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$S\widetilde{O}(n)$ — Symmetric Field Equations

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Abstract. Field equations satisfied by the irreducible realizations of any inhomogeneous pseudo-orthogonal group are derived. For those representations which are characterized by the vanishing of the invariants of the inhomogeneous group, the field equations are of first order, of the form $S_{AB}\, p_B\, \psi = p_A\, \psi$. The possibility of considering $SO(q_1, q_2)$ as a higher symmetry group is discussed briefly.

1. Introduction

One main problem in particle physics is to write down irreducible field equations, which are covariant under a prescribed combined symmetry group. This is equivalent to the determination of the irreducible realizations of this group. Amongst these equations, the first order equations are of particular interest. The combined symmetry group contains the dynamical symmetry group (DG) and the internal symmetry group (TG) as subgroups. DG is an inhomogeneous group, which defines the mass and the spin of the particle, and contains the Poincaré group (PG) as subgroup. TG is a homogeneous group, which defines other internal symmetries like isospin and hypercharge, e.g. $SU(3)$. The combined symmetry groups discussed in the literature, cf. Refs. [1, 2], have a special structure. First the homogeneous group is constructed, and then one of its subgroups (e.g. the Lorentz group) is extended to an inhomogenous group. The generators of the combined homogeneous subgroup are written in a DG $\times$ TG notation. The usual field equations for DG are then generalized in a manifestly covariant DG $\times$ TG form. This is a quite general and a very convenient way of obtaining the required field equations. The search for irreducible field equations reduces actually to the determination of the equations satisfied by the irreducible realizations of DG.

The traditional DG is the Poincaré group, $SO(3, 1)$. However, its irreducible realizations with nonvanishing mass satisfy second order equations [3, 4, 5]. PURSEY [6] constructed infinitely many sets of field equations among various irreducible realizations of PG, which describe a particle with definite mass and definite spin. McKERREL [7] did the same for vanishing mass. This implies actually the enlargement of DG. Such an enlargement is of current interest. In fact, FRONSDAL [8] and ROMAN et al. [9] have proposed the groups $SO(3, 2)$ and $SO(4, 1)$ as

DG, instead of PG. They have shown how a symmetry breaking mass formula can be obtained in terms of the invariants of the subgroup $SO(3, 1)$. Contrary to PG, certain realizations of the de Sitter group $SO(4, 1)$ satisfy first order field equations [10], which describe particles with definite mass and definite spin. FLATO and STERNHEIMER [11] and HALBWACHS [12] have proposed the conformal group $SO(4, 2)$ as a possible symmetry group. The noncompact group $S\tilde{O}(6)$ plays also the role of DG in the relativistic $S\tilde{U}(12)$ theory [1]. In fact, $S\tilde{U}(12)$ is locally isomorphic to $SU(3) \times S\tilde{U}(4)$, and $S\tilde{U}(4)$ is the covering group of $S\tilde{O}(6)$. It is, therefore, of interest to consider the inhomogeneous noncompact groups $SO(q_1, q_2)$, which contain PG as subgroup.

The purpose of the present paper is to derive the field equations satisfied by the irreducible realizations of any inhomogeneous orthogonal or pseudo-orthogonal group. In general, these equations are of the second degree in the momenta. These equations are mere generalizations of WIGNER's equations for PG [3, 4, 5]. However, if the invariants of the inhomogeneous group vanish, the corresponding realizations are found to satisfy first order field equations. These are the mere generalization of the corresponding equations for the de Sitter group, derived recently by the author [10]. STEPANOVSKII [13] had obtained similar equations for PG with vanishing mass. His derivation is somewhat different from ours. It is based on little group techniques, and is restricted to WIGNER's [3] unitary representations of PG, transforming according to the representation $D(s, 0)$ of the proper Lorentz group. Our derivation is more general. It proves that Stepanovskii's equations are valid for any other realization of the massless particles, transforming according to $D(s_1, s_2)$ of the proper Lorentz group.

The generalization of the first order field equations derived in the present paper in an $S\tilde{U}(12)$-symmetric way is straightforward, and will be published separately [14]. The inhomogeneous group $SO(q_1, q_2)$ is interesting in itself. In fact, it can be considered as a combined symmetry group, by setting some of its momenta equal to zero. This will be discussed briefly in the present paper. Besides, by a special choice of its momenta, one can obtain a theory of massless particles containing another intrinsic energy, which we call the "pseudo-mass". The smallest group affording this possibility is the conformal group $SO(4, 2)$. The implications of such an enlarged symmetry of the massless particles will be published separately [15].

