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Semimodularity and the Logic of Quantum Mechanics*

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Abstract. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then the events form an orthomodular ortholattice $(\mathscr{E}, \leqq, ')$ and the operations, mappings from the set of states $\mathscr{S}$ into $\mathscr{S}$, form a Baer *-semigroup $(S_\Omega, \circ, *, ')$. Additional axioms are adopted which yield the existence of a homomorphism θ from $(S_\Omega, \circ, *, ')$ into the Baer *-semigroup $(S(\mathscr{E}), \circ, *, ')$ of residuated mappings of $(\mathscr{E}, \leqq, ')$ such that $x \in S_\Omega$ maps states while $\theta_x \in S(\mathscr{E})$ maps supports of states. If $(\mathscr{E}, \leqq, ')$ is atomic and there exists a correspondence between atoms and pure states, then the existence of θ provides the result: $(\mathscr{E}, \leqq, ')$ is semimodular if and only if every operation $x \in S_\Omega$ is a pure operation (maps pure states into pure states).

0. Introduction

The theory of orthomodular ortholattices provides the mathematical constructs for the quantum logic approach to the foundations of quantum physics. A role for the theory of Baer *-semigroup, a mathematical theory closely related to the theory of orthomodular ortholattices, was exhibited in [15]. The definitions and terminology introduced in [15] will be utilized in this paper without further explanation. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then $(\mathscr{E}, \leqq, ')$ is an orthomodular ortholattice and $(S_\Omega, \circ, *, ')$ is a Baer *-semigroup such that $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$ is an isomorphism of $(\mathscr{E}, \leqq, ')$ onto the orthomodular ortholattice $(P'(S_\Omega), \leqq, ')$ of closed projections in S_Ω. Each $x \in S_\Omega$ is a mapping, $x : \mathscr{D}_x \to \mathscr{R}_x$, with domain $\mathscr{D}_x$ and range $\mathscr{R}_x$ contained in $\mathscr{S}$.

The connection between the theories of orthomodular ortholattices and Baer *-semigroups includes the following: if $(L, \leqq, ')$ is any orthomodular ortholattice, then there exists a Baer *-semigroup $(S(L), \circ, *, ')$ where $S(L)$ is a set of mappings of L into L and there exists an injective mapping $j : L \to S(L)$. Section I is devoted to a discussion of $S(\mathscr{E})$ for the orthomodular ortholattice $(\mathscr{E}, \leqq, ')$. In particular, the relation of $(S(\mathscr{E}), \circ, *, ')$ to the Baer *-semigroup $(S_\Omega, \circ, *, ')$ of operations will be exhibited.

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One goal of the quantum logic approach to the foundations of quantum physics is to augment the axioms of an event-state structure to obtain an axiomatic characterization of von Neumann's Hilbert space model for quantum mechanics: an axiomatic characterization where each axiom has a plausible physical interpretation. The currently available "concrete representation theorems" (the identification of $(\mathscr{E}, \leqq, ')$ with a lattice of subspaces of a vector space) require the following hypothesis: $(\mathscr{E}, \leqq, ')$ is semimodular. The correspondence between $(S_\Omega, \circ, *, ')$ and $(S(\mathscr{E}), \circ, *, ')$ will be utilized to obtain a direct phenomenological interpretation of the semimodularity of $(\mathscr{E}, \leqq, ')$ in Section III. The setting for the investigation of semimodularity will be developed in Section II.

Definitions and theorems relating to orthomodular ortholattices and Baer *-semigroups were presented in the Appendix of [15]. Additional definitions and theorems concerning residuated mappings, atomicity, and semimodularity are included in an Appendix to this paper.

I. A Role for Residuated Mappings of $(\mathscr{E}, \leqq, ')$

The set $S(L)$ of residuated mappings of any orthomodular ortholattice $(L, \leqq, ')$ admits the structure of a Baer *-semigroup $(S(L), \circ, *, ')$; moreover, the orthomodular ortholattice $\big(P'(S(L)), \leqq, '\big)$ of closed projections in $S(L)$ is isomorphic to $(L, \leqq, ')$. Therefore, if $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then the residuated mappings of the orthomodular ortholattice $(\mathscr{E}, \leqq, ')$ form a Baer *-semigroup $(S(\mathscr{E}), \circ, *, ')$. The question arises whether this Baer *-semigroup is related to the Baer *-semigroup $(S_\Omega, \circ, *, ')$ of operations in a phenomenologically interpretable way. The answer to this question requires a review of the concept of support of a state.

Definition I.1. Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.

(a) If $\alpha \in \mathscr{S}$, then $\mathscr{E}_0(\alpha)$ and $\mathscr{E}_1(\alpha)$ are the subsets of $\mathscr{E}$ defined by

$$\mathscr{E}_0(\alpha) = \{p \in \mathscr{E}: P(p, \alpha) = 0\}$$
$$\mathscr{E}_1(\alpha) = \{p \in \mathscr{E}: P(p, \alpha) = 1\}.$$

(b) If $\alpha \in \mathscr{S}$, then $p \in \mathscr{E}$ is a *support* of α provided: for $q \in \mathscr{E}$, $P(q, \alpha) = 0$ if and only if $q \perp p$.

The validity of the following assertions is evident.

Theorem I.1. *Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.*

(a) *If $\alpha \in \mathscr{S}$ and $p \in \mathscr{E}$, then the following are equivalent:*

i) *p is a support of α,*

ii) $\mathscr{E}_0(\alpha) = \{q \in \mathscr{E}: q \perp p\}$,

iii) $\mathscr{E}_1(\alpha) = \{q \in \mathscr{E}: p \leqq q\}$,

iv) *p is the least element of the subset $\mathscr{E}_1(\alpha)$ of $\mathscr{E}$.*

(b) *If $\alpha \in \mathscr{S}$, then there exists at most one $p \in \mathscr{E}$ such that p is a support of α.*

Definition I.2. If $(\mathscr{E}, \mathscr{S}, P)$ is an event-state structure and $\alpha \in \mathscr{S}$, then the support of α, provided it exists, is denoted by p_α.

The investigation of the role of $S(\mathscr{E})$ will involve the adoption of another axiom to supplement the seven axioms for event-state structures presented in [15].

Axiom I.8. (a) If $\alpha \in \mathscr{S}$, then the support p_α of α exists.

(b) If $p \in \mathscr{E}$ and $p \neq 0$, then there exists an $\alpha \in \mathscr{S}$ such that p is the support of α.

