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On Gell-Mann's λ-Matrices, d- and f-Tensors, Octets, and Parametrizations of $SU(3)$

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Abstract. The algebra of $SU(3)$ is developed on the basis of the matrices λ_i of GELL-MANN, and identities involving the tensors d_{ijk} and f_{ijk} occurring in their multiplication law are derived. Octets and the tensor analysis of the adjoint group $SU(3)/Z(3)$ of $SU(3)$ are discussed. Various explicit parametrizations of $SU(3)$ are presented as generalizations of familiar $SU(2)$ results.

1. Introduction

The aim of this paper is to discuss the algebraic properties of

(I) The Gell-Mann matrices $\lambda_i, (i = 1, 2, \ldots, 8)$, [1], which play the role for $SU(3)$ that the Pauli matrices play for $SU(2)$,

(II) The Gell-Mann tensors f_{ijk} and d_{ijk}, [1], which enter the multiplication law

$$\lambda_i \lambda_j = \frac{2}{3} \delta_{ij} + (d_{ijk} + i f_{ijk}) \lambda_k \tag{1.1}$$

of the λ_i,

(III) *Octets or real octet vectors $a_i (i = 1, 2, \ldots, 8)$ which transform* according to the adjoint or octet representation of $SU(3)$,

(IV) Special unitary 3×3 matrices, i.e. elements of $SU(3)$.

The motivation for the paper stems for the need, [2], of results such as are derived, in theories of elementary particles, including current algebra, in which $SU(3)$ or chiral $SU(3) \times SU(3)$ is present as an underlying symmetry group. The paper itself is in part a review of existing knowledge, particularly under headings (I) and (II), and in part an exposition of new results.

It seems that TARJANNE [3] was the first to give identities amongst Gell-Mann d and f tensors. Also KAPLAN and RESNIKOFF [2] have considered such matters and the generalization to $SU(n)$. Our discussion of the λ_i, and of d and f tensors is given in section two, and its claimed merits, apart from various new identities, are as follows. Firstly, all results are presented in as symmetric a way as possible with a view to making manifest their entire content, how previous treatments fail to do this being explicitly indicated. Secondly, $SU(3)$ results are shown to separate into two classes: identities in one class being those with exact $SU(n)$ analogues, for all n, identities in the other being peculiar to $SU(3)$. We discuss the former, to some extent following KAPLAN and RESNIKOFF [2], at the $SU(n)$ level, so that $SU(3)$ results arise by putting $n = 3$. KAPLAN and RESNIKOFF [2] did not consider the second class of identities at all. When we consider such identities, we follow a general procedure which could indeed be applied to $SU(n)$ for any n, although results would no doubt be different in form for other n. This procedure involves the use of the characteristic equation of an arbitrary element $A = a_k \lambda_k$ of the self representation of the algebra of $SU(3)$; the important result (2.22) stems directly from use of this equation.

Our discussion under heading (III) is given in section three. It can alternatively be described as an introduction to the analysis of tensors which can be built out of octet vectors and which transform according to representations of the adjoint group $SU(3)/Z(3)$. This tensor analysis, in contrast to that used by OKUBO [4] in his discussion of $SU(3)$, is hardly discussed at all in the literature. Our discussion emphasizes such tensors, including invariants, as can be formed out of a single octet vector. Some aspects of our discussion reflect facts that are quite well-known in other formulations of $SU(3)$ theory. For example it is very familiar that, given a single octet vector, a_k say, one can build from it one and only one octet vector, $b_k = d_{ijk}\, a_i a_j$ say, linearly independent of it. What is less familiar but useful information is the explicit expression of other octet vectors as linear combinations of a_k and b_k.

In section four, we use the results of sections two and three to study the explicit parametrization of elements of $SU(3)$, i.e. special or unimodular unitary 3×3 matrices U. In other words we attempt to write such U explicitly in the form

$$U = u_0 + i\, u_k \lambda_k$$

where $u_k = \alpha a_k + \beta d_{ijk} a_j a_k$ for some real octet vector a_k, and u_0, α, and β are explicitly given functions of the invariants which can be built out of the vector a_k. Using the exponential and Cayley representations of unitary matrices we obtain two such explicit forms. In the light of experience with the corresponding problems for $SU(2)$ our results may well seem complicated. It is very probable that the nature of our results simply reflect the inherent complexity of the $SU(3)$ situation, but the possibility of finding a parametrization of more appealing appearance is not ruled out. A recent paper by CHANG and GURSEY [5] displays various parametrizations of $U \in SU(2)$ featuring Pauli in place of Gell-Mann matrices. Our discussion has been motivated in part by a desire to extend the work of these authors on chiral $SU(2) \times SU(2)$ to chiral $SU(3) \times SU(3)$.

