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Description of the Symmetry Group $SU3/Z3$ of the Octet Model

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Abstract. A study is made of the relationship of $SU3$ and its adjoint group $SU3/Z3$ to the subgroup of $R8$ which leaves invariant not only the length $a_i a_i$ of a real eight component vector a_i but also the cubic invariant $d_{ijk} a_i a_j a_k$, d_{ijk} being the totally symmetric isotropic $SU3$ tensor introduced by GELL-MANN. A formula for the rotation $R \varepsilon R8$ corresponding to $U \varepsilon SU3$, and a formula inverse to this, which provides a way of parametrizing $U \varepsilon SU3$, are derived.

1. Introduction

It has often been argued that the absence of quarks implies that it is not $SU3$, but rather its adjoint group $SU3/Z3$, which is the underlying symmetry group of strong hadronic interactions. Even if the validity of this argument is open to question, the relevance of the group $SU3/Z3$ is surely not. Accordingly it would seem desirable to have a precise description of the group $SU3/Z3$ itself. In particular, we seek a realization of it as a transformation group of some vector space, and the relation of this realization to the usual realization of $SU3$ itself. The motivation for studying these matters stems in part from an interest in the results themselves and in part from a practical interest in the parametrization of $SU3$, for one possible parametrization is achieved by specifying the relationship of $SU3$ to $SU3/Z3$. The practical interest in question involves the construction of effective Lagrangians invariant under chiral $SU3 \times SU3$; in attempting to generalize the approach of CHANG and GURSEY [1] to this problem at the $SU2$ level, the crucial technical step is the explicit parametrization of a 3×3 unimodular matrix.

The matters studied in this paper correspond at the $SU3$ level to familiar results about $SU2$ and $R3$. It is of course very well-known that $SU2$ is homomorphic to the group $R3$ of rotations in Euclidean space $E3$ of three dimensions. There are in fact two elements $\pm U$ of $SU2$ which correspond to each rotation R of $R3$. Alternatively, if one forms the factor group $SU2/Z2$ of $SU2$ with respect to its centre $Z2$[1], one sees

[1] The centre of a group G is a subgroup of G consisting of all elements of G which commute with all elements of G.

that this group, the adjoint group of $SU2$ is isomorphic to $R3$. One may also take the point of view that $R3$ is a realization of $SU2/Z2$ as a transformation group acting in $E3$, and regard the formula, expressing $U \in SU2$ in terms of the $R \in R3$ to which it corresponds, as yielding a parametrization of $SU2$. Explicitly [2], if the rotation

$$x_i \to x'_i = R_{ij} x_j \tag{1.1}$$

of $R3$ corresponds according to

$$R_{ij} = \frac{1}{2} \mathrm{Tr}(U \tau_i U^{-1} \tau_j) \tag{1.2}$$

to $\pm U \in SU2$, the inverse relationship is

$$\pm U = (1 + \mathrm{Tr} R + i\tau_i \varepsilon_{ijk} R_{jk})/[4(1 + \mathrm{Tr} R)]^{1/2} . \tag{1.3}$$

This yields a parametrization of $SU2$ in which $R \in R3$ defines a vector $\mathbf{V}$.

$$V_i = \varepsilon_{ijk} R_{jk} , \tag{1.4}$$

and knowledge of $\mathbf{V}$ determines U, in view of the identity [2]

$$\mathbf{V}^2 = 3 + 2 \mathrm{Tr} R - (\mathrm{Tr} R)^2 . \tag{1.5}$$

Of course, parametrization of $SU2$ is an easy matter. One has amongst other results the parametrizations provided by the usual exponential representation of a unitary matrix, and by the rational or Cayley representation, and one can easily relate these to each other and to the parametrization (1.3).

Here we consider the homomorphism of $SU3$ to its adjoint group $SU3/Z3$, showing, in a manner closely corresponding to the approach of [2] that the latter can be realized as a subgroup of $R8$, the group of real rotations in Euclidean space $E8$ of eight dimensions. More precisely, $SU3/Z3$ is isomorphic to the group of rotations

$$x_i \to x'_i = R_{ij} x_j,\ 1 \leqq i, j \leqq 8 , \tag{1.6}$$

in $E8$, which not only leave

$$x_i x_i \tag{1.7}$$

invariant, but also leave invariant the cubic form

$$d_{ijk} x_i x_j x_k , \tag{1.8}$$

where d_{ijk} is the totally symmetric isotropic tensor introduced by GELL-MANN [3] in his initial paper on the eightfold way. We give formulae generalizing (1.2) and (1.3) which make explicit the correspondence between $R \in SU3/Z3$, and $U \in SU3$, and show that the analog of (1.3) does indeed provide a parametrization of $SU3$. It is not a simple parametrization, but then neither are the $SU3$ versions [4] of the exponential or Cayley parametrizations.

