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Constraints on the Derivatives of the $\pi\pi$ Scattering Amplitude from Positivity

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Abstract. Conditions derived from positivity are given both above and below threshold for the derivatives of the $\pi\pi$ scattering amplitude.

1. Introduction

From the general assumptions of unitarity, crossing and analyticity in a domain derivable from axiomatic field theory, it can be shown that the $\pi\pi$ scattering amplitudes satisfy twice subtracted dispersion relations [1]. Hence D and higher partial waves have the Froissart-Gribov representation [2] so that if $f_l(s)$ are the partial waves in the s channel for even isotopic spin states where s is the centre-of-mass energy squared [1],

$$f_l(s) = \frac{4}{\pi(4-s)} \int_4^\infty A_t(s,t)\, Q_l\left(\frac{2t}{4-s}-1\right) dt; l=2,4,\ldots \tag{1.1}$$

when $0<s<4$. $A_t(s,t)$ is the absorptive part of the scattering amplitude [2] and for certain isotopic spin combinations [3] in the s channel it has the expansion

$$A_t(s,t) = \sum_{l=0}^{\infty} (2l+1)\,\alpha_l(t)\, P_l\left(1+\frac{2s}{t-4}\right) \tag{1.2}$$

where the $\alpha_l(t) \geqq 0$ from unitarity.

It follows from (1.2) that for $t \geqq 4$ and $0<s<4$,

$$A_t(s,t) \geqq 0\,. \tag{1.3}$$

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[1] We take units such that the mass of the pion is unity.

[2] We will use the suffices s and t to indicate physical quantities in the s channel and t channel respectively.

[3] See Eq. (2.1).

In previous papers [3, 4] it was shown that the condition (1.3) puts strong constraints on the $f_l(s)$ when $0<s<4$ and in fact they are sufficient to prove that the f_l's are solutions of a Stieltjes moment problem.

However, these results did not take into account the full content of positivity which requires $\alpha_l(s)\geqq 0$ for all l or equivalently

$$\int_{-1}^{+1} \cos\theta_t P_l(\cos\theta_t)\, A_t(s,t)\, d\cos\theta_t \geqq 0,\; l=0,1,\ldots;\; 0\leqq s\leqq 4\,. \tag{1.4}$$

In this article we consider the problem of finding the constraints that this positivity places on the derivatives $(\partial^m T(s,t))/((\partial \cos\theta_s)^m)$ of the $\pi\pi$ scattering amplitude. Three types of constraints are considered.

i) Firstly conditions on the derivatives of $T(s,t)$ at any angle when s is restricted to $0\leqq s\leqq 4$, which are derived from $A_t(s,t)\geqq 0$.

ii) Secondly constraints on the quantities

$$\left.\frac{\partial^{n+m} T(s,t)}{\partial s^n (\partial\cos\theta_s)^m}\right|_{\cos\theta_s=0}, \qquad n,m=0,1,2\ldots$$

which ensure that

$$\frac{\partial^n A_t(s,t)}{(\partial\cos\theta_t)^n}\geqq 0, \qquad n=0,1,2,\ldots. \tag{1.5}$$

Although the conditions (1.5) are weaker than positivity, they are already sufficient to obtain many of the consequences of the full positivity [5].

iii) Finally we derive constraints on the quantities

$$\left.\frac{\partial^n A_t(s,t)}{(\partial\cos\theta_t)^n}\right|_{\cos\theta_t=0}$$

with t in the physical region which are necessary for the full positivity (1.4) This set of constraints can be extended to give a set of sufficient conditions for (1.4) to hold.

These three different types of constraints may prove useful for the parametrization of the $\pi\pi$ amplitudes, as they can be used both above and below threshold and thus may be directly inserted into dispersion relations. Also it is easier to handle "crossing" when dealing with derivatives of $T(s,t)$ than when working with the partial wave amplitudes $f_l(s)$ and it is therefore useful to have constraints on the former quantities.