2. Basic Results on d and f Tensors

We are here principally concerned with deriving identities involving the GELL-MANN [1] $SU(3)$ tensors d_{ijk} and f_{ijk}, which arise in the multiplication law (1.1) of the Gell-Mann matrices λ_i. The identities in question are of two distinct types, those which are special ($n = 3$) cases of results valid for $SU(n)$, and those which are specific to the $SU(3)$ situation. We discuss the former class of identities first at the $SU(n)$ level, in such a way that the required $SU(3)$ results can be read off directly.

We consider the algebra of $SU(n)$, which consists of all $n \times n$ traceless hermitian matrices, and choose as a basis a set of $N = n^2 - 1$ matrices V_i, $i = 1, \ldots, N$, such that

$$(V_i, V_j) = \mathrm{Tr}(V_i V_j) = 2\delta_{ij} . \tag{2.1}$$

The normalization fixed by (2.1) means that for $SU(2)$ the V_i are the Pauli matrices τ_i, and for $SU(3)$ the V_i are the Gell-Mann λ_i.

Since V_i, $i V_i$, I and $i I$ together span the space of all complex $n \times n$ matrices[1], it follows that we have a multiplication law of the type

$$V_i V_j = \frac{2}{n} \delta_{ij} + (d_{ijk} + i f_{ijk}) V_k \tag{2.2}$$

where (2.1) has been used to fix the coefficient of the identity. From (2.2) we obtain

$$[V_i, V_j] = 2i f_{ijk} V_k , \tag{2.3}$$

$$\{V_i, V_j\} = \frac{4}{n} \delta_{ij} + 2 d_{ijk} V_k , \tag{2.4}$$

and

$$4i f_{ijk} = \mathrm{Tr}[V_i, V_j] V_k , \tag{2.5}$$

$$4 d_{ijk} = \mathrm{Tr}\{V_i, V_j\} V_k . \tag{2.6}$$

It follows easily that f_{ijk} and d_{ijk} are respectively totally antisymmetric and totally symmetric in i, j and k, and, since $\mathrm{Tr}\, V_k = 0$ and $V_i V_i$ is a multiple of the identity, that

$$d_{iik} = 0 . \tag{2.7}$$

The associative property $(V_i V_j) V_k = V_i (V_j V_k)$ of matrix multiplication gives rise to various d, f identities. To obtain economically a minimal independent set of such identities, we use

$$[[V_i, V_j], V_k] + [[V_j, V_k], V_i] + [[V_k, V_i], V_j] = 0 ,$$

$$[\{V_i, V_j\}, V_k] + [\{V_j, V_k\}, V_i] + [\{V_k, V_i\}, V_j] = 0 ,$$

$$[V_k, [V_i, V_j]] = \{V_j, \{V_k, V_i\}\} - \{V_i, \{V_j, V_k\}\} ,$$

[1] This fact can be expressed by the useful identity

$$(V_i)_{\alpha\beta} (V_i)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{2}{n} \delta_{\alpha\beta}\delta_{\gamma\delta} .$$

and obtain

$$f_{ilm} f_{mjk} + f_{jlm} f_{imk} + f_{klm} f_{ijm} = 0 , \tag{2.8}$$

$$f_{ilm} d_{mjk} + f_{jlm} d_{imk} + f_{klm} d_{ijm} = 0 , \tag{2.9}$$

$$f_{ijm} f_{klm} = \frac{2}{n} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (d_{ikm} d_{jlm} - d_{jkm} d_{ilm}) . \tag{2.10}$$

Of these, the last is the generalization of the familiar $SU(2)$ result

$$\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} . \tag{2.11}$$

An alternative derivation of (2.8) and (2.9) is illuminating. Since (2.2) to (2.6) are invariant under the change

$$V_i \to U V_i U^{-1} \qquad U \in SU(3)$$

of basis, it follows that d_{ijk} and f_{ijk} are isotropic tensors. However, in the case of $U = 1 + i a_i V_i$ for real infinitesimal a_i, this invariance of (2.3) and (2.4) gives rise directly to (2.8) and (2.9), which are thus expressions of the isotropy of the f and d tensors.

