Commun. math. Phys. 13, 68—72 (1969)

Uniqueness of the Hamiltonian in Quantum Field Theories

STEPHEN PARROTT
Department of Physics, University of Mass.-Boston

Received June 25, 1968

Abstract. In most quantum field theories, one defines the Hamiltonian (energy) operator H as a limit of "cutoff" operators H_s: $H = \lim_{s \to \infty} H_s$. (The operator H_s would be the correct Hamiltonian for a world in which all momenta are smaller than s.) Since the cutoff operators seldom converge in any of the standard operator topologies, it is often necessary to invent more subtle notions of "convergence". For some of the these, it is not obvious that the "limit" operator H is unique. In this note we point out that for one such method of obtaining convergence, the "limit" operator is *not* unique. In fact, (under mild assumptions about the operators H_s), if H_s converges to H, then H_s also converges to $H + R$, where R is an arbitrary bounded positive operator.

0. Notation

Let $\mathscr{H}$ be a separable Hilbert space with inner product $(\cdot\,,\cdot)$. An *operator* H on $\mathscr{H}$ is a densely-defined linear transformation from $\mathscr{H}$ into $\mathscr{H}$ with domain $\mathscr{D}(H)$. We write $H' \subset H$ to mean $\mathscr{D}(H') \subset \mathscr{D}(H)$ and $H'f = Hf$ for all f in $\mathscr{D}(H')$. A *symmetric* operator satisfies: $(Hf, g) = (f, Hg)$ for all f, g in $\mathscr{D}(H)$. A symmetric operator is *essentially self-adjoint* if it has a unique self-adjoint extension. We assume that the reader is familiar with the basic facts concerning unbounded self-adjoint operators [cf. 4, Chap. 8].

1. Statement of the Problem

Suppose we are given a family of self-adjoint operators H_s, $(0 < s < \infty)$. Here are two related methods for obtaining a symmetric operator H as a "limit" of the family of operators H_s:

Method A. Find a dense linear manifold $\mathscr{D}$ and bounded invertible operators T_s, T such that:

(i) For all s, $T_s \mathscr{D} \subset \mathscr{D}(H_s)$.

(ii) For each f in $\mathscr{H}$, $\lim_{s \to \infty} T_s f = Tf$.

(iii) For all f in $\mathscr{D}$, $\lim_{s \to \infty} H_s T_s f$ exists.

Define the limit operator H with $\mathscr{D}(H) = T\mathscr{D}$ by:

$$HTf = \lim_{s \to \infty} H_s T_s f, \; f \quad \text{in} \quad \mathscr{D}\,.$$

The operator H is symmetric, but need not be self-adjoint.

Notice that, in principle, this method may yield a limit operator H with $\mathscr{D}(H) \cap \mathscr{D}(H_s) = \{0\}$ (in which case one could not hope to obtain $Hf = \lim_{s\to\infty} H_s f$).

Method B. Replace condition (iii) in Method A by:

(iii′) For all f, g in $\mathscr{D}$, $\lim_{s\to\infty} (H_s T_s f, T_s g)$ exists.

If the bilinear form defined on $T\mathscr{D} \times T\mathscr{D}$ by $Tf, Tg \to \lim_{s\to\infty} (H_s T_s f, T_s g)$ is continuous in either variable, then there is a unique operator H with $\mathscr{D}(H) = T\mathscr{D}$ such that $(HTf, Tg) = \lim_{s\to\infty} (H_s T_s f, T_s g)$.

A weaker version of Method B was used in [3], and a combination of both methods was used in [1, 2]. (See comment 4, Section 2.)

Notice that *given* $\mathscr{D}$ and T_s, the symmetric operator H defined by Method A or B is unique. However, it is conceivable that one could choose a different family T_s satisfying (i) and (ii) and obtain a different limit operator H. We shall see that this is, in fact, the case with Method B.

Let the family of self-adjoint operators H_s be given. If there exists a dense linear manifold $\mathscr{D}$ and operators T_s, T satisfying (i), (ii), and (iii) (or (iii′)), we shall say that the operator H defined by Method A (or B) is *obtainable* from H_s by Method A (or B).

The limit operator H is symmetric, but not necessarily self-adjoint, so one must pass to a self-adjoint extension of H (if there is one) to obtain a self-adjoint operator as a "limit" of the operators H_s. Of course, H may have more than one self-adjoint extension, so nonuniqueness can arise at this stage. However, suppose that the operator H obtained from H_s by Method A (or B) has a *unique* self-adjoint extension $\tilde{H}$. Let H' be another operator obtainable from H_s by Method A (or B). We can now pose the main question: Is $H' \subset \tilde{H}$?

