Commun. math. Phys. 14, 315—325 (1969)

# A Remark Concerning the Charge Operator in Quantum Electrodynamics

HELMUT REEH
Max-Planck-Institut für Physik und Astrophysik, München

Received June 25, 1969

**Abstract.** The convergence of the integral over the local charge density toward the global charge is investigated within the framework of quantum electrodynamics.

## 1. Introduction

In relativistic quantum field theories one frequently considers operators which are formal integrals over the entire three dimensional space of the zero component of a conserved quantity. In particular, one writes e.g. for the charge formally

$$Q = \int j_0(x)\, d^3\boldsymbol{x}\,, \qquad \partial^\nu j_\nu(x) = 0\,.$$

Recently one has learned that in case the theory does not contain states of arbitrarily small energy-momentum above the vacuum state this expression is to be understood in the sense

$$(\psi \,|\, Q\, \varphi) = \lim_{r\to\infty} (\psi\,|\, Q_r\, \varphi)\,, \tag{1}$$

$$Q_r = \int j_0(x)\, f_r(\boldsymbol{x})\, \alpha(x^0)\, d^3\boldsymbol{x}\,, \tag{2}$$

$$f_r(\boldsymbol{x}) = f_0\left(\frac{\boldsymbol{x}}{r}\right), \qquad r \geqq 1\,, \tag{3}$$

$$f_0(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| \leqq 1\,, \\ 0, & |\boldsymbol{x}| \geqq 2\,, \end{cases}$$

$$\int \alpha(x^0)\, dx^0 = 1 \tag{4}$$

with $f_0(\boldsymbol{x}) \in \mathscr{D}(\boldsymbol{R}^3)$, $\alpha(x^0) \in \mathscr{D}(\boldsymbol{R}^1)$. (The notation is explained at the end of this introduction.) $\psi$ and $\varphi$ are not arbitrary vectors in the Hilbert space but are generated from the vacuum state by arbitrary local operators [1–4]. This is from the mathematical point of view a rather weak kind of convergence. Strong or weak convergence in the usual sense cannot occur as is shown e.g. in [3] and [4]. On the other hand the result seems to be rather reasonable from the point of view of physics.

The question then arises of whether a similar statement holds in case the theory contains states of arbitrarily small energy-momentum and one may suggest that the same result holds provided one does not have the case of a spontaneously broken symmetry [2,4]. It is the intention of the present note to show that this indeed is true for the electric charge in quantum electrodynamics.

### *Mathematical Notations*

$\boldsymbol{R}^n$: with $n=1,3,4:n$ dimensional real Euclidean space.

$\boldsymbol{R}^1_+$: positive real axis including the origin.

$\mathscr{D}(\boldsymbol{R}^n)$: test function space of arbitrarily often differentiable complex functions over $\boldsymbol{R}^n$ with compact support.

$\mathscr{D}(K)$: subspace of $\mathscr{D}(\boldsymbol{R}^n)$ of functions with support in a subset $K \subset \boldsymbol{R}^n$.

$\mathscr{D}(\boldsymbol{R}^4-0)$: Subspace of $\mathscr{D}(\boldsymbol{R}^4)$ of functions the support of which does not contain the origin.

$\mathscr{S}(\boldsymbol{R}^n)$: test function space of arbitrarily often differentiable complex functions on $\boldsymbol{R}^n$ which as well as all their derivatives vanish faster than any power for increasing arguments.

$\mathscr{D}'(\boldsymbol{R}^n)$, $\mathscr{D}'(K)$, $\mathscr{D}'(\boldsymbol{R}^4-0)$, $\mathscr{S}'(\boldsymbol{R}^n)$: the corresponding spaces of distributions.

$x=(x^0,\boldsymbol{x})=(x^0,x^1,x^2,x^3)$: four vectors in $\boldsymbol{R}^4$.

$x^2=x^{02}-\boldsymbol{x}^2$

$g_{\nu\nu'}$: $g_{00}=1$, $g_{ii}=-1$ for $i=1,2,3$, $g_{\nu\nu'}=0$ for $\nu \neq \nu'$.

$L^1(d\mu)$: space of $\mu$-Lebesgue integrable complex functions.

$\not{C}$ : in front of a term occurring in an equation this symbol means that the equation becomes true if that term is multiplied with a certain finite and non zero constant number.