From (2.3), it can be seen that the f_{ijk} are the structure constants of $SU(n)$, and (2.8) is, of course, a well-known general property of structure constants. Also, since $SU(n)$ is compact, a further general property of structure constants of compact semi-simple Lie groups[2] yields the result

$$f_{ijk} f_{ljk} = n \delta_{il} . \tag{2.12}$$

Now, from (2.10), we obtain

$$d_{ijk} d_{ljk} = \frac{n^2 - 4}{n} \delta_{il} , \tag{2.13}$$

and along with these last results, we have the obvious result

$$d_{ijk} f_{ljk} = 0 . \tag{2.14}$$

From here, we can proceed systematically to results involving threefold products of d and f tensors, and beyond. We shall however be content to record the following easily verifiable identities

$$f_{piq} f_{qjr} f_{rkp} = -\frac{n}{2} f_{ijk} , \tag{2.15}$$

$$d_{piq} f_{qjr} f_{rkp} = -\frac{n}{2} d_{ijk} , \tag{2.16}$$

$$d_{piq} d_{qjr} f_{rkp} = \left(\frac{n^2 - 4}{2n}\right) f_{ijk} , \tag{2.17}$$

$$d_{piq} d_{qjr} d_{rkp} = \left(\frac{n^2 - 12}{2n}\right) d_{ijk} . \tag{2.18}$$

[2] See, for example, G. RACAH [6]. The actual multiple of the Kronecker delta involved in (2.18) depends on the normalization (2.1) of the V_i.

We now specialise to $n = 3$ to obtain two important relations that are peculiar to this case. The method to be used, however, is of considerable generality and can be applied to $SU(n)$, for any n, to yield results which will correspond to those below but whose form will be specific to the value of n under consideration.

The method is based on the characteristic equation of a general element of the algebra of $SU(3)$. Writing such an element in the form

$$A = a_i \lambda_i ,$$

we obtain its determinant as

$$\begin{aligned} \det A &= \frac{1}{3!} \varepsilon_{\alpha\beta\gamma} \varepsilon_{\lambda\mu\nu} (a_i\lambda_i)_{\alpha\lambda} (a_j\lambda_j)_{\beta\mu} (a_k\lambda_k)_{\gamma\nu} \\ &= \frac{1}{3!} \mathrm{Tr}(\lambda_i\lambda_j\lambda_k + \lambda_i\lambda_k\lambda_j)\, a_i a_j a_k \\ &= \frac{2}{3} d_{ijk}\, a_i a_j a_k \end{aligned} \tag{2.19}$$

using a well-known expansion of $\varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\lambda\mu\nu}$. Similarly we obtain its characteristic equation as

$$A^3 - a_i a_i A - \frac{2}{3} d_{ijk}\, a_i a_j a_k = 0 . \tag{2.20}$$

Since the a_i are arbitrary we can equate to zero the coefficients of $a_i a_j a_k$, symmetrized with respect to i, j, k, to obtain[3]

$$\frac{1}{6} (\lambda_i\lambda_j\lambda_k + \text{five perms}) = \frac{1}{3} (\delta_{ij}\lambda_k + \delta_{jk}\lambda_i + \delta_{ki}\lambda_j) + \frac{2}{3} d_{ijk} . \tag{2.21}$$

We now use (2.2) and (2.8) to grind Eq. (2.21) into the form $K_{ijkl}\lambda_l = 0$, where K_{ijkl} is an $SU(3)$ tensor. The linear independence of the λ_i now implies that $K_{ijkl} = 0$; the real part of this equation is

$$d_{ilm} d_{mjk} + d_{jlm} d_{imk} + d_{klm} d_{ijm} = \frac{1}{3} (\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}); \tag{2.22}$$

the imaginary part just reproduces (2.9).

Note that the distribution of indices on the left-hand side of (2.22) is identical to that on the left-hand sides of (2.8) and (2.9). It is to be stressed that the remaining quantity of this nature,

$$d_{ilm} f_{mjk} + d_{jlm} f_{imk} + d_{klm} f_{ijm} ,$$

is one for which no simple result exists. That the quantity is nonvanishing can be checked explicitly; its symmetry properties under per-

[3] Results equivalent to (2.21) in other formulations of $SU(3)$ theory, but of less appealing appearance, have occurred in various important contexts. In connection with the mass formula, see Okubo [4]; in connection with the non leptonic decays of baryons, and Lee-Sugawara triangle relationship, see Okubo [7] and Dalitz [8]; in connection with the Ademollo-Gatto theorem and related matter, see Ademollo and Gatto [9], and Zakharov and Kobzarev [10].

mutations of i, j, k, l prevent it from having an expansion in terms of Kronecker deltas.

