

$$\lim_{\delta\to0}\tilde{w}^\delta_{(\pm)}(\xi,k)=\tilde{w}_{(\pm)}(\xi,k)\,.$$

For $\delta>0$

$$\tilde{w}^\delta_{(\pm)}(\zeta,k)=O\left(\frac{1}{|k^2|^{2+\lambda}}\right)$$

as $k^2\to\pm\infty$. Here $0<\lambda<1$.

The function $w^\delta_{(\pm)}(\xi,x)$ is continuous and bounded for real x. At the point $x=0$ it is

$$w^\delta_{(\pm)}(\xi,0)=\frac{\pi^2}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}\frac{d\zeta\,e^{\delta\zeta^2}}{(\sin\pi\zeta)(1+\zeta)\,\zeta}\int_{m^2}^{\infty}d\kappa^2 R(\xi,\kappa^2,\zeta)<\infty\,.$$

5.2. Continuity in the Parameters $\delta_j\,(j=1,2,3)$

Following Bogolubov and Shirkov [1] one can show that the following statement holds for the regularization chosen:

If $K_1^\delta(x_1,\dots,x_n)$ and $K_2^\delta(y_1,\dots,y_m)$ are translationally invariant coefficient functions and in improper sense

$$\lim_{\delta\to0}K_1^\delta(x_1,\dots,x_n)=K_1(x_1,\dots,x_n)$$

$$\lim_{\delta\to0}K_2^\delta(y_1,\dots,y_m)=K_2(y_1,\dots,y_m)$$

where $K_1(x_1,\dots,x_n)$ and $K_2(y_1,\dots,y_m)$ are generalized functions defined on the space Z_a and the arguments $x_1,\dots,x_n;y_1,\dots,y_m$ are independent, then the limit

$$\lim_{\delta_1\to0}\lim_{\delta_2\to0}\lim_{\delta_3\to0}K_1^{\delta_1}(x_1,\dots,x_n)\,K_2^{\delta_2}(y_1,\dots,y_m)\,\Pi\,w^{\delta_3}_{(-)}(x_s-y_t)$$
$$=K_1(x_1,\dots,x_n)\,K_2(y_1,\dots,y_m)\,\Pi\,w_{(-)}(x_s-y_t)$$

exists as a generalized function on Z_a and is independent of the manner of going to the limit $\delta_1 = \delta_2 = \delta_3 = 0$.

The proof is based on the simple fact that the boundedness of the sum of negative frequencies implies the boundedness of every separate frequency.

Indeed, consider the functional

$$B(\delta_1, \delta_2, \delta_3) = \int dx_1 \dots \int dx_n \int dy_1 \dots \int dy_m f(x_1, \dots, x_n; y_1, \dots, y_m) \cdot K_1^{\delta_1}(x_1, \dots, x_n)\, K_2^{\delta_2}(y_1, \dots, y_m) \prod_{s,t} w_{(-)}^{\delta_3}(x_s - y_t)$$

where $f \in Z_a$. Let us go over to a momentum space. The translational invariance of the functions $K_j^{\delta}(\dots)$ $(j = 1, 2)$ yields

$$\int dx_1 \dots \int dx_n e^{i(p_1 x_1 + \dots + p_n x_n)} K_1^{\delta}(x_1, \dots, x_n) = \delta(p_1 + \dots + p_n)\, \tilde{K}_1^{\delta}(p_1, \dots, p_n)\,.$$

After simple transformations we obtain for

$$B(\delta_1, \delta_2, \delta_3) = \int d^4p_1 \dots \int d^4p_n \int d^4k_1 \dots \int d^4k_m \cdot \delta\left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} k_i\right) \tilde{f}(p_1, \dots, p_n; k_1, \dots, k_m) Q^{\delta_1 \delta_2 \delta_3}(p_1, \dots, p_n; k_1, \dots, k_m),$$

$$Q^{\delta_1 \delta_2 \delta_3}(p_1, \dots, p_n, k_1, \dots, k_m) = \prod_{s,t} \int dq_{st}\, \delta\left(\sum_{i=1}^{n} p_i - \sum_{st} q_{st}\right) \prod_{st} \tilde{w}_{(-)}^{\delta_3}(q_{st}) \cdot \tilde{K}_1^{\delta_1}\left(\dots, p_j - \sum_t q_{it}, \dots\right) \tilde{K}_2^{\delta_2}\left(\dots, k_j + \sum_s q_{sj}, \dots\right). \tag{5.11}$$