## 2. Assumptions and Statement

**2.1.** In quantum electrodynamics we have for the photon field $A_\nu(x)$ $(\nu=0,1,2,3)$ and the electric current $j_\nu(x)$ the equations

$$\Box A_\nu(x)=j_\nu(x)\,, \tag{5}$$

$$\partial^\nu j_\nu(x)=0\,. \tag{6}$$

We assume that $A_\nu(x)$ is an operator valued tempered distribution transforming like a vector under a unitary representation of the inhomogeneous Lorentz group on the representation space $G$. Unitary is ment with respect to the indefinite metric $(\cdot\,|\,\cdot)_G$ on $G$ ($G$ stands for Gupta) which may be expressed by a metric operator $\eta$ and a scalar product $(\cdot\,|\,\cdot)$ for which $G$ is a Hilbert space

$$(\psi\,|\,\chi)_G = (\psi\,|\,\eta\,\chi)\,, \qquad \eta^+ = \eta\,, \qquad \eta^2 = 1\,.$$

$\eta$ commutes with the four translations $P_\mu$. (However, it does not commute with the Lorentz transformations.) Hence the translations are unitarily represented on $G$ which respect to its Hilbert space metric and give rise to a decomposition of unity.

$G$ is assumed to contain a subspace $\mathscr{H}_1$ spanned by the states on which the auxiliary condition holds. $\mathscr{H}_1$ contains the unique Lorentz invariant vacuum state $\Omega$, $\mathscr{H}_1$ is invariant under Lorentz transformations as well as under all gauge invariant operators (but there are, of course, also non gauge invariant operators leaving $\mathscr{H}_1$ invariant). On $\mathscr{H}_1$ we have

$$(\psi\,|\,\psi)_G \geqq 0\,.$$

If one denotes by $\mathscr{H}_0$ the set of vectors in $\mathscr{H}_1$ with

$$(\psi\,|\,\psi)_G = 0$$

then the quotient space

$$\mathscr{H} = \mathscr{H}_1/\mathscr{H}_0$$

is the Hilbert space of physical states. On $\mathscr{H}$ we have a unitary representation of the inhomogeneous Lorentz group, the spectrum of the translations vanishes outside the forward light cone.

In addition to the photon field there is the electron field $\psi(x)$. The fields are assumed as local, i.e. they commute resp. anticommute when smeared with test functions the supports of which are space-like to each other. When smeared with test functions, the smeared fields and all polynomials of them are assumed to be applicable to $\Omega$ thus generating a dense set in $G$.

All these assumptions seem to be true for quantum electrodynamics. However, up to now they are proved only for the case of free fields [5].

**2.2.** Consider now the $Q_r$. They are supposed to generate a one parameter group of internal symmetry transformations $\Phi \to \Phi_\tau$ for the local algebra $\mathscr{R}$ formed by the field operators smeared with test functions $\in \mathscr{D}(\boldsymbol{R}^4)$ [6]. In case of a conserved symmetry it holds on $\mathscr{H}$ that the vacuum expectation values are invariant

$$(\hat{\Omega}\,|\,\hat{\Phi}\,\hat{\Omega}) = (\hat{\Omega}\,|\,\hat{\Phi}_\tau\,\hat{\Omega})$$

for all local operators $\hat{\Phi}$ on $\mathscr{H}$. (The elements of $\mathscr{H}$ as well as the induced operators on $\mathscr{H}$ carry a hat in order to distinguish them from those of $\mathscr{H}_1$ and $G$.) Hence there exists a uniquely determined family of unitary operators

$$\hat{U}(\tau) = e^{i\hat{Q}\tau},$$
$$\hat{U}(\tau)\,\hat{\Omega} = \hat{\Omega}, \qquad \hat{Q}\hat{\Omega} = 0,$$
$$\hat{\Phi}_\tau = \hat{U}(\tau)\,\Phi\,\hat{U}^{-1}(\tau).$$

The condition on the $\hat{Q}_r$ for generating a conserved symmetry is

$$\lim_{r\to\infty} (\hat{\Omega} \mid [\hat{Q}_r, \hat{\Phi}]\,\hat{\Omega}) = 0$$

for all local operators on $\mathscr{H}$ [7, 1–4]. The connection between $Q$ and $Q_r$ then is given by [1, 4]

$$\lim_{r\to\infty} (\hat{\Omega} \mid \hat{\Phi}_1\,\hat{Q}_r\,\hat{\Phi}_2\,\hat{\Omega}) = \lim_{r\to\infty} (\hat{\Omega} \mid \hat{\Phi}_1\,[\hat{Q}_r, \hat{\Phi}_2]\,\hat{\Omega}) + \lim_{r\to\infty} (\hat{\Omega} \mid \hat{\Phi}_1\,\hat{\Phi}_2\,\hat{Q}_r\,\hat{\Omega})$$
$$= (\hat{\Omega} \mid \hat{\Phi}_1\,\hat{Q}\,\hat{\Phi}_2\,\hat{\Omega}) + \lim_{r\to\infty} (\hat{\Omega} \mid \hat{\Phi}_1\,\hat{\Phi}_2\,\hat{Q}_r\,\hat{\Omega})$$