Equation (2.22) can now be used, in conjunction with (2.10), to obtain the relation

$$3 d_{ijk} d_{pqk} = \delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} - \delta_{ij}\delta_{pq} + f_{ipm} f_{jqm} + f_{iqm} f_{jpm}\,, \tag{2.23}$$

an expression for the product of two d-tensors analogous to (2.10) for the product of two f-tensors.

It seems fairly clear that no further identities for d and f tensors exist that are independent of those given above. We could obtain further results by considering more complex products; but we leave the matter here, having obtained results sufficient for our own immediate purposes and hopefully most others.

TARJANNE [3] and KAPLAN and RESNIKOFF [2] have expressed results in matrix notation with F_j and D_k defined by

$$(F_j)_{ik} = i\, f_{ijk}\,, \qquad (D_j)_{ik} = d_{ijk}\,. \tag{2.24}$$

Accordingly (2.12) to (2.14) translate into

$$\mathrm{Tr} F_i F_l = n\,\delta_{il}\,, \tag{2.25a}$$

$$\mathrm{Tr} F_i D_l = 0\,, \tag{2.25b}$$

$$\mathrm{Tr} D_i D_l = \frac{n^2 - 4}{n}\,\delta_{il} \tag{2.25c}$$

while (2.15) to (2.18) become

$$\mathrm{Tr} F_i F_j F_k = i\,\frac{n}{2}\, f_{ijk}\,, \tag{2.26a}$$

$$\mathrm{Tr} D_i F_j F_k = \frac{n}{2}\, d_{ijk}\,, \tag{2.26b}$$

$$\mathrm{Tr} D_i D_j F_k = i\,\frac{n^2 - 4}{2n}\, f_{ijk}\,, \tag{2.26c}$$

$$\mathrm{Tr} D_i D_j D_k = \frac{n^2 - 12}{2n}\, d_{ijk}\,. \tag{2.26d}$$

It would appear that considerable simplification has been achieved, but this is in fact not really so. For example, (2.26b) hardly makes manifest the entire content of (2.16), which also translates, using (2.24), into the identities

$$\begin{aligned} \frac{n}{2} D_i &= d_{ijk} F_j F_k\,, \\ \frac{n}{2} D_j &= D_p F_j F_p\,. \end{aligned} \tag{2.27}$$

If one translates (2.8) and (2.9) into matrix notation, obtaining

$$[F_i, F_j] = i\, f_{ijk} F_k\,, \tag{2.28}$$

$$[F_i, D_j] = i\, f_{ijk} D_k\,, \tag{2.29}$$

then, on the one hand, their symmetric appearance has been destroyed, but, on the other hand, (2.28) implies the well-known general result that the matrices F_j are the matrices of the adjoint representation of $SU(n)$, and (2.29) implies that the matrices D_j, related to the F_j by (2.27), also transform according to this representation. Finally, we note that the important identities (2.10) and (2.23) do not translate into matrix notation without the introduction of further independent matrices.

3. Tensor Analysis of the Octet Group

It is evident that tensor indices which take on eight values as do those of the d and f tensors are tensor indices associated with the adjoint group $SU(3)/Z(3)$ of $SU(3)$ rather than with $SU(3)$ itself. In other words, just as ordinary Cartesian tensors, with indices taking on three values, refer to $R(3) \equiv SU(2)/Z(2)$, and spinors, whose indices take on only two values, refer to $SU(2)$, so here we have tensors associated with $SU(3)/Z(3)$, which are in fact Cartesian tensors in eight real dimensions, since $SU(3)/Z(3)$ is isomorphic [11] to a subgroup of $R(8)$, and tensors associated with $SU(3)$. The tensor analysis based on the latter is well discussed in the literature [4], [12], whereas that based on the former is not, probably for lack of the algebraic tools. We wish now to employ the results of the last section to discuss briefly some aspects of the analysis of tensors belonging to the octet group $SU(3)/Z(3)$.

We begin by considering the "vectors" of the $SU(3)/Z(3)$ group, which are real eight-vectors, called octets, and which transform according to the adjoint representation of $SU(3)$. First of all consider the $SU(3)$ invariants and octets that can be formed if one has at one's disposal only a single octet vector a_i and the tensors d and f. It is of course well known that at most two linearly independent octets can be formed. We shall take these to be a_i itself and $(a * a)_i$:

$$(a * a)_i = d_{ijk}\, a_j a_k \,. \tag{3.1}$$

It is equally well known that at most two independent $SU(3)$ invariants can be formed, which we take to be