Since $\tilde{w}_{(-)}^{\delta}(q) \sim \theta(q_0)\, \theta(q^2 - m^2)$, the integration in (5.11) is over a finite region. Therefore, the limit with $\delta_j \to 0$ $(j = 1, 2, 3)$ exists independently in every parameter δ_j, since each of the functions $\tilde{K}_1^{\delta_1}$, $\tilde{K}_2^{\delta_2}$ and $\tilde{w}_{(-)}^{\delta_3}$ tends to a finite limit.

5.3. Relations between the Regularized Propagators

One can easily see that the regularized functions

$$w^{\delta}(\xi, x), \qquad w^{\delta +}(\xi, x), \qquad w_{(\pm)}^{\delta}(\xi, x)$$

are related (with $\delta > 0$) by

$$\begin{aligned} w^{\delta}(\xi, x) &= \theta(x_0)\, w_{(-)}^{\delta}(\xi, x) + \theta(-x_0)\, w_{(+)}^{\delta}(\xi, x) \\ w^{\delta +}(\xi, x) &= \theta(-x_0)\, w_{(-)}^{\delta}(\xi, x) + \theta(x_0)\, w_{(+)}^{\delta}(\xi, x) \\ w_{(-)}^{\delta}(\xi, x) &= \theta(x_0)\, w^{\delta}(\xi, x) + \theta(-x_0)\, w^{\delta +}(\xi, x) \\ w_{(+)}^{\delta}(\xi, x) &= \theta(-x_0)\, w^{\delta}(\xi, x) + \theta(x_0)\, w^{\delta +}(\xi, x) \end{aligned} \tag{5.12}$$

where

$$\theta(u)=\begin{cases}1, & u>0\\ 0, & u<0.\end{cases}$$

6. Proof of the Unitarity of *S*-Matrix in the *n*-th Order of Perturbation Theory

6.1. Unitarity in Perturbation Theory

Using the representation (3.9) for the *S*-matrix and the unitarity relation

$$SS^+=S^+S=1,$$

we obtain

$$\sum_{n=1}^{\infty}\frac{i^n}{n!}\int\int\cdots\int:d\mu_1\ldots d\mu_n:A_n(1,2,\ldots,n)=0. \tag{6.1}$$

Here

$$A_n(1,2,\ldots,n) \tag{6.2}$$
$$=\sum_{n_1+n_2=n}(-)^{n_2}P(1,\ldots,n_1|n_1+1,\ldots,n)\,A_{n,n_1}(1,\ldots,n_1|n_1+1,\ldots,n),$$

$$A_{n,n_1}(1,\ldots,n_1|n_1+1,\ldots,n)\equiv A_{n,n_1}(\xi_1x_1,\ldots,\xi_{n_1}x_{n_1}|\xi_{n_1+1}x_{n_1+1},\ldots,\xi_nx_n)$$
$$=\prod_{1\leqq i<j\leqq n_1}\{1+w(\xi_i\xi_j,x_i-x_j)\}\prod_{\substack{1\leqq k\leqq n_1\\ n_1+1\leqq l\leqq n}}\{1+w_{(-)}(\xi_l\xi_k,x_k-x_l)\} \tag{6.3}$$
$$\cdot\prod_{n_1+1\leqq s<t\leqq n}\{1+w^+(\xi_s\xi_t;x_s-x_t)\},$$

$$P(1,\ldots,n_1|n_1+1,\ldots|\ldots n)\equiv P(\xi_1x_1,\ldots,\xi_{n_1}x_{n_1}|\xi_{n_1+1}x_{n_1+1}\cdots|\ldots\xi_nx_n)$$

is the symmetrization operator denoting the sum over arbitrary decompositions of n points into $\frac{n!}{n_1!\ldots n_k!}$ sets, each containing $n_1,n_2,\ldots,n_k$ points $\left(\sum_{i=1}^{k}n_i=n\right)$.