for all local operators on $\mathscr{H}$. Hence we have to show

$$\lim_{r\to\infty} (\hat{\Phi}\,\hat{\Omega} \mid \hat{Q}_r\,\hat{\Omega}) = \lim_{r\to\infty} (\Phi\,\Omega \mid Q_r\,\Omega)_G = 0$$

for all local operators $\hat{\Phi}$ on $\mathscr{H}$ resp. for all $\Phi \in \mathscr{R}_1$ when $\mathscr{R}_1$ denotes the operators from $\mathscr{R}$ which leave $\mathscr{H}_1$ invariant. This proves at the same time that $\Phi \to \Phi_\tau$ is a conserved symmetry as well as the connection between $Q$ and $Q_r$.

**Theorem.** *Under the assumptions mentioned in* 2.1 *it follows for every* $\Phi \in \mathscr{R}_1$

$$\lim_{r\to\infty} (\Phi\,\Omega \mid Q_r\,\Omega)_G = 0\,. \tag{7}$$

## 3. Proof of the Statement

We apply an idea of Ref. [8] and [2] and make use of a Jost-Lehmann-Dyson representation derived in [9]. Before doing so, we need some preparations. In particular we extract from (5) together with the assumptions in 2.1 some information concerning the behaviour of the Fourier transform of

$$(\Phi\,\Omega \mid j_0(x)\,\Omega)_G$$

near the origin. Together with the spectrum condition and the relative locality of $j_0(x)$ and $\Phi$ this will enable us to prove the statement.

**3.1.** The assumptions imply a Källen-Lehmann representation

$$(A_\nu(x)\,\Omega \mid A_{\nu'}(y)\,\Omega)_G = \int_{p\in\boldsymbol{R}^4} e^{ip(x-y)} \{g_{\nu\nu'}\,d\mu_1(p) + p_\nu p_{\nu'}\,d\mu_2(p)\} \tag{8}$$

where $d\mu_2(p)$ and $d\mu_1(p)$ define tempered Lorentz invariant measures on $\boldsymbol{R}^4$.

[*Proof*. As usual one concludes that the Fourier transform of (8) is a distribution of the form

$$g_{\mu\nu}\varrho_1(p)+p_\mu p_\nu \varrho_2(p) \tag{9}$$

where $\varrho_1$ and $\varrho_2$ are Lorentz invariant. We take that for granted and concentrate on showing that $\varrho_2(p)\,d^4p$ defines a measure on $\boldsymbol{R}^4$. The commutativity of $\eta$ with $P_\mu$ implies that (9) is a measure on $\boldsymbol{R}^4$. In particular that is the case for $p_1 p_2 \varrho_2(p)$. Hence $\varrho_2$ is a measure on $\boldsymbol{R}^4-0$. The Lorentz invariance of $\varrho_2$ will enable us to infer that $\varrho_2(p)$ is a measure on $\boldsymbol{R}^4$: $\varrho_2(p)$ is clearly in $\mathscr{D}'(\boldsymbol{R}^4-0)$. Hence, by a result on Lorentz invariant distributions [10] one has for $f\in\mathscr{D}(\boldsymbol{R}^4-0)$

$$\int f(p)\,p_1 p_2 \varrho_2(p)\,d^4p=\bar{\varrho}_e[\bar{f}^e]+\bar{\varrho}_0[\bar{f}^0]$$

with the uniquely determined distributions $\bar{\varrho}_e\in\mathscr{D}'(\boldsymbol{R}^1)$, $\bar{\varrho}_0\in\mathscr{D}'(\boldsymbol{R}^1_+)$ and

$$\bar{f}^e(s)=\int p_1 p_2 f(p)\,\delta(p^2-s)\,d^4p\,,$$
$$\bar{f}^0(s)=\int p_1 p_2 f(p)\,\varepsilon(p_0)\,\delta(p^2-s)\,d^4p\,.$$

Consider now $g(s)\in\mathscr{D}(K)$ with a compact $K\in\boldsymbol{R}^1$ and put

$$f_1(p)=g(p^{02}-\boldsymbol{p}^2)\,p_0 p_1 p_2\,F(\boldsymbol{p}^2)\,,$$
$$f_2(p)=g(p^{02}-\boldsymbol{p}^2)\,p_0^2 p_1 p_2\,F(\boldsymbol{p}^2)\,.$$