2. Constraints on the Derivatives of the Scattering Amplitude $T(s,t)$ at any Angle for Fixed s below Threshold

We work below threshold, i.e., $0<s<4$, for $\pi\pi$ scattering in any of the isospin combinations,

$$T(s,t)=(1+\lambda/3)\,T^{(0)}(s,t)+(2\lambda/3)\,T^{(2)}(s,t),\ \lambda\geqq -1\,, \tag{2.1}$$

where $T^{(I)}$ is the amplitude for scattering in the s channel state with total isospin I. These combinations are such that the absorptive part in the t channel $A_t(s, t)$ satisfies the conditions (1.4) [3].

For $0<s<4$ there exists the fixed s dispersion relation,

$$T(s,t) = (\text{polynomial of first degree in } \cos\theta_s) + \frac{1}{\pi}\int_4^\infty \frac{1}{t'^2}\left[\frac{t^2}{t'-t}+\frac{(4-s-t)^2}{t'-4+s+t}\right] A_t(s,t')\,dt'. \tag{2.2}$$

Let

$$\mu_m(s,\cos\theta_s) = \frac{\pi}{m!}\,\frac{\partial^m T(s,t)}{(\partial\cos\theta_s)^m}. \tag{2.3}$$

Then from (2.2) for $m\geqq 2$,

$$\mu_m(s,\cos\theta_s) = \int_{z_0}^\infty \left\{\frac{1}{(z-\cos\theta_s)^{m+1}} - \frac{1}{(-z-\cos\theta_s)^{m+1}}\right\} A_t[s,(z+1)(2-s/2)]\,dz \tag{2.4}$$

where

$$z_0 = \left[\frac{4+s}{4-s}\right].$$

We consider the case when $\cos\theta_s \geqq 0$. Then defining $\chi'_\pm \equiv 1/(\pm z-\cos\theta_s)$, we have that as z varies between ∞ and z_0, χ_+ varies between 0 and $\chi_+^0 \equiv 1/(z_0-\cos\theta_s)$. Furthermore for any $z\geqq z_0$ and when $\cos\theta_s\geqq 0$, $\chi_+\geqq|\chi_-|$. We are now in a position to prove the following result.

Theorem 1. *When* $4>s>0$ *and* $\cos\theta_s\geqq 0$, *then for any odd integer* $m>2$,

$$\sum_{j=0}^n \binom{n}{j}(-1)^j(\chi_+^0)^{2(n-j)}\mu_{m+2j}(s,\cos\theta^s)/(m+2j+1)\geqq 0, \quad n=0,1,2,\ldots \tag{2.5}$$

and for any even integer $m\geqq 2$,

$$\sum_{j=0}^n \binom{n}{j}(\chi_+^0)^{2(n-j)}\mu_{m+2j}(s,\cos\theta_s)\geqq 0 \quad n=0,1,2,\ldots. \tag{2.6}$$

Proof. Consider first the case when m is odd. The left-hand side of (2.5) is equal to

$$\sum_{j=0}^n \binom{n}{j}(-1)^j(\chi_+^0)^{2(n-j)}\frac{\{\chi_+^{m+2j+1}-\chi_-^{m+2j+1}\}}{(m+2j+1)} A_t[s,(z+1)(2-s/2)]\,dz$$

$$\int_{z_0}^\infty \sum_{j=0}^n \binom{n}{j}(-1)^j(\chi_+^0)^{2(n-j)}\left\{\int_0^{\chi_+}\chi^{m+2j}d\chi+(-1)^{m+2j}\int_0^{|\chi_-|}\chi^{m+2j}d\chi\right\}A_t[s,(z+1)(2-s/2)]\,dz$$

$$\int_{z_0}^\infty\left\{\int_0^{\chi_+}\chi^m[(\chi_+^0)^2-\chi^2]^n d\chi+(-1)^{m+2j}\int_0^{|\chi_-|}\chi^m[(\chi_+^0)^2-\chi^2]^n d\chi\right\}A_t[s,(z+1)(2-s/2)]\,dz. \tag{2.7}$$

Since $\chi_+^0 \geqq \chi_+ \geqq |\chi_-|$ when $\cos\theta_s \geqq 0$, the above quantity is positive for odd m so that (2.5) is proved. Similarly it can be proved that (2.6) is true for all even m.