The right-hand side of (6.3) is represented as a sum each term of which describes some ordinary graph[1] (which, in general, is disconnected) with n vertices.

In the case $n_1=n$ this sum may be written as

$$A_{n,n}=\prod_{1\leqq i<j\leqq n}\{1+w_{ij}\}$$
$$=\sum_{\{k,j_k\}}P(1,\ldots,k_1|k_1+1,\ldots,k_1+k_2|\ldots|\ldots k_1+k_2+\cdots+k_\nu) \tag{6.4}$$
$$\cdot F_{k_1j_{k_1}}(1,\ldots,k_1)F_{k_2j_{k_2}}(k_1+1,\ldots,k_1+k_2)\ldots F_{k_\nu j_{k_\nu}}(n-k_\nu+1,\ldots,n).$$

[1] The graph without loops two adjoining vertices of which are connected by one line only [8].

Here v is the number of connectivity components of a given n-vertex graph; $F_{k\,j_k}(1\ldots k)$ is the function describing some connected Feynman graph $G_{k\,j_k}$ with k vertices; and the index j_k denotes the type of a graph and ranges over $1 \leqq j_k \leqq N_k$ where N_k is the number of different connected graphs with k vertices.

As j_k we can use some index connected, for example, with the matrix of adjacency of a vertex [8], the elements of which are given with the help of w_{ij} contained in $F_{k\,j_k}$.

We shall not discriminante between the graphs obtained from one another by permuting the vertices. This implies that $F_{k\,j_k}$ with fixed k and j_k describes the sum of graphs over every possible permutations of vertices.

Consequently, every function $F_{k\,j_k}(1, \ldots, k)$ is symmetric with respect to the permutation of its arguments.

The summation in (6.4) is over all

$$k_j,\ j_{k_j},\ k_j \geqq 0,\ \sum_j k_j = n,\ 1 \leqq j_{k_j} \leqq N_k\,.$$

By definition,

$$F_{0\,j_0} \equiv 1\,.$$

As it is easily seen from (6.3), the set of graphs described by the functions A_{n,n_1} with $n_1 \neq n$ is isomorphic to the one considered ($n_1 = n$), provided that one does not discriminate between the edges of the graphs to which different superpropagators (w_{ij}, w^+_{ij}, $w_{(-)ij}$) correspond. Therefore, A_{n,n_1} may be represented by the formula similar to (6.4)

$$\begin{aligned} A_{n,n_1} = \sum_{\{k\,j_k\}} \sum_{\{p\,j_p\}} & P(1, \ldots, k_1 | k_1 + 1, \ldots, k_1 + k_2 | \ldots, k_1 + \cdots + k_v) \\ & \cdot F_{k_1 j_{k_1}, p_1 j_{p_1}}(1, \ldots, k_1) \ldots F_{k_v j_{k_v}, p_v j_{p_v}}(n - k_v, \ldots, n) \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} & F_{k\,j_k,\,p\,j_p}(1\ldots k) \\ & = P(1\ldots p | p+1, \ldots, k)\, F_{k\,j_k,\,p\,j_p}(1\ldots p | p+1 \ldots k)\,. \end{aligned}$$

Here

$$F_{k\,j_k,\,p\,j_p}(1\ldots p | p+1 \ldots k)$$

denotes the function describing the connected graph $G_{k\,j_k}$ which is decomposed into two subgraphs of the p and $(k-p)$ vertices; the index j_p indicates the decomposition technique and ranges over $1 \leqq j_p \leqq M_{k j_k, p}$, where the number $M_{k\,j_k,\,p}$ denotes the number of independent decompositions of this kind. The function $w(\xi_i \xi_j, x_i - x_j)$ corresponds to the edges of the graph $G_{k\,j_k}$ which connect the vertices of the first subgroup $(1\ldots p)$, the function $w^+(\xi_i \xi_j, x_i - x_j)$ the those connecting the vertices of the second subgroup $(p+1\ldots k)$ and the function $w_{(-)}(\xi_i \xi_j, x_i - x_j)$ to the lines connecting the vertices of the first and second subgroups. By definition,

$$F_{k\,j_k,\,p\,j_p}(1\dots p\,|\,p+1\dots k)|_{p=k} = F_{k\,j_k}(1\dots k)\,.$$