With a suitably chosen non negative $F(\boldsymbol{p}^2)\in\mathscr{D}(\boldsymbol{R}^1)$ we have $f_1, f_2\in\mathscr{D}(\boldsymbol{R}^4-0)$ and

$$\bar{f}_0^0(s)=g(s)\cdot a\,,$$
$$a=\int p_1^2 p_2^2\,F(\boldsymbol{p}^2)\,d^3\boldsymbol{p}>0\,,$$
$$\bar{f}_2^e(s)=g(s)\,h(s)\,,$$
$$h(s)=\int\sqrt{|\boldsymbol{p}|^2+s}\;p_1^2 p_2^2\,F(\boldsymbol{p}^2)\,d^3\boldsymbol{p}\,.$$

$h(s)$ is apparently infinitely often differentiable and unequal zero for all $s\geqq 0$. By choosing $F(\boldsymbol{p}^2)$ appropriately, this stays true for all values $s\in K$. We now replace $f_1$ and $f_2$ by respectively

$$f_3(p)=\frac{1}{a}f_1(p)$$

$$f_4(p)=\begin{cases}\dfrac{g(p^{0^2}-\boldsymbol{p}^2)}{h(p^{0^2}-\boldsymbol{p}^2)}\,p_0^2 p_1 p_2\,F(\boldsymbol{p}^2) & \text{for} \quad p\in\operatorname{supp} f_2\\ 0 & \text{for} \quad p\notin\operatorname{supp} f_2\,,\end{cases}$$

which are still in $\mathscr{D}(\boldsymbol{R}^4 - 0)$. Then we have

$$\begin{aligned}
\bar{f}_3^e(s) &= 0\,,\\
\bar{f}_3^0(s) &= g(s)\,,\\
\bar{f}_4^e(s) &= g(s)\,,\\
\bar{f}_4^0(s) &= 0\,,
\end{aligned}$$

and furthermore

$$\sup_{p\in\boldsymbol{R}^4} f_3(p)\, p_1\, p_2 \leqq \sup_{s\in K} g(s)\, C_K\,,$$

$$C^K = \sup_{p^{0^2} - \boldsymbol{p}^2 \in K} a\cdot p_0\, p_1^2\, p_2^2\, F(\boldsymbol{p}^2)\,,$$

$$\sup_{p\in\boldsymbol{R}^4} f_4(p)\, p_1\, p_2 \leqq \sup_{s\in K} g(s)\, C_K'\,,$$

$$C_{K'} = \sup_{p^{0^2} - \boldsymbol{p}^2 \in K} \frac{p_0^2\, p_1^2\, p_2^2}{h(p_0^2 - \boldsymbol{p}^2)}\, F(\boldsymbol{p}^2)\,.$$

Now we know that $\varrho_2(p)$ is a measure, i.e. a distribution of order zero on $\mathscr{D}(\boldsymbol{R}^4 - 0)$. Hence

$$\begin{aligned}
|\bar{\varrho}_e[g]| = |\textstyle\int f_4(p)\, p_1\, p_2\, \varrho_2(p)\, d^4p| &\leqq \sup_{p\in\boldsymbol{R}^4} f_4(p)\, p_1\, p_2\, C_1\\
&\leqq \sup_{s\in K} g(s)\, C_1 \cdot C_K'
\end{aligned}$$

and

$$\begin{aligned}
|\bar{\varrho}_0[g]| = |\textstyle\int f_3(p)\, p_1\, p_2\, \varrho_2(p)\, d^4p| &\leqq \sup f_3(p)\, p_1\, p_2\, C_2\\
&\leqq \sup_{s\in K} g(s)\, C_2 \cdot C_K
\end{aligned}$$

($C_1$ and $C_2$ depend on $K$ too).
Hence [11] $\bar{\varrho}_e$ and $\bar{\varrho}_0$ are Radon measures on $\boldsymbol{R}^1$ and $\boldsymbol{R}^1_+$ respectively.