Corollary. *When* $\cos\theta_s \leqq 0$, *the inequality* (2.5) *is reversed and* (2.6) *holds as it is.*

The second result is an immediate consequence of the fact that $\mu_m(s, \cos\theta_s) \geqq 0$ for even m when $\cos\theta_s \leqq 0$, while the first follows from the relation

$$\mu_m(s, \cos\theta_s) = -\mu_m(s, -\cos\theta_s) \tag{2.8}$$

when m is odd.

Although conditions (2.5) and (2.6) on the $\mu_n(s, \cos\theta_s)$ are very stringent, they are only necessary but not sufficient conditions for $A_t[s, (z+1) \times (2 - s/2)] \geqq 0$. However, when $\cos\theta_s = 0$ they coincide with the conditions of Ref. [4] on the quantities $\mu_m \equiv \mu_s(s, 0)$ and are then also sufficient to ensure the positivity of A_t.

3. Constraints on the Derivatives of $T(s, t)$ with Respect to s

We will in this Section obtain sets of conditions on the quantities $\partial^{m+j} Ts, t)/\partial^m s(\partial \cos\theta_s)^j$ which ensure that Martin's positivity

$$\frac{\partial^n A_t(s, t)}{\partial s^n} \geqq 0, \qquad n = 0, 1, 2, \ldots \tag{3.1}$$

holds for $0 < s < 4$.

In fact we will consider only the case when $\cos\theta_s = 0$, as it is only then that simple conditions can be obtained even for the first condition of (3.1), viz., $A_t \geqq 0$ to hold. Taking (2.4) for $\cos\theta_s = 0$ and defining for even $m \geqq 2$,

$$k_m(s) = \tfrac{1}{4}(4 - s)^{-m} \mu_m(s, 0), \tag{3.2}$$

we obtain the representation

$$k_m^{(s)} = \int_4^\infty dt\, A_t(s, t)(2t - 4 + s)^{-m-1}, \tag{3.3}$$

where the change of variables $z \to t$ has been made and we take this to be also the definition of $k_m(s)$ for m odd and $\geqq 3$.

Differentiating with respect s,

$$k_m^{(1)}(s) = \int_4^\infty dt\, A_m^{(1)}(s, t)(2t - 4 + s)^{-m-1} - (m+1)\int_4^\infty dt\, A_t(s, t)(2t - 4 + s)^{-m-2} \tag{3.4}$$

where $k_m^{(1)} \equiv \frac{\partial k_m}{\partial s}$ and $A_m^{(1)} \equiv \frac{\delta}{\partial s} A_t(s,t)$. Hence we see that the quantity $h_m^{(1)} \equiv k_m^{(1)} + (m+1)\,k_{m+1}$ has the representation

$$h_m^{(1)}(s) = \int_4^\infty dt\, A_t^{(1)}(s,t)\,(2t-4+s)^{-m-1}. \tag{3.5}$$

Quite generally we will have

$$\begin{aligned} h_m^{(n)}(s) &\equiv k_m^{(n)}(s) - \sum_{l=1}^{n} \frac{\Gamma(n+1)\,\Gamma(m+l+1)\,(-1)^l}{\Gamma(n-l+1)\,\Gamma(l+1)\,\Gamma(m+1)}\, h_{l+m}^{(n-l)}(s) \\ &= \int_4^\infty A_t^{(n)}(s,t)\,(2t-4+s)^{-m-1}\,dt\,. \end{aligned} \tag{3.6}$$

By a straightforward application of the methods of Refs. [2, 6], the following theorem may be proved.