2. The Group Invariants

The infinitesimal generators of the inhomogeneous pseudo-orthogonal group $SO(q_1, q_2)$ are the $n = q_1 + q_2$ generators p_A of the translation and the $n(n-1)/2$ generators $J_{AB} = -J_{BA}$ of the rotation. This group will be denoted hereafter by $iS\tilde{O}(n)$. For convenience, we use a euclidian metric $p_A p_A$ = invariant, such that q_1 components of p_A are hermitian and the other q_2 components are anti-hermitian. The corresponding Lie algebra is

$$\begin{aligned} i[J_{AB}, p_C] &= \delta_{BC} p_A - \delta_{AC} p_B, \quad [p_A, p_B] = 0, \\ i[J_{AB}, J_{CD}] &= \delta_{AD} J_{BC} + \delta_{BC} J_{AD} - \delta_{AC} J_{BD} - \delta_{BD} J_{AC}. \end{aligned} \tag{2.1}$$

The generators Σ_{AB} of the homogeneous subgroup $(hS\tilde{O}(n))$ satisfy the latter commutation relation among themselves, together with $[\Sigma_{AB}, p_C] = 0$. Its finite-dimensional representations are non-unitary, and are obtained from those of the compact group $hSO(n)$, $q_1 = n$ and $q_2 = 0$, by the well-known unitary tick. The generators Σ_{AB} are hermitian in these representations. The other unitary representations are all infinite-dimensional.

In order to construct the invariants of $hSO(n)$, we introduce the following completely anti-symmetric pseudo-tensors for $n \geqq 3$:

$$\Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k}} = (1/2)\, \Sigma_{BC}\, \Gamma^{(k-1)}_{BCA_1 A_2 \cdots A_{n-2k}}, \tag{2.2}$$

where

$$\Gamma^{(1)}_{A_1 A_2 \cdots A_{n-2}} = (1/2)\, \varepsilon_{A_1 A_2 \cdots A_{n-2} BC}\, \Sigma_{BC}. \tag{2.3}$$

$\varepsilon_{A_1 A_2 \cdots A_n}$ is the Levi-Civita symbol. For $n = 2N+1$ odd, $1 \leqq k \leqq N$ while for $n = 2N$ even, $1 \leqq k \leqq N-1$. By successive application of (2.2), we arrive at the following useful identity:

$$\begin{aligned} &\Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k-1} C}\, \Sigma_{BC} - ik\, \Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k-1} B} \\ &\qquad = \frac{(-1)^n}{k+1} \sum_{\{A\}} (-1)^P\, \delta_{BA_1}\, \Gamma^{(k+1)}_{A_2 A_3 \cdots A_{n-2k-1}}. \end{aligned} \tag{2.4}$$

The summation extends over all permutations of the A's, such that $(-1)^P = +1$ for an even permutation, and -1 for an odd permutation. (2.4) holds for $1 \leqq k \leqq N-1$ if $n = 2N+1$ and for $1 \leqq k \leqq N-2$ if $n = 2N$. In the latter case we have in addition:

$$\Gamma^{(N-1)}_{AB}\, \Sigma_{BC} = i(N-1)\, \Gamma^{(N-1)}_{AB} + \frac{1}{N}\, \delta_{AB}\, \Gamma^{(N)}. \tag{2.5}$$

Further, for $n = 2N+1$ we have

$$\Gamma^{(N)}_{A}\, \Sigma_{BA} = iN\, \Gamma^{(N)}_{B}. \tag{2.6}$$

With the help of these pseudo-tensors we construct the invariants of $SO(n)$:

$$D^{(k)} = \frac{1}{(n-2k)!}\, \Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k}}\, \Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k}}. \tag{2.7}$$

The summation extends over lower indices alone. In particular, $D^{(1)} = (1/2)\, \Sigma_{AB} \Sigma_{AB}$. These invariants are also invariants of $O(n)$. For $n = 2N+1$, $D^{(k)}$, $1 \leqq k \leqq N$, are all the invariants of $SO(n)$. On the other hand, for $n = 2N$, we have only $N-1$ invariants of the type

$D^{(k)}$, $1 \leqq k \leqq N-1$. The remaining invariant is a pseudo-scalar:

$$D^{(N)} = \Gamma^{(N)} = (1/2)\, \Sigma_{AB} \Gamma^{(N-1)}_{AB} \,. \tag{2.8}$$