The matter presented below depends on knowledge of the algebraic properties of the GELL-MANN [3] matrices λ_i of $SU3$, which play the role for $SU3$ that the Pauli matrices play for $SU2$, and of the tensors d_{ijk} and f_{ijk}, which enter the multiplication law of the λ_i. All the necessary information is presented in the previous paper [4] where references are given and notation is explained[2].

The result that $SU3/Z3$ can be realised as the subgroup of $R8$ which leaves invariant a cubic form like (1.8) is not new, having been noted by DULLEMOND [5] and ESTEVE [6]. The group $SU3/Z3$ has also been further discussed by ESTEVE and collaborators [7], but most of the explicit results describing the relationship of this group to $SU3$ are new.

2. Homomorphism of $SU3$ to a Subgroup of $R8$

Let $a_i (i = 1, 2, \ldots, 8)$ be any point of real Euclidean space $E8$ of eight dimensions. Rotations in $E8$ are real linear transformations of the type

$$a_i \to a_i' = R_{ij} a_j \, , \tag{2.1}$$

which leave $I_2(a) = a_i a_i$ invariant, so that R obeys

$$R\tilde{R} = 1 \, , \tag{2.2}$$

or

$$R_{ij} R_{ik} = \delta_{ij} \, , \tag{2.3}$$

as well as $\det R = 1$.

In order to relate $SU3$ to a subgroup of $R8$, the group of rotations (2.1) in $E8$, we associate with each point of a of $E8$, a 3×3 traceless hermitian matrix A, i.e. an element of the algebra of $SU3$, by writing

$$A = a_i \lambda_i \, , \tag{2.4}$$

where the λ_i are the Gell-Mann matrices of $SU3$. Now we consider the transformations of $E8$ induced by the transformation

$$A \to A' = U A \, U^{-1} \, , \tag{2.5}$$

for any $U \in SU3$. It is clear that (2.5) leaves invariant not only

$$\operatorname{Tr} A^2 = 2 a_i a_i = I_2(a) \, , \tag{2.6}$$

but also

$$\det A = \frac{2}{3} d_{ijk} a_i a_j a_k = \frac{2}{3} I_3(a) \, . \tag{2.7}$$

Accordingly it follows that (2.5) induces a rotation of $R8$ in $E8$ which also leaves invariant the cubic form $I_3(a)$. Since (2.5) applies to any point a of $E8$, we can write it as

$$U \lambda_j U^{-1} = R_{kj} \lambda_k \tag{2.8}$$

[2] Hereafter, we refer to this paper as MSW.

with the aid of (2.1) and (2.4). Hence we see that the rotation $R \in R8$ induced by $U \in SU3$ according to (2.5) is given explicitly by

$$R_{ij} = \frac{1}{2} \operatorname{Tr}(\lambda_i \, U \lambda_j \, U^{-1}) \tag{2.9}$$

upon using MSW (2.1). To complete proof of the claim that $SU3$ is homomorphic to the subgroup of $R8$ which leaves invariant not only $I_2(a)$ but also I_3(a), various points have to be attended to. Firstly, the reality of R defined by (2.9) easily follows unitarity of U and elementary properties of the trace. The group property of R follows that of $SU3$ according to

$$R(U_1 U_2) = R(U_1)\, R(U_2) \,, \tag{2.10}$$

in the same way as does the corresponding result for $R3$ and $SU2$, [2], upon using the completeness property of the Gell-Mann matrices, i.e.

$$(\lambda_k)_{ab}\,(\lambda)_{cd} = 2\,\delta_{ad}\,\lambda_{bc} - \frac{2}{3}\,\delta_{ab}\,\lambda_{cd} \,. \tag{2.11}$$