We know that $p_1\, p_2\, \varrho_2(p)$ is a measure on $\boldsymbol{R}^4$ with value zero at the origin. Hence it is fixed uniquely by its values on testfunctions from $\mathscr{D}(\boldsymbol{R}^4 - 0)$ and we have for $f\in\mathscr{D}(\boldsymbol{R}^4)$

$$\int f(p)\, p_1\, p_2\, \varrho_2(p)\, d^4p = \bar{\varrho}_e[\bar{f}^e] + \bar{\varrho}_0[\bar{f}^0]\,.$$

We extend now $\varrho_2(p)$ to all of $\mathscr{D}(\boldsymbol{R}^4)$ by

$$\int f(p)\, \varrho_2(p)\, d^4p = \bar{\varrho}_e[\bar{\bar{f}}^e] + \bar{\varrho}_0[\bar{\bar{f}}^0]$$

with

$$\begin{aligned}
\bar{\bar{f}}^e(s) &= \int f(p)\, \delta(p^2 - s)\, d^4p\,,\\
\bar{\bar{f}}^0(s) &= \int f(p)\, \varepsilon(p^0)\, \delta(p^2 - s)\, d^4p
\end{aligned}$$

which is possible, since $\bar{\bar{f}}^e(s)$, $\bar{\bar{f}}^0(s)$ are continuous on $\boldsymbol{R}^1$ and $\boldsymbol{R}^1_+$ respectively. Furthermore we have for $f\in\mathscr{D}(K^1)$ with a compact $K^1\subset\boldsymbol{R}^4$

and with suitable numbers $C^1_{K^1}$, $C^2_{K^1}$

$$\sup_{s\in\boldsymbol{R}^1} \bar{\bar{f}}^e(s) \leqq C^1_{K^1} \sup_{p\in K^1} f(p)\,,$$

$$\sup_{s\in\boldsymbol{R}^1_+} \bar{\bar{f}}^0(s) \leqq C^2_{K^1} \sup_{p\in K^1} f(p)\,.$$

Hence $\varrho_2(p)\,dp \equiv d\mu_2(p)$ defines a Radon measure on $\boldsymbol{R}^4$, and so does $p_\nu p_{\nu'} \varrho_2\, dp = p_\nu p_{\nu'}\, d\mu_2(p)$ and $\varrho_1(p)\,dp \equiv d\mu_1(p)$. The temperedness is implied by the temperedness of $A_\nu(x)$].

From (5) and (6) it follows that

$$(j_\nu(x)\,\Omega \mid j_{\nu'}(y)\,\Omega)_G = \int_{p\in\boldsymbol{R}^4} e^{ip(x-y)}(p^2)^2(p_\nu p_{\nu'} - p^2 g_{\nu\nu'})\, d\mu_2(p)$$

and in particular

$$(j_0(x)\,\Omega \mid j_0(y)\,\Omega)_G = \int e^{ip(x-y)}(p^2)^2\, |\boldsymbol{p}|^2\, d\mu_2(p)\,. \tag{10}$$

Since the current is gauge invariant, it follows that $\mu_2$(p) is a positive measure off the light cone. The spectrum condition on $\mathscr{H}$ implies that $p^2\, d\mu_2(p)$ vanishes outside the forward light cone.

**3.2.** Consider $\Phi \in \mathscr{R}_1$, define $\Phi(x) = U(x)\,\Phi\, U^{-1}(x)$ where

$$U(x) = \int e^{-ipx}\, dE(p)$$

is the unitary representation of the space-time translations, $x\in\boldsymbol{R}^4$, $p\in\boldsymbol{R}^4$, $px = p^0x^0 - \boldsymbol{x}\boldsymbol{p}$. In

$$(\Phi\,\Omega \mid j_0(x)\,\Omega)_G = \int e^{-ipx}\, d\sigma(p)\,.$$

$\sigma(p)$ is a tempered complex measure on $\boldsymbol{R}^4$ which is locally finite. (This follows like the next equations immediately from the translation invariance and from $j_0(x)$ being an operator valued tempered distribution.) $\sigma(p)$ vanishes outside the forward light cone due to the spectrum condition. Hence $\tilde{\alpha}(p^0)\, d\sigma(p)$ with $\tilde{\alpha}(p^0)\in\mathscr{S}(\boldsymbol{R}^1)$ is a finite complex measure on $\boldsymbol{R}^4$. With the notation $\Phi[g] = \int \Phi(x)\, g(x)\, d^4x$, etc., we have due to translational invariance ($\check{g}(x) \equiv g(-x)$):

$$\begin{aligned}(\Phi[g]\,\Omega \mid j_0(x)\,\Omega)_G &= (\Phi(-x)\,\Omega \mid j_0[\check{g}]\,\Omega)_G \\ &= \int e^{-ipx}(\Phi\,\Omega \mid \eta\, dE(p)\, j_0[\check{g}]\,\Omega) \\ &= \int e^{-ipx}\tilde{g}(p)\, d\sigma(p)\,,\end{aligned}$$

with $\tilde{g}(p) = \int e^{ipy} g(y)\, d^4y$. This holds for all test functions $g\in\mathscr{S}(\boldsymbol{R}^4)$ and all $x\in\boldsymbol{R}^4$. Therefore, it follows for every Borel set $\Delta\in\boldsymbol{R}^4$ that

$$\int_{p\in\Delta} \tilde{g}(p)\, d\sigma(p) = \int_{p\in\Delta} (\Phi\,\Omega \mid dE(p)\, j_0[\check{g}]\,\Omega)_G\,.$$