Axiom I.8 is valid for von Neumann's Hilbert space model of quantum mechanics; indeed, this axiom is valid for a wide class of event-state structures [21].

Example I.1. (See Example I.1 of [15]). Let $(\mathscr{P}(H), \mathscr{S}, P)$ be the event-state structure for von Neumann's Hilbert space model of quantum mechanics. If $\alpha \in \mathscr{S}$, then the support of α is the operator-theoretic support projection of the density operator D_α corresponding to α, since Tr is a faithful normal trace on the von Neumann algebra $\mathscr{L}_c(H)$ of all continuous linear operators on H (see, for example, [4], [5] and [16]). The operator-theoretic support projection of D_α is the projection on the orthogonal complement of the null space of D_α; hence, in terms of the Baer *-semigroup $(\mathscr{L}_c(H), \circ, {}^*, {}')$ (see the Appendix of [15]), the support of α is $(D_\alpha)''$.

Theorem I.2. *Let $(\mathscr{E}, \mathscr{S}, P, \Omega)$ be an event-state-operation structure satisfying Axiom I.8. If $p \in \mathscr{E}$, $\alpha \in \mathscr{S}$, $P(p, \alpha) \neq 0$ and $\beta = \Omega_p \alpha$, then the support p_β of β and the support p_α of α satisfy*

$$p_\beta \leqq (p_\alpha \vee p') \wedge p .$$

Proof. p_α is the support of α and $p'_\alpha \perp p_\alpha$; hence, $P(p'_\alpha, \alpha) = 0$ by the defining property of support. Since $p'_\alpha \wedge p \leqq p_\alpha$, it follows that

$$0 \leqq P(p'_\alpha \wedge p, \alpha) \leqq P(p'_\alpha, \alpha) = 0$$

and, since $p'_\alpha \wedge p \leqq p$, Axiom II.6 of [15] implies

$$P(p'_\alpha \wedge p, \Omega_p \alpha) = \frac{P(p'_\alpha \wedge p, \alpha)}{P(p, \alpha)} ;$$

hence, $P(p'_\alpha \wedge p, \Omega_p \alpha) = 0$. Since $\beta = \Omega_p \alpha$, one has

$$P(p_\alpha \vee p', \beta) = 1 - P(p'_\alpha \wedge p, \beta) = 1 .$$

By assertion a) of Theorem I.1, $P(p_\alpha \vee p', \beta) = 1$ implies $p_\beta \leqq p_\alpha \vee p'$. Axiom II.2 of [15] asserts $P(p, \Omega_p \alpha) = 1$; hence, $P(p, \beta) = 1$, since $\beta = \Omega_p \alpha$, and $p_\beta \leqq p$ again by Theorem I.1. Therefore, $p_\beta \leqq p_\alpha \vee p'$ and $p_\beta \leqq p$; hence, $p_\beta \leqq (p_\alpha \vee p') \wedge p$. Q.E.D.

The assertion of Theorem I.2 may be expressed in terms of residuated mappings as follows: if $p \in \mathscr{E}$, $\alpha \in \mathscr{D}_p$ and $\beta = \Omega_p \alpha$, then $p_\beta \leqq \phi_p(p_\alpha)$, where ϕ_p is the following residuated mapping of $\mathscr{E}$ into $\mathscr{E}$:

$$\phi_p(q) = (q \vee p') \wedge p , \quad q \in \mathscr{E} .$$

The example of von Neumann's Hilbert space model of quantum mechanics and the case of a compatible logic provide stronger results than Theorem I.2.

Example I.2. For the event-state-operation structure $(\mathscr{P}(H), \mathscr{S}, P, \Omega)$ (see Example II.1 of [15]), Ω is defined as follows: if $P \in \mathscr{P}(H)$, $\alpha \in \mathscr{S}$, $P(P, \alpha) \neq 0$, and D_α is the density operator for α, then $\beta = \Omega_p \alpha$ is the state with density operator D_β,

$$D_\beta = \frac{P D_\alpha P}{Tr(D_\alpha P)} .$$

The supports of α and β are $P_\alpha = (D_\alpha)''$ and $P_\beta = (D_\beta)''$, respectively, in terms of the Baer *-semigroup $(\mathscr{L}_c(H), \circ, *, ')$. The support $(D_\beta)''$ coincides with $(PD_\alpha P)''$, since the positive number $(T_r(D_\alpha P))^{-1}$ is immaterial for supports. D_α is a positive operator on H; hence, there exists a $T_\alpha \in \mathscr{L}_c(H)$ such that $D_\alpha = T_\alpha^* T_\alpha$. Consequently, Theorem A.5 of the Appendix asserts

$$(D_\beta)'' = (PD_\alpha P)'' = ((D_\alpha)'' \vee P') \wedge P .$$

Therefore, the inequality of the conclusion of Theorem I.2 is replaced by an equality

$$P_\beta = (P_\alpha \vee P') \wedge P$$

for the special case of von Neumann's Hilbert space model for quantum physics.

Theorem I.3. *Let $(\mathscr{E}, \mathscr{S}, P, \Omega)$ be an event-state-operation structure such that*

i) *Axiom I.8 is satisfied, and*

ii) *if $p, q \in \mathscr{E}$, then $p \,\mathrm{C}\, q$.*

If $p \in \mathscr{E}$, $\alpha \in \mathscr{S}$, $P(p, \alpha) \neq 0$ and $\beta = \Omega_p \alpha$, then

$$p_\beta = p_\alpha \wedge p = (p_\alpha \vee p') \wedge p .$$

Proof. Because of hypothesis ii), the relation of orthogonality may be characterized as follows (see, for example, [14]): for $p, q \in \mathscr{E}$, $p \perp q$ if and only if $p \wedge q = 0$. If $q \in \mathscr{E}$, then $p \,\mathrm{C}\, q$ and

$$P(q, \beta) = P(q, \Omega_p \alpha) = \frac{P(q \wedge p, \alpha)}{P(p, \alpha)}$$

by Theorem IV.1 of [15]; hence, for $q \in \mathscr{E}$, $P(q, \beta) = 0$ if and only if $P(q \wedge p, \alpha) = 0$. $P(q \wedge p, \alpha) = 0$ if and only if $q \wedge p \perp p_\alpha$ by the definition of the support of α. $q \wedge p \perp p_\alpha$ if and only if $q \perp p \wedge p_\alpha$, since $(q \wedge p) \wedge p_\alpha = q \wedge (p \wedge p_\alpha)$ and $\perp$ has the above characterization. Consequently, for $q \in \mathscr{E}$, $P(q, \beta) = 0$ if and only if $q \perp p \wedge p_\alpha$. This, however, is the defining property for the support p_β of β; hence, $p_\beta = p_\alpha \wedge p$. Since $p \,\mathrm{C}\, p_\alpha$, the characteristic properties of the relation C (see Section III of [15]) imply $(p_\alpha \vee p') \wedge p = p_\alpha \wedge p$. Q.E.D.