$$I_2(a) = a_i a_i \,, \tag{3.2}$$

$$I_3(a) = (a * a)_i\, a_i = d_{ijk}\, a_i a_j a_k \,. \tag{3.3}$$

In terms of the notation

$$\langle abc \rangle = d_{ijk}\, a_i b_j c_k \,, \tag{3.4}$$

we write $I_3(a) = \langle a^3 \rangle$ quite often. While the two statements made answer the questions of principle, it is clear that other octets and $SU(3)$ invariants can be written down. To exhibit explicitly how such quantities can be expressed in terms of the selected basis octets and $SU(3)$ invariants is in fact essential for practical purposes. For octets, we note

$$f_{ijk}\, a_j a_k = 0\,, \tag{3.5}$$

$$f_{ijk}\, a_j (a * a)_k = 0\,, \tag{3.6}$$

$$f_{ijk} (a * a)_j (a * a)_k = 0\,, \tag{3.7}$$

$$d_{ijk}\, a_j (a * a)_k = \frac{1}{3} I_2(a)\, a_i\,, \tag{3.8}$$

$$d_{ijk} (a * a)_j (a * a)_k = \frac{2}{3} I_3(a)\, a_i - \frac{1}{3} I_2(a)\, (a * a)_i\,. \tag{3.9}$$

Of these two are obvious, and three require straightforward use of identities given in section two. Turning to $SU(3)$ invariants, we use results of section two, along with (3.8) and (3.9), to derive

$$(a * a)_i (a * a)_i = \langle a\, a (a * a)\rangle = \frac{1}{3} [I_2(a)]^2\,, \tag{3.10}$$

$$\langle a (a * a)\, (a * a)\rangle = \frac{1}{3} I_2(a)\, I_3(a)\,, \tag{3.11}$$

$$\langle (a * a)^3 \rangle = \frac{2}{3} [I_3(a)]^2 - \frac{1}{9} [I_2(a)]^3\,. \tag{3.12}$$

It is to be noted that unless $I_3(a) = 0$ the octets a_i and $(a * a)_i$ are not orthogonal. For some purposes it is desirable to replace $(a * a)_i$ by O_i such that $a_i O_i = 0$, and it follows from (3.2) and (3.3) that the choice

$$O_i = I_3(a)\, a_i - I_2(a)\, (a * a)_i\,, \tag{3.13}$$

satisfies the orthogonality condition. We note that

$$O_i O_i = I_2(a) \left\{ \frac{1}{3} [I_2(a)]^3 - [I_3(a)]^2 \right\}. \tag{3.14}$$

Now both $O_i O_i$ and $I_2(a)$, being the norms-squared of real octet vectors, are positive definite; hence (3.14) implies[4]

$$\frac{1}{3} [I_2(a)]^3 - [I_3(a)]^2 \geqq O\,. \tag{3.15}$$

A further consequence of (3.5) to (3.7) that may be worth noting is as follows. In contexts in which one has a single real octet vector a_i, all 3×3 matrices are of the form $M = \alpha + [\beta a_i + \gamma (a * a)_i]\, \lambda_i$ where α, β, γ are functions of $I_2(a)$ and $I_3(a)$, and accordingly commutative. To see this, note $[M, M']$ involves only terms like $\beta\gamma'\, 2\, i f_{ijk}\, a_i (a * a)_j\, \lambda_k$ which vanishes in view of (3.6). We now go on to consider the general tensor representation of $SU(3)/Z(3)$. In general, a tensor $T_{ijk}\ldots$ will carry an irreducible representation of $SU(3)/Z(3)$ if and only if

1. its indices have irreducible permutation symmetry,
2. all contractions which can be formed using Kronecker deltas and d and f tensors, are zero.

[4] The fact that the cubic (2.20) necessarily has three real roots for any octet vector a_i likewise yields the result (3.15).

We do not attempt a general discussion, but illustrate using simple examples. Given a general tensor T_{ij} of second rank, we know, of course, on the basis of

$$8 \times 8 \to 1 + 8 + 8 + 10 + \overline{10} + 27 ,$$

what irreducible tensors can be built from its components. Explicitly we find, using the results of section two in the consideration of contractions, that the irreducible constituents of the symmetric part of T_{ij} are

$$T_{ii} , \qquad \text{(1 component)},$$
$$\frac{1}{2}(T_{ij} + T_{ji})\, d_{ijk} , \qquad \text{(8 components)},$$
$$\frac{1}{2}(T_{ij} + T_{ji}) - \frac{3}{5} d_{ijl}\, d_{lmk} \frac{1}{2}(T_{mk} + T_{km}) - \frac{1}{8} \delta_{ij}\, T_{pp} , \qquad \text{(27 components)},$$

while those of the antisymmetric part are

$$\frac{1}{2}(T_{ij} - T_{ji})\, f_{ijk} , \qquad \text{(8 components)},$$
$$\frac{1}{2}(T_{ij} - T_{ji}) - \frac{1}{3} f_{ijl}\, f_{lmk} \frac{1}{2}(T_{mk} - T_{km}) , \qquad \text{(20 components)} .$$