From this we get by Schwarz' inequality on $\mathscr{H}_1$

$$\left|\int_{\Delta} \tilde{g}(p)\, d\sigma(p)\right| \leqq \|\Phi\,\Omega\|_G \left\{\int_{\Delta} (j_0[\check{g}]\,\Omega|\mathrm{d}E(p) j_0[\check{g}]\,\Omega)_G\right\}^{1/2}$$
$$= \|\Phi\,\Omega\|_G \left\{\int_{\Delta} |\tilde{g}(p)|^2 (p^2)^2 |\boldsymbol{p}|^2\, d\mu_2(p)\right\}^{1/2}.$$

We now put $\tilde{g}(p)=\tilde{g}_1(p)\,\tilde{\alpha}(p^0)$ with $\tilde{\alpha}(p^0)\in\mathscr{S}(\boldsymbol{R}^1)$, $\tilde{g}_1(p)\in\mathscr{S}(\boldsymbol{R}^4)$ and write $\tilde{g}(p)$ instead of $\tilde{g}_1(p)$

$$\left|\int_{\Delta} \tilde{g}(p)\,\tilde{\alpha}(p^0)\, d\sigma(p)\right| \leqq \|\Phi\,\Omega\|_G \left\{\int_{\Delta} |\tilde{g}(p)|^2 (p^2)^2 |\boldsymbol{p}|^2 |\tilde{\alpha}(p^0)|^2\, d\mu_2(p)\right\}^{1/2}. \tag{11}$$

$\tilde{\alpha}(p^0)\, d\sigma(p)$ and $|\tilde{\alpha}(p^0)|^2\, d\mu_2(p)$ define finite measures on $\boldsymbol{R}^4$. (11) holds for every continuous function $\tilde{g}(p)$ bounded by a polynomial for large $p$, in particular for $\tilde{g}(p)=1$. Application of the Radon-Nikodym theorem [12] implies

$$\tilde{\alpha}(p^0)\, d\sigma(p) = m(p)\,(p^2)^2\, |\boldsymbol{p}|^2\, |\tilde{\alpha}(p^0)|^2\, d\mu_2(p)$$

with $m(p)\in L^1((p^2)\,|\boldsymbol{p}|^2\,|\tilde{\alpha}(p^0)|^2\, d\mu_2(p))$. Inserting this into (11), we get

$$\begin{aligned}&\left|\int_{\Delta} \tilde{g}(p)\, m(p)\,(p^2)^2\, |\boldsymbol{p}|^2\, |\tilde{\alpha}(p^0)|^2\, d\mu_2(p)\right| \\ &\qquad \leqq \|\Phi\,\Omega\|_G \left\{\int_{\Delta} |\tilde{g}(p)|^2\, (p^2)^2\, |\boldsymbol{p}|^2\, |\tilde{\alpha}(p^0)|^2\, d\mu_2(p)\right\}^{1/2}.\end{aligned} \tag{12}$$

Let now $\Delta$ be bounded, contain the origin and let $\tilde{g}(p)\geqq 0$ on $\Delta$. If $m(p)$ is decomposed into its real and imaginary parts, then (12) holds for each part separately. By decomposing $\Delta$ into $\Delta_r^+\cup\Delta_r^-$ (or $\Delta_i^+\cup\Delta_i^-$ respectively) with $\mathrm{Re}\, m(p)\geqq 0$ on $\Delta_r^+$, $\mathrm{Re}\, m(p)<0$ on $\Delta_r^-$ (respectively for $\mathrm{Im}\, m(p)$) [14], and by applying (12) on $\Delta_r^+$ and $\Delta_r^-$ (or $\Delta_i^+$, $\Delta_i^-$) separately, one shows that

$$\begin{aligned}&\int_{\Delta} \tilde{g}(p)\, |m(p)|\, (p^2)^2\, |\boldsymbol{p}|^2\, |\tilde{\alpha}(p^0)|^2\, d\mu_2(p) \\ &\qquad \leqq 2\sqrt{2}\, \|\Phi\,\Omega\|_G \left\{\int_{\Delta} |\tilde{g}(p)|^2\, (p^2)^2\, |\boldsymbol{p}|^2\, |\tilde{\alpha}(p^0)|^2\, d\mu_2(p)\right\}^{1/2}.\end{aligned}$$

