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# An Algebraic Spectrum Condition

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**Abstract.** A condition, necessary and sufficient for the existence of a vacuum representation with positive energy of the quasilocal algebra, is formulated.

Most postulates of axiomatic quantum field theory can be translated easily into the language of $C^*$-algebras [1]. A remarkable exception is the usual spectrum condition. The required algebraic formulation has to assure the existence of a vacuum representation of the quasilocal algebra, for which the energy-momentum spectrum is contained in the future cone $\bar{V}_+$. Such a representation will be called a "positive vacuum representation" [4]. Algebraic spectrum conditions have been formulated by Doplicher [2], Montvay [3], and Borchers [4]. In this note, we will propose another condition of this type.

Consider a $C^*$-algebra $\mathfrak{A}$, called quasilocal algebra [1], and a representation $x \to \alpha_x$ of four-dimensional translations $x$ by automorphisms $\alpha_x$ of $\mathfrak{A}$. This representation shall be strongly continuous, i.e.,

$$\lim_{x \to 0} \|\alpha_x A - A\| = 0$$

for any $A \in \mathfrak{A}$. Then $\mathfrak{A}$ contains, as a norm-dense invariant sub-$*$-algebra $\mathfrak{A}^{(1)}$, the set of all $A \in \mathfrak{A}$ for which

$$\operatorname*{norm-lim}_{\tau \to 0} \frac{1}{\tau} (\alpha_{\tau a} A - A) \underset{\text{df.}}{=} D_a A$$

exists for all four-vectors $a$ [5].

A positive linear functional $\varphi$ on $\mathfrak{A}$, normalized to $\|\varphi\| = 1$, is called a state. Then, for arbitrary state $\varphi$ and $A \in \mathfrak{A}^{(1)}$, the functions

$$\hat{\varphi}(\tau \mid A, a) \underset{\text{df.}}{=} \varphi(A^* \alpha_{\tau a} A)$$

are differentiable with respect to $\tau$. Denote by $E_+$ the set of all states $\varphi$ for which

$$\frac{1}{i} \frac{d}{d\tau} \hat{\varphi}(\tau \mid A, a) \bigg|_{\tau = 0} = \frac{1}{i} \varphi(A^* D_a A) \geqq 0$$

for all $a \in \bar{V}_+$ and all $A \in \mathfrak{A}^{(1)}$.

[1] The local structure of $\mathfrak{A}$, however, will not be used here.

Since $\mathfrak{A}^{(1)}$ is invariant with respect to the mappings $\alpha_x$, $E_+$ is invariant with respect to the adjoint mappings $\alpha_x^*$ of the dual space $\mathfrak{A}^*$ of $\mathfrak{A}$, defined by

$$(\alpha_x^* \varphi)(A) = \varphi(\alpha_x A).$$

The representation $\alpha_x^*$ is continuous in the $\mathfrak{A}$ topology of $\mathfrak{A}^*$, i.e.,

$$\lim_{x\to 0} |(\alpha_x^* \varphi)(A) - \varphi(A)| = 0$$

for all $\varphi \in \mathfrak{A}^*$, $A \in \mathfrak{A}$.

Obviously $E_+$ is convex. Moreover, it is compact with respect to the $\mathfrak{A}$ topology of $\mathfrak{A}^*$. By Ref. [6], Part I, V.4.3. it is sufficient to show that $E_+$ is closed in this topology, since $E_+$ is bounded in norm.

For fixed $A \in \mathfrak{A}^{(1)}$ and fixed $a$, the set $E_+(A, a)$ of all states $\varphi$ with

$$\frac{1}{i}\,\varphi(A^* D_a A) \geqq 0$$

is closed, since $\frac{1}{i}\,\varphi(A^* D_a A)$ is a continuous function of $\varphi$. By definition, $E_+$ is the intersection of all sets $E_+(A, a)$ with $A \in \mathfrak{A}^{(1)}$ and $a \in \bar{V}_+$, and is therefore closed too.

In terms of $E_+$ an algebraic spectrum condition may then be formulated with the help of the following theorem.