If R defined by (2.1) is to leave invariant $I_3(a)$, it must obey

$$d_{ijk}\,R_{ip}\,R_{jq}\,R_{kr} = d_{pqr} \,. \tag{2.12}$$

It can be directly proved that R defined by (2.9) obeys (2.12) either by using an identity for

$$d_{ijk}\,(\lambda_i)_{ad}\,(\lambda_j)_{be}\,(\lambda_k)_{cf}$$

in terms of Kronecker deltas in $a, \ldots, f$ which follows (2.11), or else more easily by using (2.8). Also the relationship in question is not an isomorphism: if U belonging to $SU(3)$ defines $R \in R8$ according to (2.9), then ωU and $\omega^2 U$, where ω, ω^2 are complex cube roots of unity, each also are elements of $SU3$ and each gives rise to the same $R \in R8$. Accordingly we have a homomorphism involving a three valued mapping of $SU3$ on to the subgroup of $R8$. Now the centre Z_3 of $SU3$, i.e. the group of elements of $SU3$ which commute with all elements of $SU3$, consists of the multiples 1, ω, ω^2 of the unit matrix, and $SU3$ is related by a three valued homomorphism to the factor group $SU3/Z3$ of $Z3$ in $SU3$. Hence $SU3/Z3$ and the subgroup of $R8$ which leaves $I_3(a)$ invariant are isomorphic, and the latter can be said to realise the former as a transformation group in $E8$.

We turn next to the inversion of (2.9). One form of the formula for U in terms of R is not hard to obtain, but this will not be the simplest form. Two steps are involved.

a) From (2.8) with the aid of (2.11), we obtain

$$R_{lk}\,\lambda_l \lambda_k = U \lambda_k\, U^{-1} \lambda_k = 2\, U (\operatorname{Tr} U^{-1}) - \frac{2}{3} \,. \tag{2.13}$$

b) For any 3×3, $V = V_0 + V_k \lambda_k$ we have

$$\det V = V_0{}^3 - V_0 I_2(V) + \frac{2}{3} I_3(V) \,, \tag{2.14}$$

by direct computation using explicit representations [5] of λ_k. Writing (2.13) with the aid of MSW (2.2) as

$$2U(\mathrm{Tr}\,U^{-1}) = \frac{2}{3}(1 + \mathrm{Tr}R) + K_i\lambda_i\,, \tag{2.15}$$

where[3]

$$K_i = k_{ijk}R_{jk} = (d_{ijk} + i\,f_{ijk})\,R_{jk}\,, \tag{2.16a}$$

$$= d_i + i\,f_i\,, \tag{2.16b}$$

we find

$$8(\mathrm{Tr}\,U^{-1})^3 = \frac{8}{27}(1 + \mathrm{Tr}R)^3 - \frac{2}{3}(1 + \mathrm{Tr}R)\,I_2(K) + \frac{2}{3}I_3(K)\,, \tag{2.17}$$

since, of course, $\det U = 1$.

Now (2.15) and (2.17) yield a formula for computing U given R. Eq. (2.17) is not in its simplest form, and is hard to simplify directly. The fact that three distinct $U \in SU3$ correspond to given R, reflects itself in the fact that a cube root has to be taken in (2.17) to compute (2.17). This step cannot be avoided — it seems [4] that explicit parametrizations of $SU3$ always involve the solving of a cubic equation or what is roughly equivalent the extraction of a cube root.

To obtain a simpler expression for $\mathrm{Tr}\,U^{-1}$ and also to discuss how the inverse formula to (2.9) provides a parametrization of $SU3$ in the sense of MSW Section 3, we must extract more information from (2.13). Firstly, we see by taking its trace, that (2.13) yields

$$(\mathrm{Tr}\,U)(\mathrm{Tr}\,U^{-1}) = (1 + \mathrm{Tr}R)\,. \tag{2.18}$$

Secondly, we multiply (2.13) by itself to obtain

$$4U^2(\mathrm{Tr}\,U^{-1})^2 = \left(\frac{2}{3} + R_{lk}\,\lambda_l\lambda_k\right)\left(\frac{2}{3} + R_{pq}\,\lambda_p\lambda_q\right), \tag{2.19}$$

and hence

$$\mathrm{Tr}\,U^2(\mathrm{Tr}\,U^{-1})^2 = \frac{1}{3}(1 + \mathrm{Tr}R)^2 + \frac{1}{2}I_2(K)\,, \tag{2.20}$$

upon using Eqs. (2.2) and (2.1) of MSW to compute traces on the right. Eq. (2.18) and (2.20) yield a formula for $(\mathrm{Tr}\,U^{-1})$ much simpler than (2.17). To obtain it, we note that