The following axiom for event-state-operation structures is, therefore, motivated by the general result of Theorem I.2 and the results of the special cases of Example I.2 and Theorem I.3.

Axiom II.8. If $p \in \mathscr{E}$, $\alpha \in \mathscr{S}$, $P(p, \alpha) \neq 0$, $\beta = \Omega_p \alpha$, and the supports p_α and p_β of α and β exist, then

$$p_\beta = \phi_p(p_\alpha) = (p_\alpha \vee p') \wedge p .$$

Theorem I.4. *Let* $(\mathscr{E}, \mathscr{S}, P, \Omega)$ *be an event-state-operation structure satisfying Axioms I.8 and II.8 with* $x \in S_\Omega$, $p_1, p_2, \ldots, p_n \in \mathscr{E}$, *and* $x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n}$.

a) *If* $\alpha \in \mathscr{D}_x$, *then*

$$p_{x\alpha} = \phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n}(p_\alpha) .$$

b) $\alpha \in C\mathscr{D}_x$ *if and only if*

$$\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n}(p_\alpha) = 0 .$$

c) $\alpha \in C\mathscr{D}_x$ *if and only if*

$$p_\alpha \leqq (\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n})^*(1)' .$$

d) $q_x = (\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n})^*(1)'$.

Proof. Assertion a) follows from Axiom II.8 by induction on n. Let $p_{n+1} = 1$. The following characterization of $C\mathscr{D}_a$ was obtained in Section II of [15]: $\alpha \in C\mathscr{D}_x$ if and only if there exists an i, $1 \leqq i \leqq n$, such that

$$P(p_j, \Omega_{p_{j+1}} \circ \cdots \circ \Omega_{p_n} \circ \Omega_{p_{n+1}} \alpha) \neq 0$$

for $i \leqq j \leqq n$ and

$$P(p_i, \Omega_{p_{i+1}} \circ \cdots \circ \Omega_{p_n} \circ \Omega_{p_{n+1}} \alpha) = 0 .$$

Since the supports of the states involved in these two expressions may be determined by utilizing a), this characterization of $C\mathscr{D}_x$ yields the following: $\alpha \in C\mathscr{D}_x$ if and only of there exists an i, $1 \leqq i \leqq n$, such that

$$\phi_{p_{j+1}} \circ \cdots \circ \phi_{p_n} \circ \phi_{p_{n+1}}(p_\alpha) \perp p_j$$

for $i < j \leqq n$ and

$$\phi_{p_{i+1}} \circ \cdots \circ \phi_{p_n} \circ \phi_{p_{n+1}}(p_\alpha) \not\perp p_i .$$

For any $p, q \in \mathscr{E}$, $\phi_p(q) = 0$ if and only if $p \perp q$. Hence, the characterization of $C\mathscr{D}_x$ may be expressed as follows: $\alpha \in C\mathscr{D}_x$ if and only if $\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n}(p_\alpha) = 0$. Thus, assertion b) is established. For any $q \in \mathscr{E}$ and $\phi \in S(\mathscr{E})$, $\phi(q) = 0$ if and only if $q \leqq \phi^*(1)'$; hence, assertion c) follows immediately from b). Because of the properties of p_α, c) asserts

$$C\mathscr{D}_x = \{\alpha \in \mathscr{S} : P((\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n})^*(1)', \alpha) = 1\} .$$

Since q_x is the unique element of $\mathscr{E}$ such that

$$C\mathscr{D}_x = \mathscr{S}_1(q_x) = \{\alpha \in \mathscr{S} : P(q_x, \alpha) = 1\}$$

(see Definition II.3 of [15]), assertion d) is valid. Q.E.D.

Theorem I.5. *Let* $(\mathscr{E}, \mathscr{S}, P, \Omega)$ *be an event-state-operation structure satisfying Axioms I.8 and II.8.*

a) *The mapping* $\theta : S_\Omega \to S(\mathscr{E})$ $(x \in S_\Omega \to \theta_x \in S(\mathscr{E}))$ *defined as follows is well-defined: if* $x \in S_\Omega$, *then select* $p_1, p_2, \ldots, p_n \in \mathscr{E}$ *such that* $x = \Omega_{p_1} \circ \circ\, \Omega_{p_2} \circ \cdots \circ \Omega_{p_n}$ *and define* θ_x *by*

$$\theta_x = \phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n} .$$

b) θ *is a homomorphism of the Baer *-semigroup* $(S_\Omega, \circ, *, ')$ *of operations into the Baer *-semigroup* $(S(\mathscr{E}), \circ, *, ')$ *of residuated mappings of* $(\mathscr{E}, \leqq, ')$.

c) *If* $x \in S_\Omega$ *and* $\alpha \in \mathscr{D}_x$, *then*

$$p_{x\alpha} = \theta_x(p_\alpha) .$$

d) *If* $x \in S_\Omega$ *and* $\alpha \in \mathscr{S}$, *then the following are equivalent:*

i) $\alpha \notin \mathscr{D}_x$,

ii) $\theta_x(p_\alpha) = 0$,

iii) $p_\alpha \leqq \theta_x^*(1)'$.