We let now $\tilde{g}(p)$ converge toward $(p^2\,|\boldsymbol{p}|)^{-1}$ pointwise on $\Delta$. Fatou's lemma [13] shows that

$$m(p)\in L^1(p^2\,|\boldsymbol{p}|)\, |\alpha(p^0)|^2\, d\mu_2(p)\,. \tag{14}$$

Since $\mu_2(p)$ is tempered and $p^2\, d\mu_2(p)$ vanishes outside the forward light cone, this stays true (compare the last inequality!) if $m(p)$ is multiplied by any power of the components of $p$.

**3.3.** Consider now with $\alpha(x^0) \in \mathscr{D}(\boldsymbol{R}^1)$

$$(\Phi[\check{\alpha}]\,\Omega|\,j_0(x)\,\Omega)_G$$

the Fourier transform of which, $\tilde{\alpha}(p^0)\,d\sigma(p)$, is discussed above. According to [9] one has for this a Jost-Lehmann-Dyson representation

$$\begin{aligned}
&(\Phi[\check{\alpha}]\,\Omega\,|\,j_0(x)\,\Omega)_G \\
&= \complement \int_{D_1} d^3\xi' \int d\sigma(p)\,\tilde{\alpha}(p_0)\,e^{-ip\xi'}\left(\frac{\partial}{\partial x^0} - ip^0\right)\Delta^+_{\sqrt{p^2}}(x-\xi') \\
&= \complement \int_{D'} d^3\xi' \int d\mu_2(p)\,m(p)\,(p^2)^2\,|\boldsymbol{p}|^2\,|\tilde{\alpha}(p^0)|^2\,e^{-ip\xi'}\left(\frac{\partial}{\partial x^0} - ip^0\right)\Delta^+_{\sqrt{p^2}}(x-\xi'),
\end{aligned}$$

where $D_1$ denotes a compact region in $\boldsymbol{R}^3$. We keep now $x^0$ finite (in fact, we shall put it zero) and let $\boldsymbol{x}$ increase. Since $D_1$ is compact, $x-\xi'$ will become space-like for sufficiently large $\boldsymbol{x}$, and we get (observe the factor $p^2$!)

$$\begin{aligned}
&(\Phi[\check{\alpha}]\,\Omega\,|\,j_0(x)\,\Omega)_G \\
&\quad = \complement \int_{D_1} d^3\xi' \int d\mu_2(p)\,m(p)\,p^2\,|\boldsymbol{p}|^2\,|\tilde{\alpha}(p^0)|^2\,e^{-ip\xi'}\left(\frac{\partial}{\partial x^0} - ip^0\right) \\
&\qquad\qquad \cdot \sqrt{p^2}^{\,3}\,\frac{K_1\left(\sqrt{p^2}\sqrt{-(x-\xi)^2}\right)}{\sqrt{-(x-\xi)^2}}
\end{aligned}$$

with the cylindrical function $K_1$. Hence for $|\boldsymbol{x}|$ sufficiently large

$$\begin{aligned}
&|(\Phi[\check{\alpha}]\,\Omega\,|\,j_0(x)\,\Omega)_G| \\
&\quad \leqq \complement \left|\int_{D_1} d^3\xi'\right| \cdot \int |m(p)|\,p^2\,|\boldsymbol{p}|^2\,p^0\,|\tilde{\alpha}(p^0)|^2\,d\mu_2(p) \cdot \sup_{y\in\boldsymbol{R}^1}(y^3\,K_1(y))\,\frac{1}{|\boldsymbol{x}|^4} \\
&\quad + \complement \left|\int_{D_1} d^3\xi'\right| \cdot \int |m(p)|\,p^2\,|\boldsymbol{p}|^2\,|\tilde{\alpha}(p^0)|^2\,d\mu_2(p) \cdot \sup_{y\in\boldsymbol{R}^1}(y^3\,K_1(y))\,\frac{x^0}{|\boldsymbol{x}|^6} \\
&\quad + \complement \left|\int_{D_1} d^3\xi'\right| \cdot \int |m(p)|\,p^2\,|\boldsymbol{p}|^2\,|\tilde{\alpha}(p^0)|^2\,d\mu_2(p) \cdot \sup_{y\in\boldsymbol{R}^1}\left(y^4\,\frac{\partial}{\partial y}K_1(y)\right)\frac{x^0}{|\boldsymbol{x}|^6}\,.
\end{aligned}$$

Since for small $y$ $K_1(y) \approx 1/y$, and since for large $y$ $K_1$ drops exponentially, it follows for $x^0=0$ and $|\boldsymbol{x}|$ large

$$|(\Phi[\check{\alpha}]\,\Omega\,|\,j_0(x)\,\Omega)_G| \leqq \complement\,\frac{1}{|\boldsymbol{x}|^4}\,.$$

In particular, we see now that

$$\lim_{r\to\infty} (\Phi\Omega \mid Q_r\Omega)$$

exists.