We introduce additional functions

$$F_{k\,j_k,\,p}(1\dots p\,|\,p+1\dots k) = \sum_{j_p=1}^{M_{k\,j_k},\,p} F_{k\,j_k,\,p\,j_p}(1\dots p\,|\,p+1\dots k)$$

$$F_{k\,j_k,\,p}(1\dots k) = P(1\dots p\,|\,p+1\dots k)\,F_{k\,j_k,\,p}(1\dots p\,|\,p+1\dots k)\,.$$

Taking (6.4) and (6.5) into account, the function $A_n(1\dots n)$ may be written as

$$\begin{aligned} A_n(1\dots n) &= \sum_{n_1+n_2=n} (-)^{n_2} \sum_{\{k,\,j_k\}} \sum_{\{p,\,j_p\}} P(1\dots k_1\,|\,k_1+1\dots k_1+k_2\,|\dots|\,n-k_\nu+1\dots n) \\ &\quad \cdot F_{k_1 j_{k_1},\,p_1 j_{p_1}}(1\dots k_1)\dots F_{k_\nu j_{k_\nu},\,p_\nu j_{p_\nu}}(n-k_\nu+1\dots n) \\ &= (-)^n \sum_{\{k,\,j_k\}} P(1\dots k_1\,|\dots|\,n-k_\nu+1\dots n) \\ &\quad \cdot \prod_{i=1}^{\nu} \left\{ \sum_{\{p_i,\,j_{p_i}\}} (-)^{p_i} F_{k_i j_{k_i},\,p_i j_{p_i}} \left(\sum_{l=1}^{i-1} k_l + 1 \dots \sum_{l=1}^{i} k_l \right) \right\}. \end{aligned} \tag{6.6}$$

We insert (6.6) into (6.1) obtaining that the S-matrix is unitary subject to

$$\int\dots\int : d\mu_1\dots d\mu_k : \sum_{\{p,\,j_p\}} (-)^p F_{k\,j_k,\,p\,j_p}(1\dots k) = 0 \tag{6.7}$$

or more specifically,

$$\begin{aligned} &\int dx_1\dots\int dx_n\, g(x_1)\dots g(x_n) \int d\sigma(\xi_1)\dots\int d\sigma(\xi_n) \\ &\cdot : e^{i(\xi_1\phi(x_1)+\dots+\xi_n\phi(x_n))} : \sum_{\{p,\,j_p\}} (-)^p F_{k\,j_k,\,p\,j_p}(x_1\,\xi_1,\dots,x_n\,\xi_n) = 0\,. \end{aligned} \tag{6.8}$$

The formulas (6.7) and (6.8) represent the unitarity relation for any connected Feynman graph.

So the problem of proving the unitarity of S matrix reduces to proving the relation (6.8) for an arbitrary connected graph.

6.2. "Unitarity" of the Regularized S-Matrix ($S^\delta \circledast^\delta S^{\delta+} \equiv 1$)

Following Veltman [9], we will show that for the functions $F^\delta_{k\,j_k,\,p\,j_p}$ constructed from the regularized superpropagators $w^\delta, w^{\delta+}$ and $w^\delta_{(-)}$ there holds an identity

$$\int\dots\int : d\mu_1\dots d\mu_k : \sum_{\{p,\,j_p\}} (-)^p F^\delta_{k\,j_k,\,p\,j_p}(1\dots k) \equiv 0 \tag{6.9}$$

which provides for the condition $S^\delta \circledast^\delta S^{\delta +} \equiv 1$ to be fulfilled (see Section 2).