2. Uniqueness of the Limit

It is not difficult to see that an operator obtainable by Method A is unique in the sense described above. An elementary computation (see [5]) shows that if H is obtainable from H_s by Method A, then

$$\lim_{s\to\infty} (H_s - i)^{-1} f = (H - i)^{-1} f \quad \text{for all } f \text{ in } \quad (H - i)\,\mathscr{D}(H)\,. \qquad (1)$$

If H has a unique self-adjoint extension $\tilde{H}$, then the Cayley transform $(H - i)(H + i)^{-1}$ is a unitary operator and hence $(H - i)\,\mathscr{D}(H)$ is dense in $\mathscr{K}$. Since the family $(H_s - i)^{-1}$ is uniformly bounded

$$\lim_{s\to\infty} (H_s - i)^{-1} f = [\mathrm{cl}\,(H - i)^{-1}]\, f = (\tilde{H} - i)^{-1} f \quad \text{for all} \quad f \quad \text{in} \quad \mathscr{K}\,.$$

If H' is also obtainable from H_s by Method A, equation (1) holds with H replaced by H'. Thus $(H' - i)^{-1} f = (\tilde{H} - i)^{-1} f$ for all f in $(H' - i)\,\mathscr{D}(H')$, and this shows that $H' \subset \tilde{H}$.

This result says that Method A yields a unique limit modulo the usual difficulties with uniqueness of extensions of symmetric operators. Thus the uniqueness of Method A is as good as one could hope. The result belows says that the uniqueness of Method B is as bad as one might fear.

Theorem. *Let the family* H_s, $(0 < s < \infty)$ *of self-adjoint operators be given, and assume that the spectrum of each* H_s *contains the interval* $(0, \infty)$. *Suppose* H *is obtainable from* H_s *by Method* B, *and let* R *be any bounded positive operator. Then there is an operator* H' *with* $H' \subset H$ *such that* $H' + R$ *is also obtainable from* H_s *by Method* B.

Proof. Since H is obtainable from H_s by Method B, we are given a dense linear manifold $\mathscr{D}$ and bounded invertible operators T_s, T, such that the conditions (i), (ii) and (iii′) of Method B are satisfied. We shall show that there exists a dense linear manifold $\mathscr{D}' \subset \mathscr{D}$ and bounded invertible operators T_s such that:

(i) For all s, $T'_s \mathscr{D}' \subset \mathscr{D}(H_s)$.

(ii) $\lim\limits_{s \to \infty} T'_s f = T f$ for all f in $\mathscr{K}$.

(iii′) $\lim\limits_{s \to \infty} (H_s T'_s f, T'_s g) = ((H + R)\, T f, T g)$ for all f, g in $\mathscr{D}'$.

The operator H' will, of course, be the restriction of H to $T\mathscr{D}'$.

We shall set $T'_s = T_s + s^{-1/2} U_s Q$, where the operators U_s are isometries to be defined below, and $Q = (T * R T)^{1/2}$.

Let $P_s(\cdot)$ be the spectral measure associated with H_s. Then for each f in $\mathscr{K}$,

$$\lim_{\varepsilon \to 0} P_s((a, a + \varepsilon))\, f = 0\,. \tag{2}$$

Further, the condition that the spectrum of H_s contains $(0, \infty)$ implies:

If $0 < a < b$, then the range of $P_s((a, b))$ is infinite dimensional. (3)

Let $f_1, f_2, \ldots$ be a countable set of elements in $\mathscr{D}$ whose linear span is dense in K. Let $\mathscr{D}'$ be the set of all finite linear combinations of the vectors f_k. By (2), given any fixed s, there is a number $\varepsilon = \varepsilon(s)$, $0 < \varepsilon(s) < 1$ such that

$$\|P_s\big((s, s + \varepsilon(s))\big)\, T_s f_k\| < \frac{1}{s} \quad \text{for} \quad 1 \leq k < s\,. \tag{4}$$

Let $E_s = P_s\big((s, s + \varepsilon(s))\big)$. By (3), the range of E_s is infinite dimensional, and hence there exists an isometry U_s which maps the entire Hilbert space $\mathscr{K}$ onto the range of E_s. Now

$$\begin{aligned}(H_s T'_s f_j, T'_s f_k) = {} & (H_s T_s f_j, T_s f_k) + s^{-1/2}(H_s T_s f_j, U_s Q f_k) \\ & + s^{-1/2}(H_s U_s Q f_j, T_s f_k) + s^{-1}(H_s U_s Q f_j, U_s Q f_k)\,.\end{aligned} \tag{5}$$

By hypothesis, the limit of the first term as $s \to \infty$ is $(H T f_j, T f_k)$. We shall show that the limits of the second and third terms are 0, and the limit of the last term is (RTf_j, Tf_k). From this, the theorem follows immediately by linearity.