**3.4.** We may write

$$(\Phi[\check{\alpha}]\,\Omega \mid j_0(x)\,\Omega)_G = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} F_i(x)$$

with

$$F_i(x) = ¢ \int\limits_{D_1} d^3\xi' \int d\mu_2(p)\, m(p)(p^2)^2\, p_i |\tilde{\alpha}(p^0)|^2 e^{-ip\xi'} \left(\frac{\partial}{\partial x^0} - ip^0\right) \Delta^+_{\sqrt{p^2}}(x-\xi').$$

As in 3.3 it follows for $x^0 = 0$ and $|\boldsymbol{x}|$ large

$$|F_i(x)| \leqq ¢ \frac{1}{|\boldsymbol{x}|^4}.$$

Hence the Fourier transform $\tilde{F}_i(\boldsymbol{p})$ of $F_i(\boldsymbol{x}, 0)$ is bounded and continuous in $\boldsymbol{p}$ and

$$\begin{aligned}(\Phi\Omega \mid Q_r\Omega)_G &= \int \tilde{f}_r(\boldsymbol{p})\, p_i\, \tilde{F}_i(\boldsymbol{p})\, d^3\boldsymbol{p} \\ &= \frac{1}{r} \int \tilde{f}_0(\boldsymbol{q})\, q_i\, \tilde{F}_i\left(\frac{\boldsymbol{q}_i}{r}\right) d^3\boldsymbol{q}\end{aligned}$$

which converges toward zero for $r \to \infty$ as it was stated above.

*Acknowledgement.* I benefitted from discussions with Dr. P. Breitenlohner, Dr. D. Maison, E. Rudolph and Dr. S. Schlieder.

*Note added in proof.* As it was pointed out to the author by J. A. Swieca, perturbation theory indicates that $\varrho_2(p)$ has a contribution of the first derivative of a delta function on the light cone. The preceding proof works also if such a contribution is present because it does not show up in the two point function of the current, and our statement stays true. However, the assumption that $\eta$ commutes with $P_\mu$ then has to be modified. — I thank Prof. Swieca for this information.

## References

1. Robinson, D. W.: Istanbul lectures 1966.
2. Swieca, J. A.: Nuovo Cimento **52**, 242 (1967).
3. Schroer, B., and P. Stichel: Commun. Math. Phys. **3**, 258 (1966).
4. Reeh, H.: Fortschr. Physik **16**, 687 (1968).
5. cf. Gårding, L., and A. S. Wightman: Arkiv Fys. **28**, 129—184 (1965) and the standard textbooks on quantum electrodynamics.
6. See e.g. [4]. "Internal" means that the space-time support of the operators is not changed. Compare in this connection also: Maison, D.: Symmetry Transformations from local currents, to be published.

7. Kastler, D., D. W. Robinson, and A. Swieca: Commun. Math. Phys. **2**, 108 (1966).
8. Ezawa, H., and J. A. Swieca: Commun. Math. Phys. **5**, 330 (1967).
9. Araki, H., K. Hepp, and D. Ruelle: Helv. Phys. Acta **35**, 164 (1962).
10. Gårding, L., and J.L.Lions: Nuovo Cimento **14**, Suppl. 9 (1959), Sect. 8; Rieckers, A., and W. Güttinger: Commun. Math. Phys. **7**, 190 (1967).
11. cf. e.g. Treves, F.: Topological vector spaces, distributions and kernels, Sect. 21. New York-London: Academic Press 1967.
12. cf. e.g. Berberian, S. K.: Measure and integration, Sect. 52, Theorem 1. New York: Macmillan Comp. 1965.
13. cf. e.g. Berberian, S. K.: l.c. [12], Sect. 32, Theorem 1.
14. Compare e.g. Berberian, S. K.: l.c. [12], Sect. 49 for the decomposition into positive and negative parts.


H. Reeh
Max-Planck-Institut für Physik und Astrophysik
8000 München 23, Föhringer Ring 6