Proof. $\theta : S_\Omega \to S(\mathscr{E})$ is well-defined provided: if $x \in S_\Omega, p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m \in \mathscr{E}$, and

$$x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n} = \Omega_{q_1} \circ \Omega_{q_2} \circ \cdots \circ \Omega_{q_m} ,$$

then

$$\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n} = \phi_{q_1} \circ \phi_{q_2} \circ \cdots \circ \phi_{q_m} ,$$

that is, if $p \in \mathscr{E}$, then

$$\text{(I)} \qquad \phi_{p_1} \circ \phi_{p_2} \circ \cdots \phi_{p_n}(p) = \phi_{q_1} \circ \phi_{q_2} \circ \cdots \phi_{q_m}(p) .$$

Both sides of (I) are equal to 0, if $p = 0$. If $p \in \mathscr{E}$ and $p \neq 0$, then there exists an $\alpha \in \mathscr{S}$ such that p is the support p_α of α, $p = p_\alpha$, by Axiom I.8. If $\alpha \in \mathscr{D}_x$, then both sides of (I) are equal to the support of $x\alpha$ by assertion a) of Theorem I.4. If $\alpha \notin \mathscr{D}_x$, then both sides of (I) are equal to 0 by assertion b) of Theorem I.4. Consequently, θ is well-defined.

θ obviously preserves $\circ$ and $*$,

$$\theta_{x \circ y} = \theta_x \circ \theta_y, \qquad \theta_{x^*} = (\theta_x)^*$$

for $x, y \in S_\Omega$. If $x \in S_\Omega, p_1, p_2, \ldots, p_n \in \mathscr{E}$ and $x = \Omega_{p_1} \circ \Omega_{p_2} \circ \cdots \circ \Omega_{p_n}$, then $x' = \Omega_{q_x}$ (see Definition II.3 of [15]). By the definition of $'$ for $(S(\mathscr{E}), \circ, *, ')$, $(\theta_x)' = \phi_\alpha$ where $q \in \mathscr{E}$ is given by

$$q = \theta_x^*(1)' = (\phi_{p_1} \circ \phi_{p_2} \circ \cdots \circ \phi_{p_n})^*(1)' .$$

d) of Theorem I.4 asserts that $q = q_x$; hence, $(\theta_x)' = \phi_{q_x}$. Since $x' = \Omega_{q_x}$, $(\theta_x)' = \theta_{x'}$ and θ preserves $'$.

Assertion c) is an immediate consequence of assertion a) of Theorem I.4 while assertion d) is an immediate consequence of b) and c) of Theorem I.4. Q.E.D.

Consequently, if $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure satisfying Axioms I.8 and II.8, then, for each element x of the Baer *-semigroup $(S_\Omega, \circ, *, ')$ of operations, there is an element θ_x of the Baer *-semigroup $(S(\mathscr{E}), \circ, *, ')$ of residuated mappings of $(\mathscr{E}, \leqq, ')$. x maps states while θ_x maps supports of states; specifically, if $\alpha \in \mathscr{D}_x$ then the support p_α of the state α is mapped into the support $p_{x\alpha}$ of the state $x\alpha$ by θ_x,

$$p_{x\alpha} = \theta_x(p_\alpha) .$$

II. Atoms and Pure States

The purpose of this section is to augment the axioms of an event-state structure to provide a setting for investigating the role of semi-modularity.

Definition II.1. Let $(\mathscr{E}, \mathscr{S}, P)$ be an event-state structure.

a) If $\alpha_1, \alpha_2, \ldots, \in \mathscr{S}$, $t_1, t_2, \ldots, \in [0, 1]$, and $\sum_i t_i = 1$, then the unique $\alpha \in \mathscr{S}$ such that

$$P(p, \alpha) = \sum_i t_i P(p, \alpha_i) ,$$

for all $p \in \mathscr{E}$ (see Axioms I.6 and I.7 of [15]) is denoted by

$$\alpha = \sum_i t_i \alpha_i$$

and called the *mixture* of $\alpha_1, \alpha_2, \ldots$ with respective weights $t_1, t_2, \ldots$

b) A state $\alpha \in \mathscr{S}$ is *pure* provided: if $t \in (0, 1)$, $\alpha_1, \alpha_2 \in \mathscr{S}$, and

$$\alpha = t\alpha_1 + (1 - t)\alpha_2 ,$$

then $\alpha_1 = \alpha_2$; otherwise, α is *mixed*. The set of all pure states is denoted by $\tilde{S}$.

c) The set of all atoms in $\mathscr{E}$ is denoted by $\tilde{\mathscr{E}}$.

For a general event-state structure, $\mathscr{E}$ is not atomic; indeed there may be no atoms in $\mathscr{E}$ (for the definitions of atom and atomic, see the Appendix). Furthermore, there may exist no pure states in $\mathscr{S}$. An additonal axiom is necessary [12].

Axiom I.9. a) If $p \in \mathscr{E}$ and $p \neq 0$, then there exists a pure state $\alpha \in \mathscr{S}$ such that $P(p, \alpha) = 1$.

b) $\alpha \in \mathscr{S}$ is a pure state if and only of there exists a $p \in \mathscr{E}$ such that, for $\beta \in \mathscr{S}$, $P(p, \beta) = 1$ is equivalent to $\beta = \alpha$.

If $p \in \mathscr{E}$ and $p \neq 0$, then there exists an $\alpha \in \mathscr{S}$ such that $P(p, \alpha) = 1$ by assertion f) of Theorem I.1 of [15]. Part a) of Axiom I.9 asserts that this state may be selected to be a pure state. Part b) of Axiom I.9 asserts that a state is pure if and only if it may be prepared (see Theorem III.1) and identified by observing a single event.

Theorem II.1. *If* $(\mathscr{E}, \mathscr{S}, P)$ *is an event-state structure satisfying Axiom I.8, then the following are equivalent statements:*

a) $(\mathscr{E}, \mathscr{S}, P)$ *satisfies Axiom I.9,*

b) $(\mathscr{E}, \mathscr{S}, P)$ *satisfies the following:*

  i) *if* $p \in \mathscr{E}$ *and* $p \neq 0$, *then there exists a pure state* $\alpha \in \mathscr{S}$ *such that* $P(p, \alpha) = 1$, *and*

  ii) $\alpha \in \mathscr{S}$ *is a pure state if and only if* p_α *is an atom and* α *is the unique state in* $\mathscr{S}$ *with support equal to* p_α; *and*

c) $(\mathscr{E}, \mathscr{S}, P)$ *satisfies the following:*

  i) $(\mathscr{E}, \leqq, ')$ *is atomic and*

  ii) *there exists a mapping* $p \to \alpha_p$ *of the set of atoms,* $\tilde{\mathscr{E}}$, *onto the set of pure states,* $\tilde{\mathscr{S}}$, *such that, for* $p \in \tilde{\mathscr{E}}$, α_p *is the unique state* $\alpha \in \mathscr{S}$ *with* $P(p, \alpha) = 1$.