$$\mathrm{Tr}\,U^{-1} = \frac{1}{2}(\mathrm{Tr}\,U)^2 - \frac{1}{2}\mathrm{Tr}\,U^2\,.$$

which can be proved by diagonalizing U. Hence

$$\begin{aligned}(\mathrm{Tr}\,U^{-1})^3 &= \frac{1}{2}(\mathrm{Tr}\,U)^2(\mathrm{Tr}\,U^{-1})^2 - \frac{1}{2}\mathrm{Tr}\,U^2(\mathrm{Tr}\,U^{-1})^2 \\ &= \frac{1}{3}(1 + \mathrm{Tr}R)^2 - \frac{1}{4}I_2(K)\,.\end{aligned} \tag{2.21}$$

[3] In Terms of the matrix notation of MSW, where we defined D_j, F_j by $(D_j)_{ik} = d_{ijk}$ and $(F_j)_{ik} = if_{ijk}$, we have $d_i = \mathrm{Tr}(D_iR)$, $if_i = \mathrm{Tr}(F_iR)$.

Now (2.13) and (2.21) yield the required formula inverse to (2.9)

$$U = \frac{\frac{1}{3}(1 + \mathrm{Tr} R) + \frac{1}{2}\lambda_i K_i}{\left[\frac{1}{3}(1 + \mathrm{Tr} R)^2 - \frac{1}{4} I_2(K)\right]^{1/3}}, \tag{2.22}$$

where

$$K_i = d_{ijk} R_{jk} + i f_{ijk} R_{jk} .$$

One might well ask, if (2.17) reduces to (2.21). The answer not surprisingly is that it does if one extracts all the information conveyed by (2.13). Because we need much of this information in section three, we give some of the details of the reduction of (2.17). From (2.13) and its hermitian conjugate

$$2 U^+ (\mathrm{Tr}\, U) = \frac{2}{3} + R_{pq} \lambda_q \lambda_p$$

by direct multiplication and use of $U U^+ = 1$, we obtain

$$4 \,\mathrm{Tr}\, U \,\mathrm{Tr}\, U^{-1} = \left(\frac{2}{3} + R_{lk} \lambda_l \lambda_k\right)\left(\frac{2}{3} + R_{pq} \lambda_q \lambda_p\right). \tag{2.23}$$

This can be rearranged as a matrix equation of the form $\alpha_0 + \alpha_i \lambda_i = 0$, from which $\alpha_0 = 0$ and $\alpha_i = 0$ follow. Taking the trace of each side of (2.23), whose left side is a multiple of the unit matrix, and using (2.18), we get the scalar equation implied by (2.23) in the form

$$3\,(1 + \mathrm{Tr} R) = \frac{1}{3}(1 + \mathrm{Tr} R)^2 + \frac{1}{2}[I_2(d) + I_2(f)]. \tag{2.24}$$

Multiplying by λ_i and taking traces yields the octet vector equation in the form[4]

$$\frac{4}{3}(1 + \mathrm{Tr} R)\, d_i + (d * d)_i + (f * f)_i = 0 . \tag{2.25}$$

To extract the scalar content of (2.25), we operate on it with octet vectors d_i, f_i, $(d * d)_i$, $(d * f)_i$ and $(f * f)_i$ in turn, and grind down terms like $(d * d)_i (d * f)_i$ using identities involving d_{ijk} and f_{ijk} tensors displayed in section two of MSW. After rather long but straightforward calculations, one derives[4]

$$\frac{4}{3}(1 + \mathrm{Tr} R)\, I_2(d) + \langle d^3 \rangle + \langle d f^2 \rangle = 0 , \tag{2.26a}$$

$$\langle d^2 f \rangle = \left[\frac{1}{6}(1 + \mathrm{Tr} R) - \frac{3}{2}\right] d_i f_i , \tag{2.26b}$$

$$\langle f^3 \rangle = -\frac{3}{2} \mathrm{Tr} R\, d_i f_i , \tag{2.26c}$$

$$\langle d f^2 \rangle - \langle d^3 \rangle = \left[\frac{1}{6}(1 + \mathrm{Tr} R) - \frac{3}{2}\right][I_2(f) - I_2(d)] , \tag{2.26d}$$

$$\frac{4}{3}(1 + \mathrm{Tr} R)\,[\langle d^3 \rangle + \langle d f^2 \rangle] + \frac{1}{3}[I_2(d) - I_2(f)]^2 + \frac{4}{3}(d_i f_i)^2 = 0 . \tag{2.26e}$$