Note that $E_s U_s = U_s$ and that E_s commutes with H_s. Thus

$$\begin{aligned} s^{-1/2}(H_s T_s f_j, U_s Q f_k) &= s^{-1/2}(H_s T_s f_j, E_s U_s Q f_k) \\ &= s^{-1/2}(H_s E_s T_s f_j, U_s Q f_k) \end{aligned} \tag{6}$$

The norm of the restriction of H_s to the range of E_s is at most $s + \varepsilon(s) < s + 1$, and Eq. (4) says that for large s, $\|E_s T_s f_j\| < \frac{1}{s}$. Hence the modulus of (6) is less than $s^{-1/2}(s+1)\left(\frac{1}{s}\right)\|Qf_k\|$, which goes to 0 as $s \to \infty$. Similarly, the limit of the third term of (5) is 0.

As for the last term, we have

$$\begin{aligned} s^{-1}(H_s U_s Q f_j, U_s Q f_k) &= s^{-1}((H_s - s)\, U_s Q f_j, U_s Q f_k) \\ &\quad + (U_s Q f_j, U_s Q f_k). \end{aligned} \tag{7}$$

Since $\|(H_s - s)\, U_s\| \leqq \varepsilon(s) < 1$, the first term of (7) tends to 0 as $s \to \infty$. And, $(U_s Q f_j, U_s Q f_k) = (Qf_j, Qf_k) = (RTf_j, Tf_k)$ because U_s is an isometry.

Comments

1. The hypothesis that the spectrum of H_s contains $(0, \infty)$ simplifies the proof, but is stronger than is necessary. It is sufficient to assume that there exist numbers $m(s)$ with $\lim_{s \to \infty} m(s) = \infty$ such that $P_s\big((m(s), m(s)+1)\big)$ has infinite-dimensional range. Stated loosely, the only case in which the H_s are unbounded and the conclusion of the theorem may be false is when some of the H_s have spectrum whose unbounded part consists of a discrete sequence of eigenvalues of finite multiplicity. In this case, one can show that the limit is never unique, but it is probably not true that the conclusion of the theorem holds.

2. There are examples in which the conclusion of the theorem holds with $\mathscr{D}' = \mathscr{D}$. This is true even if $\mathscr{D}$ has uncountable Hamel dimension.

3. It is worth noting that in case the operators H_s are uniformly bounded, the convergence defined by Method A (respectively, B) is actually convergence in the strong (resp. weak) operator topology.

4. In [1], the author defines the renormalized Hamiltonian H_{ren} as a limit (in the sense of Method B) of the renormalized cutoff operators $H_{\text{ren}}(s)$. The nonuniqueness of Method B noted above indicates that this procedure is undesirable, and we shall briefly describe here how one can obtain the renormalized Hamiltonian of [1] without recourse to Method B. All page numbers refer to [1].

The operators $H_{\text{ren}}(s)$ can be written as a sum $H_{\text{ren}}(s) = H_{1s} + V_{2s}$ (pages 344, 369, 383). The author proves:

(a) H_{1s} converges in the sense of Method A to an operator H_1 (p. 381).

(b) There is an operator V_2 such that $\mathscr{D}(H_1) \subset \mathscr{D}(V_2)$ (p. 382, 383, 358) and for g_1, g_2 in $\mathscr{D}(H_1)$, $\lim_{s \to \infty} (g_1, V_{2s} g_2) = (g_1, V_2 g_2)$ (p. 383). Thus H_1 and V_2 are both defined on $\mathscr{D}(H_1)$, and each is a (unique) limit of the corresponding cutoff operators H_{1s}, V_{2s}.

It is natural to define the renormalized Hamiltonian H_{ren} by: $H_{\text{ren}} = H_1 + V_2$.

I wish to thank Professor JAMES GLIMM for many helpful and stimulating conversations.

References

1. GLIMM, J.: Yukawa coupling of quantum fields in two dimensions, I. Commun. Math. Phys. **5**, 343 (1967).
2. — Yukawa coupling of quantum fields in two dimensions, II. Commun. Math. Phys. **6**, 61 (1967).
3. — Boson fields with the $:\Phi^4:$ interaction in three dimensions, Commun. Math. Phys. (to appear).
4. RIESZ, F., and SZ-NAGY, B.: Functional analysis. New York: Fred. Ungar 1955.
5. GLIMM, J., and JAFFEE, A.: Singular perturbations of self adjoint operators. Comm. Pure Appl. Math. Phys. (to appear).

S. K. PARROTT
Department of Mathematics
University of Massachusetts at Boston
Boston, Mass. 02116, USA