Moreover, if Axiom I.9 is satisfied and $p \in \tilde{\mathscr{E}}$, *then* p *is the support of* α_p.

Proof. First, consider part b) of Axiom I.9. Suppose $\alpha \in \mathscr{S}$ and $p \in \mathscr{E}$ has the following property: for $\beta \in \mathscr{S}$, $P(p, \beta) = 1$ if and only if $\beta = \alpha$. Since $P(p, \alpha) = 1$, one has $p_\alpha \leqq 1$; consequently for $\beta \in \mathscr{S}$, $P(p_\alpha, \beta) = 1$ implies $P(p, \beta) = 1$ and, hence, $\beta = \alpha$. Conversely, if $\beta \in \mathscr{S}$ and $\beta = \alpha$, then $P(p_\alpha, \beta) = P(p_\alpha, \alpha) = 1$. Consequently, part b) of Axiom I.9 is equivalent to the following: $\alpha \in \mathscr{S}$ is pure if and only if, for $\beta \in \mathscr{S}$, $P(p_\alpha, \beta) = 1$ is equivalent to $\beta = \alpha$.

Suppose $\alpha \in \mathscr{S}$ and p_α satisfies the property: (I) for $\beta \in \mathscr{S}$, $P(p_\alpha, \beta) = 1$ is equivalent to $\beta = \alpha$. It is asserted first that p_α is an atom. Indeed, let $q \in \mathscr{E}$, $q \neq 0$, and $q \leqq p_\alpha$. By Axiom I.8, there exists a state $\beta \in \mathscr{S}$ such that the support p_β of β equals q; hence, $P(p_\alpha, \beta) = 1$, since $p_\beta = q \leqq p_\alpha$. Therefore, $\beta = \alpha$ and $q_\beta = p = p_\alpha$; consequently, p_α is, indeed, an atom. Let $\beta \in \mathscr{S}$ and suppose the support of β equals p_α. Then $P(p_\alpha, \beta) = P(p_\beta, \beta) = 1$ and $\beta = \alpha$. Therefore, if p_α satisfies (I), then p_α satisfies: (II) p_α is an atom and α is the unique state with support equal to p_α. Suppose now that p_α satisfies (II). If $\beta \in \mathscr{S}$ and $P(p_\alpha, \beta) = 1$, then $p_\beta \leqq p_\alpha$ and, hence, $p_\beta = p_\alpha$, since p_α is an atom; consequently, $\beta = \alpha$ since α is the unique state with support equal to p_α. Therefore, if p_α satisfies (II), then p_α satisfies (I).

Consequently, a) and b) are equivalent. Assume now the validity of b) for $(\mathscr{E}, \mathscr{S}, P)$. $\mathscr{E}$ is atomic provided: if $p \in \mathscr{E}$ and $p \neq 0$, then there exists an atom $q \leqq p$. If $p \in \mathscr{E}$ and $p \neq 0$, then there exists a pure state $\alpha \in \mathscr{S}$ such that $P(p, \alpha) = 1$; consequently, p_α is an atom such that $p\alpha \leqq p$. Therefore, $\mathscr{E}$ is atomic. If $p \in \tilde{\mathscr{E}}$, then select a pure state $\alpha \in \mathscr{S}$ such that $P(p, \alpha) = 1$. Since p is an atom and $p_\alpha \leqq p$, p coincides with the support of α; consequently, there exists exactly one $\alpha \in \mathscr{S}$ such that $P(p, \alpha) = 1$. Denote this α by α_p. The mapping $p \to \alpha_p$ from $\tilde{\mathscr{E}}$ into $\tilde{\mathscr{S}}$

is surjective; indeed, if $\alpha \in \mathscr{S}$ is pure, then $\alpha = \alpha_p$ where p is the support of α. Consequently, b) implies c).

Assume the validity of c). If $p \in \tilde{\mathscr{E}}$, then p is the support of α_p. Indeed, $P(p, \alpha_p) = 1$ implies $p_\alpha \leqq p$ and, hence, $p_\alpha = p$, since p is an atom. Suppose $p \in \mathscr{E}$ and $p \neq 0$. Since $\mathscr{E}$ is atomic, there exists an atom $q \in \tilde{\mathscr{E}}$ such that $q \leqq p$. The pure state α_q satisfies $P(p, \alpha_q) = 1$ since q is the support of α_q and $q \leqq p$. Therefore, c) implies b). Q.E.D.

Consequently, for an event-state structure $(\mathscr{E}, \mathscr{S}, P)$ satisfying Axiom I.8, Theorem II.1 asserts that Axiom I.9 is satisfied if and only if $\mathscr{E}$ is atomic and the restriction of the mapping $\alpha \to p_\alpha$ from $\mathscr{S}$ onto $\mathscr{E}$ to the set of pure states, $\tilde{\mathscr{S}}$, is a one-to-one mapping of $\tilde{\mathscr{S}}$ onto the set of atoms, $\tilde{\mathscr{E}}$, such that if $\alpha \in \tilde{\mathscr{S}}$ and $\beta \in \mathscr{S}$ with $p_\alpha = p_\beta$, then $\beta = \alpha$.

The event-state structure $(\mathscr{P}(H), \mathscr{S}, P)$ of von Neumann's Hilbert space model of quantum mechanics satisfies Axiom I.9.

Example II.1. The atoms of $\mathscr{P}(H)$ are the projections with one-dimensional range. A state $\alpha \in \mathscr{S}$ is pure if and only if the density operator D_α is a projection with one-dimensional range [10]; therefore, the support of a pure state α is the atom D_α.

III. Semimodularity and Pure Operations

The recent interest in the quantum logic approach to the foundations of quantum physics, at least partially, stems from the recognition of the desirability of an axiomatic characterization of von Neumann's Hilbert space model for quantum mechanics. Ideally, a criterion for the adoption of each axiom of this characterization would be the existence of a phenomenological interpretation of the axiom. The currently available "concrete representation theorems," the identification of $(\mathscr{E}, \leqq, ')$ with an appropriate lattice of subspaces of a vector space (although not necessarily a Hilbert space), depend upon hypothesizing the atomicity and semimodularity of $(\mathscr{E}, \leqq, ')$ (see [10], [11], [12], and [13]). The purpose of this section is to provide a direct phenomenological interpretation of the property of semimodularity when Axioms I.8 and I.9 are satisfied.