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[4] We are again using notations of MSW, namely:

$$(a * b)_i = d_{ijk} a_j b_k, \quad \langle a b c \rangle = d_{ijk} a_i b_j c_k .$$

Eq. (2.24) having been used to simplify various of these results. The first four of these results allow

$$I_3(K) = \langle d^3 \rangle - 3\,\langle d f^2 \rangle + 3i\,\langle d^2 f \rangle - i\,\langle f^3 \rangle$$

to be simplified. Taking the real and imaginary parts of (2.17) separately for convenience, one finds now that these reduce to the corresponding parts of (2.21), easily in the case of the imaginary part, but not so easily and with use of (2.24) in the case of the real part.

Another consistency question that might well be asked is whether the norm of the complex number (2.21) reduces to $(1 + \mathrm{Tr}\,R)^{3/2}$ as (2.18) demands. With the aid of the consequence

$$\frac{4}{3}\,(d_i f_i)^2 = \left[\frac{4}{3}\,(1 + \mathrm{Tr}\,R)\right]^2 I_2(d) - \frac{1}{3}\,[I_2(f) - I_2(d)]^2\,, \qquad (2.26\mathrm{f})$$

of (2.26a) and (2.26e), one finds the desired answer to this question.

3. Parametrization of $SU3$ Using the Homomorphism

A parametrization of $SU3$ in terms of a single real octet vector a_i is achieved by writing $U \in SU3$ in the form

$$U = u_0 + u_i \lambda_i \qquad (3.1)$$

where

$$u_i = x a_i + y\,(a * a)_i \qquad (3.2)$$

and giving the complex quantities u_0, x and y as explicit functions of the invariants $I_2(a)$ and $I_3(a)$. We could indeed show that (2.22) achieves such a parametrization. Or, alternatively we can say that a rotation R of $R8$ which satisfies (2.12) yields, according to (2.21), a parametrization of $SU3$ of the type described by (3.1) and (3.2), if a is a real octet vector determined by R, and u_0, x and y are given explicitly as function of the two independent scalars that can be formed from R. In the latter context, three points need to be discussed:

i that R determines only two independent scalars,

ii that R determines suitable real octet vectors for use as 'a' in (3.2),

iii that u_0, x and y are indeed known as explicit functions of the scalars selected under i,

and we discuss them in turn.

Scalar quantities that can be built from R include the following

$$\begin{aligned} &\mathrm{Tr}\,R,\ \mathrm{Tr}\,R^2, \ldots, I_2(d),\ I_2(f),\ f_i d_i\,, \\ &I_3(d) = \langle d^3 \rangle,\ \langle d^2 f \rangle,\ \langle d f^2 \rangle,\ I_3(f) = \langle f^3 \rangle\,. \end{aligned} \qquad (3.3)$$

A possible choice of independent scalars is the pair $\mathrm{Tr}\,R$ and $[I_2(f) - I_2(d)]$ From (2.24), $[(I_2(d) + I_2(f))]$ and, hence, $I_2(d)$ and $I_2(f)$ can be expressed in terms of them. Then, by (2.26f) so also can $d_i f_i$, and, by (2.26a) to

(2.26d), so also can $\langle d^3 \rangle$, $\langle d^2 f \rangle$, $\langle d f^2 \rangle$ and $\langle f^3 \rangle$. An alternative choice retains $\mathrm{Tr} R$, and replaces $[I_2(f) - I_2(d)]$ by $\mathrm{Tr} R^2$, the justification stemming from the identity

$$I_2(f) - I_2(d) = \frac{1}{3}\,[8 - 3\,\mathrm{Tr} R^2 - (\mathrm{Tr} R)^2 + 10\,\mathrm{Tr} R]\,. \tag{3.4}$$

To prove (3.4), expand $I_2(f)$ and $I_2(d)$ separately using (2.10) and (2.24) of MSW, subtract and use (2.12).

To show that $f_i = f_{ijk} R_{jk}$ can play the role of 'a' in (3.2), it is necessary only to establish the existence of a result of the type

$$d_i = \alpha f_i + \beta (f * f)_i\,, \tag{3.5}$$

with α, β functions of the invariants of R. It is clear that operation on (3.5) with f_i and d_i leads to solution for α and β in terms of invariants.

Given the answers to questions i and ii, it is now apparent that (2.21) can be cast readily into the required form.

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