For $iS\tilde{O}(n)$ we introduce, on substituting $J_{AB} = x_A p_B - x_B p_A + \Sigma_{AB}$,

$$\begin{aligned} \Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}} &= (1/2)\, J_{BC} \Pi^{(k-1)}_{BCA_1 A_2 \cdots A_{n-2k-1}} \\ &= (1/2)\, \Sigma_{BC} \Pi^{(k-1)}_{BCA_1 A_2 \cdots A_{n-2k-1}} \\ &= \Gamma^{(k)}_{A_1 A_2 \cdots A_{n-2k-1} B}\, p_B \,. \end{aligned} \tag{2.9}$$

Here

$$\begin{aligned} \Pi^{(1)}_{A_1 A_2 \cdots A_{n-3}} &= (1/2)\, \varepsilon_{A_1 A_2 \cdots A_{n-3} BCD} J_{BC} p_D \\ &= (1/6)\, \varepsilon_{A_1 A_2 \cdots A_{n-3} BCD} P_{BCD} \,, \end{aligned} \tag{2.10}$$

and

$$P_{ABC} = \Sigma_{AB} p_C + \Sigma_{BC} p_A + \Sigma_{CA} p_B \,. \tag{2.11}$$

The $N+1$ invariants of $iS\tilde{O}(n)$ are one pseudo-scalar

$$C^{(N)} = \begin{cases} \Pi^{(N)} = \Gamma^{(N)}_A p_A & \text{for} \quad n = 2N+1 \,, \\ \Pi^{(N-2)}_{A_1 A_2 A_3} P_{A_1 A_2 A_3} & \text{for} \quad n = 2N \,, \end{cases} \tag{2.12}$$

and N scalars:

$$\begin{aligned} &\Delta = p_A p_A \,, \\ &C^k = \Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}} \Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}} \qquad (1 \leqq k \leqq N-1) \,. \end{aligned} \tag{2.13}$$

Obviously all $C^{(k)}$ and Δ commute with J_{AB} and p_A.

3. Second Order Field Equations

Each irreducible representation of $iS\tilde{O}(n)$ is characterized by $N+1$ numbers $\Delta = dI$ and $C^{(k)} = c^{(k)} I$. Thus the carrier space ψ satisfies $N+1$ equations

$$\begin{aligned} \Delta \psi &= d\psi \,, \\ C^{(k)} \psi &= c^{(k)} \psi \,. \end{aligned} \tag{3.1}$$

The particular representation $d = c^{(k)} = 0$ satisfies first order equations, as we shall see in the next section. For n even, all of Eqs. (3.1) are of the second degree in p_A. On the other hand, for n odd, N of them are of the second degree, while $k = N$ is linear. For example, for $n = 3$ we have two equations

$$\begin{aligned} \Sigma \cdot \mathbf{p} \psi &= c\psi \,, \\ \mathbf{p}^2 \psi &= d\psi \,, \end{aligned} \tag{3.2}$$

where Σ is the spin vector. Hammer-Good's theory [16] of massless particles is a special case of Eqs. (3.2), with $d = c^2$ and $c = p_0$ is the energy of the particle. Eqs. (3.1) are the generalization of Wigner's equations [3] for the Poincaré group.

The physically interesting groups are those which contain $SO(3, 1)$ as subgroup. Thus we concentrate on the groups $SO(q_1, q_2)$ with $q_1 \geqq 3$ and $q_2 \geqq 1$. Let p_1, p_2 and p_3 be the hermitian cartesian components of the physical momentum $\mathbf{p}$. Let, further ,$p_4 = i p_0$, where p_0 is the hermitian energy operator of the particle. The mass m of the particle is given by

$$m^2 = p_0^2 - \mathbf{p}^2 = M^2 - d\,, \tag{3.3}$$

where

$$M^2 = \sum_{A=5}^{n} p_A^2\,. \tag{3.4}$$

M is invariant under the subgroup $SO(q_1 - 3, q_2 - 1)$. M^2 and m^2 are both real. The components p_A characterize the particular representation. In space-time, Eqs. (3.1) become field equations if $p_\mu \to -i\partial_\mu$, $\mu = 1,2,3,4$, and the other momentum components are kept constant:

$$p_{i+4} = m_i, \qquad 1 \leqq i \leqq n - 4\,. \tag{3.5}$$

$q_1 - 3$ of the m_i's are real and the other $q_2 - 1$ are imaginary. However, we can consider the symmetry group as an arbitrary pseudo-orthogonal group, such that the components m_i may be complex. If some of the m_i's are set equal to zero, the field equations become invariant under the corresponding subgroup. For example, if all $m_i = 0$ and $d \neq 0$, the field equations are invariant under $SO(q_1 - 3, q_2 - 1)$. This subgroup may serve as an internal symmetry subgroup. On the other hand, the non-vanishing of some of the "mass-components" m_i introduces more than one fundamental length in the theory. One could perhaps distinguish between strong and other interactions of the hadrons in this way. However, we shall not discuss this possibility here.

Since the mass of the particle is the physically interesting quantity, we classify the different theories according to the values of m^2 and M^2. The classification of the inequivalent theories reduces to that of the little group. The latter is the maximal symmetry subgroup which leaves a minimal number of p_A invariant, subject to the conditions $m^2 =$ cst. and $M^2 =$ cst. It depends on the signs of m^2 and M^2. For example, if m is real and $m, M \neq 0$; then we can choose all $p_A = 0$ except $p_4 = i\varepsilon m$ and $p_n = \eta M$, where $\varepsilon, \eta = \pm 1$. The little group is $SO(q_1 - 1, q_2 - 1)$ if M is real, and $SO(q_1, q_2 - 2)$ if M is pure imaginary. In the Table we give the different little groups of $SO(q_1, q_2)$ for different signs of m^2 and M^2. The original symmetry $S\tilde{O}(n)$ reduces to $S\tilde{O}(n-2)$ if m^2, $M^2 \neq 0$ or $m^2 = M^2 = 0$. On the other hand, if either m or M vanishes, but not both, the little group is $S\tilde{O}(n-1)$. The case $m = M = d = 0$ will be discussed in detail in the next section.

Table. *Different little groups of* $SO(q_1, q_2)$

sign m^2	sign M^2	nonvanishing p_A components	The little group $SO(q_1, -n_1, q_2 - n_2)$ n_1	n_2
$+$	$+$	$p_4 = i\,\varepsilon\, m, \quad p_n = \eta\, M$	1	1
$+$	$-$	$p_4 = i\,\varepsilon\, m, \quad p_n = i\,\eta\,\lvert M\rvert$	0	2
$-$	$+$	$p_1 = \varepsilon\,\lvert m\rvert, \quad p_n = \eta\, M$	2	0
$-$	$-$	$p_1 = \varepsilon\,\lvert m\rvert, \quad p_n = i\,\eta\,\lvert M\rvert$	1	1
0	$+$	$p_n = \eta\, M$	1	0
0	$-$	$p_n = i\,\eta\,\lvert M\rvert$	0	1
$+$	0	$p_4 = i\,\varepsilon\, m$	0	1
$-$	0	$p_1 = \varepsilon\,\lvert m\rvert$	1	0
0	0	$p_n = K, \quad p_{n-1} = i\,\eta\, K$	1	1

4. The Associated Lie Algebra

In order to construct the irreducible representations of $iS\tilde{O}(n)$, it is sufficient to study the Lie algebra generated by P_{ABC}, given by (2.11), over the field of complex numbers and p_A. In fact, P_{ABC} generate a finite-dimensional Lie algebra:

$$\begin{aligned} &i[P_{A'B'C'}, P_{ABC}] \\ &= \delta_{AA'}\chi_{BB'CC'} + \delta_{BB'}\chi_{CC'AA'} + \delta_{CC'}\chi_{AA'BB'} \\ &\quad + \delta_{AC'}\chi_{CB'BA'} + \delta_{CA'}\chi_{AB'BC'} + \delta_{AB'}\chi_{CA'BC} \\ &\quad + \delta_{BA'}\chi_{CB'AC'} + \delta_{CB'}\chi_{BA'AC'} + \delta_{BC'}\chi_{CA'AB'}\,, \end{aligned} \tag{4.1}$$

where

$$\chi_{AA'BB'} = P_{AA'B}\,p_{B'} + P_{BB'A}\,p_{A'}\,.$$

Introducing

$$P_A = -i\Sigma_{AB}\,p_B\,, \tag{4.2}$$

we find that

$$\begin{aligned} X_{AB} &\equiv -i P_{ABC}\,p_C \\ &= -i\Delta\Sigma_{AB} + p_A P_B - p_B P_A \\ &= [P_A, P_B]\,. \end{aligned} \tag{4.3}$$