Theorem 2. *A set of necessary and sufficient conditions for* (3.1) *to be true is that the quantities* $h_m^{(n)}(s)$ *defined in* (3.6) *satisfy for* $m \geqq 2$

$$\sum_{j=0}^{l} \binom{l}{j} (-1)^j h_{m+2j}^{(n)} (4+s)^{2(l-j)} \geqq 0 \qquad n=0,1,\ldots; \qquad 0<s<4\,. \tag{3.7}$$

We want to write the conditions (3.7) as constraints on the quantities

$$\frac{\partial^n \mu_m(s,0)}{\partial s^n} = \left.\frac{\partial^{m+n} T(s,t)}{\partial s^n (\partial \cos\theta_s)^m}\right|_{\cos\theta_s=0}. \tag{3.8}$$

However, they have the unpleasant feature of depending on $k_m^{(n)}(s)$ for both odd and even values of m. But it is only for even values of m that we can use (3.2) to get $k_m^{(n)}(s)$ in terms of the $\partial^n \mu_m(s,0)/\partial s^n$.

To obtain $k_m(s)$ as defined by (3.3) for odd m from its value for even m, we can use a very accurate method of interpolation which has been discussed by one of us in Section 5 of the second paper of Ref. [3]. Similar results are true for the quantities $k_m^{(n)}(s)$. Using these values the conditions (3.7) can be tested to a high degree of accuracy.

4. Constraints for Physical t

So far we have not derived constraints which give the full positivity conditions (1.4) and this is what we will do now. These conditions are equivalent to having the expansion

$$A_t(s,t) = \sum_{l=0}^{\infty} (2l+1)\,\alpha_l(t)\,P_l(\cos\theta_t) \tag{4.1}$$

for $t \geqq 4$, where $\alpha_l(t) = \operatorname{Im} f_l(t)$ are positive. This expansion can be compared with the Taylor's series expansion,

$$A_t(s,t) = \sum_{l=0}^{\infty} \frac{[\cos\theta_t - 1]^n}{n!} \left. \frac{\partial^n A_t(s,t)}{(\partial \cos\theta_t)^n} \right|_{\cos\theta_t = 1} \tag{4.2}$$

Since [7]

$$\left. \frac{d^n P_l(x)}{dx^n} \right|_{x=1} = \frac{(l+n)!}{2^n n!\,(l-n)!}, \tag{4.3}$$

we get

$$\left. \frac{\partial^n A_t(s,t)}{(\partial \cos\theta_t)^n} \right|_{\cos\theta_t = 1} = \frac{1}{2^n n!} \sum_{l=0}^{\infty} (2l+1) \frac{(l+n)!}{(l-n)!} \alpha_l(s). \tag{4.4}$$

Then since

$$(l+n)!/(l-n)! = (l+n)(l+n-1) \ldots (l-n+2)(l-n+1)$$

and

$$(l+n)(l-n+1) = l(l+1) - n(n-1),$$

we see that each coefficient of α_l in (4.4) is proportional to the same polynomial in $l(l+1)$. Defining the quantities $\beta_n(s)$ to be the left-hand side of (4.4), then

$$\beta_n(s) \equiv \left. \frac{\partial^n A_t(s,t)}{(\partial \cos\theta_t)^n} \right|_{\cos\theta_t = 1} = \frac{1}{2^n n!} \sum_{l=0}^{\infty} (2l+1)\alpha_l(s) \left\{ \prod_{m=0}^{n} [l(l+1) - m(m-1)] \right\}. \tag{4.5}$$

We now define $g_n(s)$ to be

$$g_n(s) \equiv \sum_{l=0}^{\infty} (2l+1)\,[l(l+1)]^n \alpha_l(s). \tag{4.6}$$

The $\beta_n(s)$ may be obtained from the $g_n(s)$ and conversely by a non-singular linear transformation of the form

$$\begin{aligned} \beta_n(s) &= a_{n,n} g_n(s) + a_{n,n-1} g_{n-1}(s) + \cdots + a_{n,0} g_0(s) \\ \beta_{n-1}(s) &= \qquad\qquad a_{n-1,n-1} g_{n-1}(s) + \cdots + a_{n-1,0} g_0(s) \\ &\vdots \\ \beta_0(s) &= \qquad\qquad\qquad\qquad\qquad + a_{0,0} g_0(s) \end{aligned} \tag{4.7}$$

where the coefficients $a_{n,m}$ can be determined from (4.5).