It is noteworthy that the last tensor cannot be split up into two parts (corresponding to the 10 and $\overline{10}$ in the antisymmetric part of 8×8) without discussion of a conjugation operation, a matter not taken up here. In the special situation wherein there is only a single octet vector at one's disposal, so that $T_{ij} = a_i a_j$, only the symmetric part exists and we have

$$\begin{aligned} &I_2(a) , && \text{(1 component)} , \\ &(a * a)_i , && \text{(8 components)} \\ &a_i a_j - \frac{3}{5} d_{ijk}(a * a)_k - \frac{1}{8} \delta_{ij} I_2(a) . && \end{aligned} \tag{3.16}$$

In further illustration, we consider a question which arose in the authors' study [2] of chiral $SU(3) \times SU(3)$ dynamics: how many independent symmetric second rank tensors can be built when only a single octet vector a_i is at one's disposal. It is not hard to convince oneself that the following are a minimal set of such tensors:

$$\begin{aligned} &\delta_{ij},\ d_{ijk} a_k,\ a_i a_j,\ d_{ijk}(a * a)_k , \\ &a_i (a * a)_j + a_j (a * a)_i,\ (a * a)_i (a * a)_j . \end{aligned} \tag{3.17}$$

Tensors such as

$$d_{ipk}\, d_{jqk}\, a_p a_q,\ f_{ipk}\, f_{jqk}\, a_p a_q ,$$

etc. can be expressed as linear combinations of those in the set (3.17), using identities given in section two. Similarly, one may show that a minimal set of antisymmetric rank two tensors built out of a single real octet vector is

$$f_{ijk}a_k,\ f_{ijk}(a*a)_k,\ a_i(a*a)_j - a_j(a*a)_i,\ d_{ipq}a_p f_{jqr}(a*a)_r\,. \tag{3.18}$$

4. Special Unitary 3 × 3 Matrices

We wish here to give generalizations to $SU(3)$ in terms of Gell-Mann matrices of certain familiar representations of elements of $SU(2)$ in terms of Pauli matrices.

It is well-known that any special unitary matrix U can be written in the form

$$U = e^{iA} \tag{4.1}$$

with A hermitian, to make U unitary, and traceless, to ensure det $U = 1$. In the 2×2 case, writing

$$A = \boldsymbol{a} \cdot \boldsymbol{\tau}\,, \tag{4.2}$$

and putting $\boldsymbol{a} = \theta\, \boldsymbol{n}$, where $\boldsymbol{n}^2 = 1$, so that $\boldsymbol{a}^2 = \theta^2$, one easily develops

$$U = \cos\theta/2 + i \sin\theta/2\, \boldsymbol{\tau} \cdot \boldsymbol{n}\,. \tag{4.3}$$

An alternative description of any unitary matrix is the Cayley or rational representation. This allows almost any unitary matrix U to be written in terms of a hermitian matrix B in the form

$$U = (1 + iB)\,(1 - iB)^{-1} \tag{4.4}$$

the two factors being commutative. In the two by two case, the corresponding U is unimodular if and only if B is traceless. In this case writing $B = \boldsymbol{b} \cdot \boldsymbol{\tau}$ one converts (4.4) into the form

$$U = \frac{1 - \boldsymbol{b}^2 + 2i\, \boldsymbol{b} \cdot \boldsymbol{\tau}}{1 + \boldsymbol{b}^2}\,. \tag{4.5}$$

At the $SU(2)$ level, relationships between different parametrizations are easily seen, and we may alternatively write any $U \in SU(2)$ in the form

$$U = c_0 + i\, \boldsymbol{c} \cdot \boldsymbol{\tau} \tag{4.6a}$$

in terms of real quantities c_0, $\boldsymbol{c}$ subject to

$$c_0^2 + \boldsymbol{c}^2 = 1\,. \tag{4.6b}$$

We wish here to consider the problem of parametrizing special unitary 3×3 matrices, i.e. elements of $SU(3)$. The problem is very much harder than the $SU(2)$ problem in view of the fact that the algebra of Gell-Mann matrices, involving d-tensors, is much more complicated than that of the Pauli matrices.

We discuss in turn the use of the exponential and Cayley representations of $U \in SU(3)$ and the (not completely successful) search for a result analogous to (4.6).