Although von Neumann's early papers on quantum mechanics involved the quantum logic approach (see [19] and [20], pp. 247—254), the first formalization of the quantum logic approach was contained in the classic work by BIRKHOFF and VON NEUMANN [2]. The hypothesis of modularity was imposed on the logic, $(\mathscr{E}, \leqq, ')$, of quantum mechanics in [2] despite the fact that $(\mathscr{P}(H), \leqq, ')$ is not modular for a separable infinite dimensional complex Hilbert space H. More recently, it has been recognized that the orthomodularity of $(\mathscr{E}, \leqq, ')$ is adequate to replace modularity for many purposes; indeed, the orthomodularity of $(\mathscr{E}, \leqq, ')$

is a consequence of the axioms for an event-state structure and possesses a phenomenological interpretation even when only the set of events is considered [13]. Definition A.4 and Theorems A.7 and A.8 of the Appendix indicate the interdependence of modularity, semimodularity, orthomodularity, and distributivity.

The semimodularity of $(\mathscr{E}, \leqq, ')$ will be discussed in terms of pure operations [9].

Definition III.1. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then $x \in S_\Omega$ is a *pure* operation provided: if $\alpha \in \mathscr{D}_x$ and α is a pure state, then $x\alpha$ is a pure state.

Theorem III.1. *Let $(\mathscr{E}, \mathscr{S}, P, \Omega)$ be an event-state-operation structure satisfying Axioms I.8 and I.9. If $p \in \mathscr{E}$ is an atom, $\alpha \in \mathscr{S}$ and $P(p, \alpha) \neq 0$, then $\Omega_p \alpha = \alpha_p$; hence, Ω_p is a pure operation.*

Proof. By Axiom II.3 of [15], $P(p, \Omega_p \alpha) = 1$; since p is an atom, $\Omega_p \alpha = \alpha_p$ by assertion c) of Theorem II.1. α_p is a pure state; hence, Ω_p is trivially a pure operation. Q.E.D.

A first indication of a connection between semimodularity and pure operations is contained in the following theorem.

Theorem III.2. *Let $(\mathscr{E}, \mathscr{S}, P, \Omega)$ be an event-state-operation structure satisfying Axioms I.8 and I.9. If $(\mathscr{E}, \leqq, ')$ is semimodular, then Ω_p is a pure operation for every $p \in \mathscr{E}$; hence, every $x \in S_\Omega$ is a pure operation.*

Proof. If $p \in \mathscr{E}$ and $\alpha \in \mathscr{D}_{\Omega_p}$, then $P(p, \alpha) \neq 0$. Consequently, by the definition of support, $p_\alpha \not\perp p$. The support p_β of $\beta = \Omega_p \alpha$ satisfies

$$p_\beta \leqq (p_\alpha \vee p') \wedge p$$

because of Theorem I.2. If, moreover, α is a pure state, then p_α is an atom since Axiom I.9 holds. Since $(E, \leqq, ')$ is atomic, orthomodular and semimodular, $(p_\alpha \vee p') \wedge p$ is an atom (see Theorem A.8 of the appendix). p_β is the support of a state; hence, $p_\beta \neq 0$. Therefore, the support p_β of $\beta = \Omega_p \alpha$ is an atom,

$$p_\beta = (p_\alpha \vee p') \wedge p \,.$$

Consequently, β is a pure state and Ω_p is a pure operation. If $x \in S_\Omega$, then there exist $p_1, p_2, \ldots, p_n \in \mathscr{E}$ such that

$$x\alpha = \Omega_{p_1}(\Omega_{p_2}(\ldots \Omega_{p_n} \alpha \ldots))$$

for all $\alpha \in \mathscr{D}_x$; therefore, if $\alpha \in \mathscr{D}_x$ is pure, then $x\alpha$ is pure. Q.E.D.

The crucial step in the proof of Theorem III.2 was noticing the fact that the semimodularity of the atomic, orthomodular ortholattice $(\mathscr{E}, \leqq, ')$ yields the following property of the residuated mapping ϕ_p: if $q \in \mathscr{E}$, $q \not\perp p$, and q is an atom, then

$$\phi_p(q) = (q \vee p') \wedge p$$

is an atom. Indeed, this is a characterization of the semimodularity of $(\mathscr{E}, \leqq, ')$ when $(\mathscr{E}, \leqq, ')$ is atomic. Consequently, if the connection between operations and residuated mappings is utilized, that is, if Axiom II.8 is assumed, then a characterization of semimodularity is obtained.

Theorem III.3. *If* $(\mathscr{E}, \mathscr{S}, P, \Omega)$ *is an event-state-operation structure satisfying Axioms I.8, I.9 and II.8, then the following are equivalent:*

a) $(\mathscr{E}, \leqq, ')$ *is semimodular,*

b) Ω_p *is a pure operation for every* $p \in \mathscr{E}$; *and*

c) *every* $x \in S_\Omega$ *is a pure operation.*

Proof. b) and c) are equivalent obviously and a) implies b) by Theorem III.2 (without the assumption of Axiom II.8). Assume Ω_p is a pure operation for every $p \in \mathscr{E}$. To prove $(\mathscr{E}, \leqq, ')$ is semimodular it suffies to prove (see Theorem A.8 of the Appendix): if $p \in \mathscr{E}$, $q \in \mathscr{E}$ and q is an atom with $q \not\perp p$, then

$$\phi_p(q) = (q \vee p') \wedge p$$

is an atom. Consider the state α_q corresponding to the atom q. Since $p \not\perp q$ and q is the support of α_q, $P(p, \alpha_q) \neq 0$ and α_q is in the domain of Ω_p; moreover, α_q is a pure state. Since Ω_p is a pure operation, $\beta = \Omega_p(\alpha_q)$ is a pure state and the support p_β is an atom. By Axiom II.8,

$$p_\beta = \phi_p(p_{\alpha_q}) = \phi p(q) = (q \vee p') \wedge p$$

and $\phi_p(q)$ is an atom. Q.E.D.