From (4.6) we notice that the $g_n(s)$ may be written in the form

$$g_n(s) = \int_0^{\infty} u^n d\phi(u), \qquad n = 0, 1, 2, \ldots. \tag{4.8}$$

where $\phi(u)$ is a bounded non-decreasing function of u with points of increase at $u = l(l+1)$ where $l = 0, 1, 2, \ldots$. The following theorem is then an immediate consequence of Theorem (1.3) given in Ref. [6] for sequences with such representations.

Theorem 3. *A necessary set of conditions for* $A_t(s, t)$ *to have the expansion* (4.1) *with* $\alpha_l \geqq 0$ *for all* l *is that the* $g_n(s)$ *defined by* (4.5) *and* (4.7) *satisfy*

$$\begin{vmatrix} g_m(s)\, g_{m+1}(s) \ldots g_{m+n}(s) \\ g_{m+}{}^1(s) \qquad \vdots \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ g_{m+n}(s) \cdots\cdots\; g_{m+2n}(s) \end{vmatrix} \geqq 0; \qquad m, n = 0, 1, 2, \ldots \tag{4.9}$$

By using the inverse of the transformation (4.7) the conditions (4.9) can be written in terms of the derivatives of $A_t(s, t)$ in the forward direction. For example we get taking $m = 0$ and $n = 1$ that

$$2\beta_0(s)\,\beta_2(s) + \beta_0(s)\,\beta_1(s) - [\beta_1(s)]^2 \geqq 0\,. \tag{4.10}$$

The conditions (4.9) are also sufficient for the $g_n(s)$ to have the representation (4.8) with $\phi(u)$ bounded and non-decreasing but other conditions have to be added to ensure that $\phi(u)$ has only points of increase at $u = l(l+1)$ with $l = 0, 1, 2, \ldots$ so that $g_n(s)$ has the representation (4.6) and $\beta_n(s)$ the corresponding representation (4.5) These extra conditions are given by the following theorem [8].

Theorem 4. *Let*

$$C_n = \frac{4^n}{n!} \sum_{m=n}^{\infty} (-\pi^2)^m \frac{m!}{(2m)!\,(m-n)!} \tag{4.11}$$

and suppose that the $g_n(s)$ *satisfy the conditions* (4.9) *and are such that the series* $\sum_{n=0}^{\infty} C_n g_n(s)$ *converges absolutely. Then* $g_n(s)$ *has the representation* (4.6) *if and only if the sum of this series is* $-g_0$.

Proof. We have the representation (4.8) for $g_n(s)$ since conditions (4.9) are satisfied and we have to prove that $\phi(u)$ increases only at the points $u = (l+1)$. Now

$$\begin{aligned} \sum_{n=0}^{\infty} C_n g_n(s) &= \sum_{n=0}^{\infty} C_n \int_0^{\infty} d\phi(u)\, u^n \\ &= \sum_{n=0}^{\infty} \frac{4^n}{n!} \left\{ \sum_{m=n}^{\infty} \frac{(-\pi^2)^m}{(2m)!} \frac{m!}{(m-n)!} \int_0^{\infty} d\phi(u)\, u^n \right\} \\ &= \int_0^{\infty} d\phi(u) \left\{ \sum_{m=0}^{\infty} \frac{(-\pi^2)^m}{(2m)!} (1+4u)^m \right\} \\ &= 2\int_0^{\infty} d\phi(u) \sin^2 \frac{\pi}{2} [\sqrt{1+4u} - 1] - g_0 \end{aligned} \tag{4.12}$$

where we have used the absolute convergence of the series to invert the order of summation and integration. The last integral vanishes if and only if $d\phi(u)$ is different from zero only when $u = l(l+1)$ with $l = 0, 1, \ldots,$ so the theorem is proved.

We have thus obtained sufficient conditions for $g_n(s)$ to have the representation (4.6). That they are not necessary is easily seen from the fact that the existence of the $g_n(s)$ for all $n = 0, 1, 2, \ldots,$ does not imply the absolute convergence of the series $\sum_{n=0}^{\infty} C_n g_n(s)$. Finally from unitarity we have the requirement that $\alpha_l(s)$ is smaller than unity. We have been unable to obtain either necessary or sufficient conditions on the derivatives of $A_t(s, t)$ for this to be so.

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