Also

$$\begin{aligned} [X_{A'B'}, P_{ABC}] &= \delta_{AA'}F_{BB'C} + \delta_{BB'}F_{AA'C} + \delta_{BA'}F_{CB'A} \\ &\quad + \delta_{AB'}F_{CA'B} + \delta_{CA'}F_{AB'B} + \delta_{CB'}F_{BA'A}\,, \\ [X_{AB}, X_{A'B'}] &= \Delta\{\delta_{AA'}X_{BB'} + \delta_{BB'}X_{AA'} - \delta_{BA'}X_{AB'} \\ &\quad - \delta_{AB'}X_{BA'} - i p_A P_{BA'B'} + i p_B P_{AA'B'}\}\,, \end{aligned}$$

where

$$F_{ABC} = i p_B X_{CA} - \Delta P_{ABC}\,.$$

Further

$$[P_D, P_{ABC}] = \delta_{AD} X_{BC} + \delta_{CD} X_{BA} + \delta_{BD} X_{AC} - \Delta\{\delta_{AD}\Sigma_{CB} + \delta_{CD}\Sigma_{BA} + \delta_{BD}\Sigma_{AC}\},$$

$$[X_{AB}, P_C] = p_C X_{BA} + \Delta\{\delta_{AC} P_B - \delta_{BC} P_A - p_C \Sigma_{AB}\}.$$

If $\Delta \neq 0$, the Lie algebra $\{P_{ABC}\}$ is semi-simple. The corresponding realizations of $iS\tilde{O}(n)$ are the solutions of Eq. (3.1). On the other hand, if $\Delta = p_A p_A = 0$, then

$$\begin{aligned} &[X_{AB}, X_{CD}] = 0\,, \\ &i[P_{ABC}, X_{A'B'}] = (p_{A'}\delta_{AB'} - p_{B'}\delta_{AA'})\, X_{BC} \\ &\quad + (p_{A'}\delta_{BB'} - p_{B'}\delta_{BA'})\, X_{CA} + (p_{A'}\delta_{CB'} - p_{B'}\delta_{CA'}) X_{AB}\,, \\ &[P_A, P_B] = X_{AB} = p_A P_B - p_B P_A\,, \\ &i[P_D, P_{ABC}] = \delta_{AD} X_{CB} + \delta_{CD} X_{BA} + \delta_{BD} X_{AC}\,, \\ &[P_C, X_{AB}] = p_C X_{AB}\,. \end{aligned} \tag{4.4}$$

X_{AB} generate an abelian ideal, and the finite-dimensional Lie algebra $\{P_{ABC}\}$ is not semi-simple. For the finite-dimensional irreducible representations, $\{P_{ABC}\}$ should be semi-simple [17]. Thus these representations are characterized by $X_{AB} = 0$. Then

$$[P_A, P_B] = p_A P_B - p_B P_A = 0\,, \tag{4.5}$$

and

$$[P_{ABC}, P_D] = 0\,. \tag{4.6}$$

In fact, for all k,

$$[\Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}, P_B] = 0\,. \tag{4.7}$$

This is proved most easily by induction: From (2.9) one proves that, if (4.7) holds for k, it holds also for $k+1$; but (4.6) implies that (4.7) holds for $k = 1$. Hence it holds for all k. This proves that P_A belong to the centrum of the algebra. From (4.5) it follows that

$$P_A = -i\Sigma_{AB} p_B = \lambda p_A\,, \tag{4.8}$$

where λ is a characteristic number of the representation. We show in what follows that λ is the maximal (real positive) eigenvalue of Σ_{AB}. Eqs. (4.8) were obtained recently by the author [10] for the de Sitter group $SO(4, 1)$.

5. First Order Field Equations

Let ψ be the carrier vector space of the preceding representation. Then ψ satisfies

$$\Sigma_{AB} p_B \psi = i\lambda p_A \psi\,. \tag{5.1}$$

Since $\Delta = p_A p_A = 0$, these representations exist only for noncompact groups $i S\tilde{O}(n)$. In this case the mass is given by

$$m^2 = p_0^2 - \mathbf{p}^2 = \sum_{k=1}^{n-4} m_k^2 . \tag{5.2}$$

$m_k = p_{k+4}$ are the "mass-components" of the "mass-vector" in an euclidian (or pseudo-euclidian) $(n-4)$-dimensional space, transforming according to the subgroup $SO(q_1 - 3, q_2 - 1)$.