**Theorem.** *$\mathfrak{A}$ possesses a positive vacuum representation if and only if $E_+$ is not empty.*

*Proof.* If $\pi$ is a positive vacuum representation of $\mathfrak{A}$ with a vacuum vector $\Omega$ and a unitary representation $U(x) = e^{iPx}$ of translations, then the state $\omega$ defined by

$$\omega(A) = (\Omega, \pi(A)\Omega)$$

belongs to $E_+$. Indeed, for $A \in \mathfrak{A}^{(1)}$ and $a \in \bar{V}_+$ it follows

$$\frac{1}{i}\frac{d}{d\tau}\,\hat{\omega}(\tau\,|\,A, a)\Big|_{\tau=0} = (\pi(A)\Omega, aP\pi(A)\Omega) \geqq 0\,.$$

This proves the necessity of $E_+ \neq \emptyset$.

In order to prove sufficiency, we note that $\alpha_x^*$ and $E_+$ fulfill the assumptions of the Markov-Kakutani fixed-point theorem (Ref. [6], Part I, V.10.6.). Therefore, if $E_+ \neq \emptyset$, there exists an invariant (vacuum) state

$$\omega \in E_+\,, \quad \alpha_x^* \omega = \omega \quad \text{for all } x\,.$$

The Gelfand-Segal construction then leads to a representation $\pi$ of $\mathfrak{A}$ in a Hilbert space $\mathscr{H}$, a strongly continuous unitary representation

$U(x) = e^{iPx}$ of translations with

$$\pi(\alpha_x A) = U(x)\,\pi(A)\,U^*(x)\,,$$

and a cyclic invariant vacuum vector $\Omega \in \mathscr{H}$ with

$$\omega(A) = (\Omega, \pi(A)\Omega)\,.$$

We will show that the self-adjoint energy-momentum operators $P_\nu$ fulfill the spectrum condition or, equivalently, that the self-adjoint operators $aP$ with arbitrary $a \in \overline{V}_+$ are positive semidefinite.

The vectors $\pi(A)\Omega$ with $A \in \mathfrak{A}^{(1)}$ form a domain $D$ which is dense in $\mathscr{H}$, and is contained in the domain of definition of all operators $aP$. For arbitrary $a \in \overline{V}_+$ and arbitrary $\Phi = \pi(A)\Omega \in D$,

$$\frac{1}{i}\frac{d}{d\tau}\hat{\omega}(\tau\,|\,A, a)\bigg|_{\tau=0} = \frac{1}{i}\frac{d}{d\tau}(\Omega, \pi(A^*)\,e^{i\tau aP}\pi(A)\Omega)\bigg|_{\tau=0}$$
$$= (\Phi, aP\Phi) \geqq 0$$

since $\omega \in E_+$. The restriction $aP|_D$ of $aP$ to the domain $D$ is therefore a positive semidefinite symmetric operator. Accordingly (Ref. [6], Part II, XII.5.2), it has a positive semidefinite self-adjoint extension $Q_a$. Of course, $aP$ itself is also a self-adjoint extension of $aP|_D$. The proof will be finished by showing that $aP|_D$ is essentially self-adjoint, which implies $aP = Q_a$.

According to Ref. [5], $\mathfrak{A}^{(1)}$ contains a sub-$*$-algebra $\tilde{\mathfrak{A}}$ of "analytic elements", which is also norm-dense in $\mathfrak{A}$. Using the methods of Ref. [5] one may show that vectors of the form $\pi(A)\Omega$ with $A \in \tilde{\mathfrak{A}}$ are analytic vectors for the operators $aP$ with arbitrary $a$. They form a dense subset of $D$. Essential self-adjointness of $aP|_D$ then follows from a theorem of Nelson [7].

The algebraic spectrum condition proposed here is relatively simple. It has, however, some serious shortcomings. A state $\varphi \in E_+$ which is not translation invariant has no immediate physical interpretation. In particular, it does not lead to a "positive representation" [4] of $\mathfrak{A}$. Therefore, the set $E_+$ seems to be of little use if one wants to formulate conditions for the existence of positive representations which do not necessarily have a vacuum.

On the other hand, one may demand the required positive vacuum representation of $\mathfrak{A}$ to be faithful. For this purpose one might try to use, instead of $E_+$, the set $E_+^f$ of states $\varphi \in E_+$ which lead to faithful representations. However, we can not simply generalize our theorem to this case since we have not been able to show that $E_+^f$ is compact in some useful topology.

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