Taking into account the relations (5.12), one can easily verify that

$$\begin{aligned} & F^\delta_{k\,j_k,\,p}(1\ldots a\ldots p\,|\,p+1\ldots k) \\ &= F^\delta_{k\,j_k,\,p-1}(1\ldots a-1,a+1\ldots p\,|\,a,p+1\ldots k) \end{aligned} \tag{6.10}$$

for $x_{a0} \leqq x_{j0}$ ($j=1,\ldots,a-1,a+1,\ldots,k$) since in this case

$$\begin{aligned} w^\delta(\xi_a\xi_s, x_a-x_s) &= w^\delta_{(-)}(\xi_a\xi_s, x_a-x_s), \qquad (s=1\ldots a-1,a+1\ldots p), \\ w^\delta_{(-)}(\xi_a,\xi_t, x_a-x_t) &= w^{\delta+}(\xi_a\xi_t, x_a-x_t), \qquad (t=p+1,\ldots,k). \end{aligned}$$

Hence

$$\begin{aligned} B(1\ldots k) &= \sum_{p=0}^{k} (-)^p F^\delta_{k\,j_k,\,p}(1\ldots k) \\ &= \sum_{p=0}^{k} (-)^p P(1\ldots p\,|\,p+1\ldots k)\, F^\delta_{k\,j_k,\,p}(1\ldots p\,|\,p+1\ldots k) = 0\,. \end{aligned} \tag{6.11}$$

The sum (6.11) contains 2^k terms. All the terms cancel in pairs because of (6.10).

Indeed, suppose that one of the arguments $(x_{10},\ldots,x_{k0})$ is the smallest. Without loss of generality we can assume

$$x_{10} \leqq x_{j0} \qquad (j=2,3,\ldots,k)\,. \tag{6.12}$$

Then the sum in (6.11) may be decomposed into two terms

$$\begin{aligned} & \sum_{p=0}^{k} (-)^p P(1\ldots p\,|\,p+1\ldots k)\, F^\delta_{k\,j_k,\,p}(1\ldots p\,|\,p+1\ldots k) \\ &= \sum_{p=1}^{k} (-)^p P(2,\ldots,p\,|\,p+1,\ldots,k)\, F^\delta_{k\,j_k,\,p}(1,\ldots,p\,|\,p+1,\ldots,k) \\ &\quad + \sum_{p=0}^{k-1} (-)^p P(2,\ldots,p+1\,|\,p+2,\ldots,k)\, F^\delta_{k\,j_k,\,p}(2,\ldots,p+1\,|\,1,p+2,\ldots,k). \end{aligned} \tag{6.13}$$

Taking (6.10) and (6.12) into account, we obtain

$$\begin{aligned} B(1\ldots k) &:= \sum_{p=1}^{k} (-)^p P(2\ldots p\,|\,p+1\ldots k)\, F^\delta_{k\,j_k,\,p-1}(2\ldots p\,|\,1,p+1,\ldots,k) \\ &\quad + \sum_{p=0}^{k-1} (-)^p P(2\ldots p+1\,|\,p+2\ldots k)\, F^\delta_{k\,j_k,\,p}(2\ldots p+1\,|\,1,p+2\ldots k) \\ &= \sum_{p=0}^{k-1} (-)^p P(2\ldots p+1\,|\,p+2\ldots k) \\ &\quad \cdot \{-F^\delta_{k\,j,\,p}(2\ldots p+1\,|\,1,p+2\ldots k) + F^\delta_{k\,j_k,\,p}(2\ldots p+1\,|\,1,p+2\ldots k)\} \equiv 0\,. \end{aligned} \tag{6.14}$$

Thus we see that $B(1\ldots k)\equiv 0$ for any relations between the arguments $(x_{10}\ldots x_{k0})$, i.e., for any choice of a minimal argument. And so there holds (6.9).

Note that our considerations are true, provided that the superpropagators w^{δ}, $w^{\delta+}$, $w^{\delta}_{(-)}$ from which the functions $F^{\delta}_{k\,j_k,\,p}$ are constructed are locally integrable and obey the relations (5.12).

6.3. Conclusions

Thus we have demonstrated that the regularization chosen possesses all the properties (1) – (4) stated in Section 2. Hence the unitarity of the S-matrix in the n-th order of perturbation theory is proved in a nonlocal quantum field theory.

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