The rule for the phenomenological interpretation of operations presented in [15] provides a heuristic reason for asserting that Ω_p should be a pure operation for each $p \in \mathscr{E}$. The phenomenological characterization of the fact that a state $\alpha \in \mathscr{S}$ is pure is the following *indecomposability* of the corresponding ensemble. The ensemble of physical systems prepared by a state-preparation procedure corresponding to α can not be decomposed into two subensembles prepared by state-preparation procedures corresponding to two distinct states α_1 and α_2 with the following property: a system of the ensemble corresponding to α may be attributed to the ensemble corresponding to α_1 with probability t and to the ensemble corresponding to α_2 with probability $1 - t$, where $0 < t < 1$. The ensemble corresponding to $\Omega_p \alpha$ is constructed by selecting the systems of the ensemble corresponding to α for which the observation procedure for p indicates that the event p occurs. It is, therefore, plausible that this ensemble is indecomposable provided the ensemble corresponding to α is indecomposable.

IV. Summary

If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then the set S_Ω of operations admits the structure of a Baer *-semigroup, $(S_\Omega, \circ, *, ')$. Since $(\mathscr{E}, \leqq, ')$ is an orthomodular ortholattice, the set $S(\mathscr{E})$ of residuated mappings of $(\mathscr{E}, \leqq, ')$ also admits the structure of a Baer *-semigroup, $(S(\mathscr{E}), \circ, *, ')$. Theorem I.1 indicated a connection between these two Baer *-semigroups: if $p \in \beta$ and $\alpha \in \mathscr{S}$ with $P(p, \alpha) \neq 0$, then the supports p_β and p_α of $\beta = \Omega_p \alpha$ and α, respectively, satisfy

$$p_\beta \leqq \phi_p(p_\alpha) ,$$

where ϕ_p is the residuated mapping

$$\phi_p(q) = (q \vee p') \wedge p , \quad q \in \mathscr{E} .$$

Axiom II.8 replaces the above inequality

$$p_\beta \leqq \phi_p(p_\alpha)$$

by an equality

$$p_\beta = \phi_p(p_\alpha) .$$

Axiom II.8 is satisfied, for example, by a compatible logic (that is, $p \mathrel{C} q$ for every pair $p, q \in \mathscr{E}$) and by von Neumann's Hilbert space model for quantum mechanics. If Axiom II.8 is satisfied, the there exists a homomorphism $\theta : S_\Omega \to S(\mathscr{E})$ of the Baer *-semigroup $(S_\Omega, \circ, *, ')$ of operations into the Baer *-semigroup $(S(\mathscr{E}), \circ, *, ')$ of residuated mappings. The operation $x \in S_\Omega$ maps states while the residuated mapping θ_a maps supports of states, specifically, if $\alpha \in \mathscr{D}_x$, then

$$\theta_x(p_\alpha) = p_{x\alpha}$$

where p_α and $p_{x\alpha}$ are the supports of α and $x\alpha$, respectively.

When the logic $(\mathscr{E}, \leqq, ')$ is atomic and there exists a correspondence between pure states and atoms, Theorems III.2 and III.3 provide a connection between semimodularity of $(\mathscr{E}, \leqq, ')$ and pure operations (operations which map pure states into pure states). Indeed, if $(\mathscr{E}, \leqq, ')$ is semimodular, then every $x \in S_\Omega$ is a pure operation. When Axiom II.8 is imposed, $(\mathscr{E}, \leqq, ')$ is semimodular if and only if every $x \in S_\Omega$ is a pure operation.

Appendix

This Appendix is devoted to the exposition of facts about the residuated mappings of an orthomodular ortholattice and about the semimodularity of lattices, specifically, atomic orthomodular ortholattices, and of the proof of one theorem for general Baer *-semigroups.

A.I. Residuated Mappings

Some definitions and theorems from the theory of posets will be needed [1, 3].

Definition A.1. Let $(X, \leqq)$ be a poset.

a) If Y is a subset of X, then $x \in Y$ is a *least* (respectively, *greatest*) element for Y provided: $x \leqq y$ (respectively, $y \leqq x$) for every $y \in Y$. (There exists at most one least (respectively, greatest) element of Y.)

b) A mapping $\phi : X \to X$ is *isotone* provided: if $x, y \in X$ and $x \leqq y$, then $\phi(x) \leqq \phi(y)$.

c) A mapping $\phi : X \to X$ is *residuated* provided:

i) ϕ is isotone, and

ii) if $x \in X$, then the subset

$$\{y \in X : \phi(y) \leqq x\}$$

is nonempty and possesses a greatest element.

d) $S(X)$ denotes the set of all residuated mappings of X.

e) If $\phi : X \to X$ and $\psi : X \to X$, then the mapping $\phi \circ \psi : X \to X$ is defined by

$$(\phi \circ \psi)(x) = \phi(\psi(x))$$

for all $x \in X$.

f) If $\phi \in S(X)$, then the mapping $\phi^+ : X \to X$ is defined as follows: for $x \in X$, $\phi(x)$ is the greatest element of the subset

$$\{y \in X : \phi(y) \leqq x\}$$

of X.

Theorem A.1. *Let* $(X, \leqq)$ *be a poset.*

a) *An isotone mapping* $\phi : X \to X$ *is residuated if and only if there exists an isotone mapping* $\psi : X \to X$ *such that*

$$(\psi \circ \phi)(x) \geqq x$$

and

$$(\phi \circ \psi)(x) \leqq x$$

for all $x \in X$; *moreover, if* ϕ *is residuated, then* ψ *is uniquely determined,* $\psi = \phi^+$.

b) *If* $\phi, \psi \in S(X)$, *then* $\phi \circ \psi \in S(X)$ *and*

$$(\phi \circ \psi)^+ = \psi^+ \circ \phi^+ .$$

c) *Let* 0 *be a least element for* $(X, \leqq)$, $\phi \in S(X)$ *and* $x \in X$. $\phi(x) = 0$ *if and only* $x \leqq \phi^+(0)$.

d) *If* $(X, \leqq)$ *has least and greatest elements,* 0 *and* 1, *respectively, then the mappings* $0 : X \to X$ *and* $1 : X \to X$ *are residuated where*

$$0(x) = 0, \quad x \in X$$

$$1(x) = x, \quad x \in X ;$$

moreover, $(S(X), \circ)$ *is a semigroup with zero,* 0, *and unit,* 1.

If $(X, \leqq)$ is not only a poset but also has an orthocomplementation. then $(S(X), \circ)$ admits an involution [6].