First consider the exponential form (4.1) with A traceless and written as

$$A = a_k \lambda_k \,. \tag{4.7}$$

We wish, in as close analogy as possible with (4.3), to express U in the form

$$U = u_0 + i\, u_k \lambda_k \tag{4.8}$$

where

$$u_k = x a_k + y (a * a)_k \tag{4.9}$$

with u_0, x and y given explicitly as functions of the invariants $I_2 = a_k a_k$, $I_3 = \langle a^3 \rangle$. A direct approach, based on manipulation of the series expansion of e^{iA} and use of the characteristic equation

$$A^3 - I_2 A - \frac{2}{3} I_3 = 0 \tag{4.10}$$

of A, quickly becomes unmanageable. An alternative approach consists of two steps, first, computation of $u_0 = \frac{1}{3} \operatorname{Tr} U$ as a function of I_2 and I_3, and second, calculation of x and y in terms of u_0. The first step is easy to perform but leads to a complicated result. Let $\varphi_\alpha (\alpha = 1, 2, 3)$ be the three real latent roots of A, i.e. solutions $\varphi = \varphi_\alpha$ of

$$\varphi^3 - I_2 \varphi - \frac{2}{3} I_3 = 0 \,, \tag{4.11}$$

given explicitly [13] in terms of I_2 and I_3 by means of

$$\varphi = 2 (I_2/3)^{1/2} \cos \frac{1}{3} (\chi + 2\pi\alpha) \,, \qquad \alpha = 1, 2, 3 \,, \tag{4.12a}$$

$$\cos \chi = \sqrt{3}\, I_3 (I_2)^{-3/2} \,. \tag{4.12b}$$

It then follows that

$$u_0 = \frac{1}{3} \operatorname{Tr} U = \frac{1}{3} \sum_\alpha e^{i\varphi_\alpha} \,. \tag{4.13}$$

To compute x and y of (4.9), we develop

$$\begin{aligned} u_k &= -\frac{1}{2} \operatorname{Tr} i \lambda_k U \\ &= -\frac{1}{2} \operatorname{Tr} \frac{\partial}{\partial a_k} (e^{iA}) \\ &= -\frac{1}{2} \frac{\partial}{\partial a_k} \operatorname{Tr} U \\ &= -\frac{1}{2} \sum_\alpha e^{i\varphi_\alpha} \frac{\partial \varphi_\alpha}{\partial a_k} \,, \end{aligned} \tag{4.14}$$

and using (4.7) get

$$\frac{\partial \varphi_\alpha}{\partial a_k} = \frac{2 (\varphi_\alpha a_k + d_{ijk} a_i a_j)}{3 \varphi_\alpha{}^2 - I_2} \,. \tag{4.15}$$

Hence the quantities x and y of (4.9) can be identified as

$$x = -\sum_\alpha \varphi_\alpha e^{i\varphi_\alpha}(3\varphi_\alpha^2 - I_2)^{-1}, \tag{4.16}$$

$$y = -\sum_\alpha e^{i\varphi_\alpha}(3\varphi_\alpha^2 - I_2)^{-1}, \tag{4.17}$$

and, in virtue of (4.12), the desired expression for U of form (4.1) with (4.7) has been obtained. While the result with u_0, x, y given as complex and complicated functions of I_2 and I_3 may seem disappointing as a generalization of (4.3) for $SU(2)$, it seems clear that there is no explicit parametrization of $SU(3)$ which will not involve the solution of a cubic (if not even a sixth order) equation.

Turning now to the rational or Cayley representation (4.4) of a 3×3 unitary matrix U, we do not assume B to be necessarily traceless but rather set

$$B = b_0 + \lambda_k b_k, \tag{4.18}$$

and see what the restriction $\det U = 1$ implies. If any 3×3 unitary matrix U is written in the form (4.8) with u_0, u_k in general complex, then directly one obtains

$$\det U = u_0^3 + u_0 u^2 - \frac{2}{3} i \langle u^3 \rangle. \tag{4.19}$$

Applying this to the consequence

$$\det(1 - iB) = \det(1 + iB)$$

of $\det U = 1$, we deduce that b_0 must be a function of the invariants I_2, I_3 which can be built out of b_k, which obeys[5]

$$b_0^3 = b_0(I_2 + 3) - \frac{2}{3} I_3, \tag{4.20}$$

and that

$$\det(1 - iB) = 1 + I_2 - 3b_0^2. \tag{4.21}$$

From (4.20), it follows that $\det U = 1$ and $b_0 = 0$ require $I_3 = 0$. In order to make practical use of the Cayley representation we need to be able to obtain $(1 - iB)^{-1}$ explicitly in the form