Definition A.2. If $' : X \to X$ is an orthocomplementation of a poset $(X, \leqq)$ and $\phi \in S(X)$, then the mapping $\phi^* : X \to X$ is defined by

$$\phi^*(x) = \phi^+(x')', \quad x \in X .$$

Theorem A.2. *Let* $':X \to X$ *be an orthocomplementation of the poset* $(X, \leqq)$.

a) *If* $\alpha \in S(X)$ *and* $x \in X$, *then*

$$\phi^{+}(x) = \phi^{*}(x')'$$

and $\phi^{*}(x)$ *is the least element of the subset*

$$\{y' : \phi(y) \leqq x'\}$$

of X.

b) *If* $\phi \in S(X)$, *then* $\phi^{*} \in S(X)$; *moreover*, ϕ^{*} *is the unique isotone mapping* $\psi : X \to X$ *such that*

$$\psi(\phi(x')') \leqq x$$

and

$$\phi(\psi(x')') \leqq x$$

for all $x \in X$.

c) *Let* $(X, \leqq)$ *have least and greatest elements* 0 *and* 1, $\phi \in S(X)$, *and* $x \in X$. $\phi(x) = 0$ *if and only if* $x \perp \phi^{*}(1)$.

d) $*$ *is an involution for the semigroup* $(S(X), \circ)$.

If $(X, \leqq, ')$ is not only an orthoposet but also an orthomodular ortholattice, then $S(X)$ admits the structure of a Baer *-semigroup [6].

Definition A.3. Let $(L, \leqq, ')$ be an ortholattice.

a) If $p \in L$, then the mapping $\phi_p : L \to L$ is defined by

$$\phi_p(q) = (q \vee p') \wedge p, \qquad q \in L\,.$$

b) If $\phi \in S(L)$, then the mapping $\phi' \colon L \to L$ is defined by $\phi' = \phi_p$ where $p = \phi^{*}(1)'$.

Theorem A.3. *Let* $(L, \leqq, ')$ *be an orthomodular ortholattice.*

a) *If* $p \in L$, *then* $\phi_p \in S(L)$; *moreover*,

$$\phi_p^{*} = \phi_p \circ \phi_p = (\phi_p)'' = \phi_p\,.$$

b) *For* $p, q \in L$, $\phi_p(q) = 0$ *if and only if* $p \perp q$.

c) $(S(L), \circ, *, ')$ *is a Baer *-semigroup such that* $p \to \phi_p$ *is an isomorphism of the orthomodular ortholattice* $(L, \leqq, ')$ *onto the orthomodular ortholattice* $\left(P'(S(L)), \leqq, '\right)$ *of closed projections in* $S(L)$.

*A.II. Two Theorems on Baer *-semigroups*

The following theorem is from the general theory of Baer *-semigroups [6, 7, 8].

Theorem A.4. *If* $(S, \circ, *, ')$ *is a Baer *-semigroup, then*

a) *for* $x \in S$, $(x^{*} \circ x)'' = x''$,

b) *for* $x, y \in S$, $(x \circ y)'' = (x'' \circ y)''$,

c) *for* $e, f \in P'(S)$, $(e \circ f)'' = (e \vee f') \wedge f$.

The computation of the operator-theoretic support projection of PDP, where D is a positive operator and P is a projection on a Hilbert space H, is an application of the following theorem to the Baer *-semigroup $(\mathscr{L}_c(H), \circ, *, ')$ of continuous linear operators on H.

Theorem A.5. *If* $(S, \circ, *, ')$ *is a Baer *-semigroup,* $y \in S$, $e \in P'(X)$ *and* $x = y^* \circ y$, *then* $(e \circ x \circ e)'' = (x'' \vee e') \wedge e$.

Proof. First, one notes

$$e \circ x \circ e = (y \circ e)^* \circ (y \circ e)$$

and, by a) of Theorem A.4,

$$(e \circ x \circ e)'' = (y \circ e)'' .$$

Utilizing b) of Theorem A.4, one has

$$(y \circ e)'' = (y'' \circ e)'' ;$$

hence,

$$(e \circ x \circ e)'' = (y'' \circ e)'' .$$

Since $y'' \in P'(S)$ and $e \in P'(S)$, c) of Theorem A.4 asserts

$$(y'' \circ e)'' = (y'' \vee e') \wedge e .$$

Since $x = y^* \circ y$, another application of a) of Theorem A.4 yields

$$x'' = (y^* \circ y)'' = y'' ;$$

consequently,

$$(e \circ x \circ e)'' = (x'' \vee e') \wedge e ,$$

the assertion of the theorem. Q.E.D.

A.III. Semimodularity and Atomicity

The following definition and theorem indicate the interdependence of distributivity, orthomodularity, semimodularity, and modularity [1, 11].

Definition A.4. Let $(L, \leqq)$ be a lattice.

a) If $p, q, r \in L$, then (p, q, r) is a *distributive triple*, written $(p, q, r)D$, provided:

$$(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r) .$$

$(L, \leqq)$ is a *distributive* lattice provided: if $p, q, r \in L$, then $(p, q, r)D$.

b) If $q, r \in L$, then (q, r) is a *modular pair*, written $(q, r)M$, provided: if $p \in L$ and $p \leqq r$, then $(p, q, r)D$. $(L, \leqq)$ is a *modular lattice* provided: if $p, q \in L$, then $(p, q)M$.

c) $(L, \leqq)$ is *semimodular* provided: if $p, q \in L$ and $(p, q)M$, then $(q, p)M$.

Theorem A.6. *An ortholattice* $(L, \leqq, ')$ *is orthomodular if and only if every orthogonal pair is a modular pair, i.e.,* $p, q \in L$ *and* $p \perp q$ *implies* $(p, q)M$.

Consequently, every distributive lattice is modular and every modular ortholattice is orthomodular. It should be noted that the projection lattice of every von Neumann algebra is not only orthomodular but also semimodular [18]. Semimodularity admits a useful characterization in an atomic orthomodular ortholattice [17].

Definition A5. Let $(X, \leqq)$ be a poset with a least element 0.

a) An element $x \in X$ is an *atom* provided:

i) $x \neq 0$ and

ii) if $y \in X$ and $y \leqq x$, then either $y = 0$ or $y = x$.

b) $(X, \leqq)$ is *atomic* provided: if $x \in X$ and $x \neq 0$, then there exists an atom $y \in X$ such that $y \leqq x$.

Theorem A.7. *A necessary and sufficient condition for an atomic orthomodular ortholattice $(L, \leqq, ')$ to be semimodular is the following: if $p \in L$, $q \in L$ is an atom and q is not orthogonal to p, $q \not\perp p$, then $(q \vee p') \wedge p$ is an atom.*

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