$$(1 - iB)^{-1} = c_0 + i c_k \lambda_k, \tag{4.22}$$

where

$$c_i = \alpha b_i + \beta d_{ijk} b_j b_k. \tag{4.23}$$

Directly one obtains

$$\alpha = (1 - i b_0)\,\Omega^{-1}, \tag{4.24}$$

$$\beta = i\Omega^{-1}, \tag{4.25}$$

$$c_0 = \left[\frac{2}{3}(1 - 3 i b_0) + \frac{1}{3}\Omega\right]\Omega^{-1}, \tag{4.26}$$

[5] The condition that (4.20) have three real roots can be directly shown to be satisfied, since the condition that $x^3 = I_2 x + \frac{2}{3} I_3$ have three real roots is satisfied.

where $\Omega = \det(1 - iB) = 1 + I_2 - 3b_0^2$. Now, since

$$U = 2(1 - iB)^{-1} - 1$$

it follows that

$$U = \frac{4}{3}(1 - 3ib_0)\,\Omega^{-1} - \frac{1}{3} + 2i\lambda_k \cdot [(1 - ib_0)\,b_k + i d_{ijk} b_i b_j]\,\Omega^{-1}\,. \tag{4.27}$$

Thus again we are lead to a parametrization of $SU(3)$ based on a single real vector b_k wherein scalar quantities involved are given explicitly in terms of the invariants formed from b_k only after a cubic equation has been solved. It would seem however that the Cayley approach leads to a more manageable final result than the exponential form. Result (4.27) shows clearly how restrictive on U is the condition $b_0 = 0$ in (4.14), since $b_0 = 0$ implies $\operatorname{Tr} U$ real.

Finally we consider the possibility of expressing a matrix $U \in SU(3)$ in the form

$$U = f_0 + ig_0 + i\lambda_k(f_k + ig_k)\,, \tag{4.28}$$

where

$$g_i = xf_i + y d_{ijk} f_j f_k \tag{4.29}$$

and expressing f_0, g_0, x and y in terms of the invariants J_2, J_3 which can be built out of f_k. To handle this problem, we write unitary U as

$$U = v_0 + iv_k\lambda_k \tag{4.30}$$

and, as for $(1 - iB)^{-1}$, find

$$U^{-1} = (\det U)^{-1}\left[v_0^2 + \frac{1}{3}v_kv_k - \lambda_k(iv_0v_k + d_{ijk}v_iv_j)\right].$$

Putting $\det U = 1$, and equating U^{-1} to

$$U^\dagger = v_0^* - iv_k^*\lambda_k$$

yields the equations

$$v_0^* = v_0^2 + \frac{1}{3}v_kv_k\,, \tag{4.31 a}$$

$$v_k^* = v_0v_k - i\,d_{ijk}\,v_i\,v_j \tag{4.31 b}$$

When we put $v_0 = f_0 + ig_0$, $v_k = f_k + ig_k$, with g_k given by (4.29), (4.31) yields in fact six equations for f_0, g_0, x, y in terms of $J_2 = f_kf_k$, $J_3 = \langle f^3\rangle$. Of these four are independent:

$$\begin{aligned} 1 &= f_0 - g_0x + \frac{2}{3}y\,J_2\,,\\ 0 &= -\,g_0y + 2x\,,\\ -\,x &= f_0x + g_0 + \frac{2}{3}y(x\,J_2 + y\,J_3)\,,\\ -\,g_0 &= 2f_0g_0 + \frac{2}{3}(xJ_2 + yJ_3)\,, \end{aligned} \tag{4.32}$$

and the other two are consequences thereof. No better result than the following was obtained: x and y are given by

$$x = g_0(1 + 3f_0)^{-1}, \quad y = 2(1 + 3f_0)^{-1} \tag{4.33}$$

in terms of f_0, g_0 which are related to J_2, J_3 by

$$(1 + 3f_0)(1 - f_0) = \frac{4}{3} J_2 - g_0^2, \tag{4.34a}$$

$$3g_0(1 + 3f_0) = -2\left[g_0^3 - J_2 g_0 + \frac{2}{3} J_3\right]. \tag{4.34b}$$

Elimination of f_0, g_0 from (4.34) leads to equations of sixth order for f_0 or g_0 (and hence for x, y). It is of course possible that a more attractive answer exists — various other possibilities have been considered. Eq. (4.33) provides a useful check on results for the exponential or Cayley representations.

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