If $m^2 \neq 0$, we can choose $p_A = 0$ for all A except $p_n = m_{n-4} = m$ and $p_4 = i\varepsilon m$, where $\varepsilon = \pm 1$. Then (5.1) becomes

$$\begin{gathered} \Sigma_{n4}\psi = \varepsilon\lambda\psi , \\ (\Sigma_{4k} + i\varepsilon\Sigma_{kn})\psi = 0, \qquad k \neq 4, n . \end{gathered} \tag{5.3}$$

ψ is an eigenfunction of Σ_{n4} of eigenvalue $\varepsilon\lambda$. Since for the finite-dimensional representations of $hS\tilde{O}(n)$, Σ_{AB} is hermitian, and λ is real. For each triad n, 4, $k \neq n$, 4,

$$\Sigma_1 = \Sigma_{4k}, \qquad \Sigma_2 = \Sigma_{kn}, \qquad \Sigma_3 = \Sigma_{n4}$$

are generators of $SO(3)$. As is well-known from the theory of angular momentum, if $\Sigma_3\psi_{\varepsilon\lambda} = \varepsilon\lambda\psi_{\varepsilon\lambda}$, then $(\Sigma_1 + i\varepsilon\Sigma_2)\psi_{\varepsilon\lambda} = 0$ if and only if λ is the maximal eigenvalue of Σ_3. Eqs. (5.3) show that λ is the maximal eigenvalue of Σ_{n4} (for all $n \neq 4$). By arguments similar to those of Hepner [18] for $SO(5)$, it can be shown that for any $SO(n)$, all generators Σ_{AB} have the same eigenvalues. Hence λ is the maximal real positive eigenvalue of Σ_{AB} in the representation.

If $m = 0$, we define the real "pseudo-mass" K such that

$$K^2 = \sum_{k=1}^{q_2-1} |m_k|^2 = \sum_{k=q_2}^{n-4} m_k^2 , \tag{5.4}$$

where m_k is pure imaginary for $1 \leqq k \leqq q_2 - 1$, and real for $q_2 \leqq k \leqq n - 4$. In this case we can choose $p_{q_2+3} = m_{q_2-1} = i\varepsilon K$, $p_n = m_{n-4} = K$, and all other $p_A = 0$. We can prove in the same way that λ is the maximal real positive eigenvalue of Σ_{AB}. We note that, in any case, the little group is $SO(q_1 - 1, q_2 - 1)$. It is interesting to note that for theories of massless particles ($m = 0$) with a nonvanishing pseudo-mass K, we can define a rest-system in which the energy and the physical momentum of the particle both vanish.

We remark that Eqs. (5.1) are derivable from generalized Bargmann-Wigner equations in n dimensions. Let $\gamma_A^{(k)}$, $k = 1, 2, \ldots, 2\lambda$, constitute 2λ commutative Clifford algebras in n dimensions:

$$\begin{gathered} \gamma_A^{(k)}\gamma_B^{(k)} + \gamma_B^{(k)}\gamma_A^{(k)} = 2\delta_{AB} I , \\ \{\gamma_A^{(k)}, \gamma_B^{(j)}] = 0 \quad \text{for} \quad k \neq j , \end{gathered} \tag{5.5}$$

$A, B = 1, 2, \ldots, n$. As is well-known, this Clifford algebra (for each k) possesses one irreducible representation of 2^N dimensions, if $n = 2N$. If $n = 2N + 1$, there are two inequivalent representations $\pm\gamma_A$ of 2^N dimensions. The generalized Bargmann-Wigner equations are

$$\gamma_B^{(k)} p_B \psi = 0, \quad 1 \leqq k \leqq 2\lambda . \tag{5.6}$$

ψ may be constructed as the direct product

$$\psi = \prod_{k=1}^{2\lambda} \otimes \psi^{(k)} . \tag{5.7}$$

Here $\psi^{(k)}$ is the carrier vector space in 2^N dimensions, belonging to the irreducible representation of $\gamma_A^{(k)}$. It satisfies

$$\gamma_B^{(k)} p_B \psi^{(k)} = 0, \quad 1 \leqq k \leqq 2\lambda . \tag{5.8}$$

The matrices

$$\gamma_A = (1/2) \sum_{k=1}^{2\lambda} \gamma_A^{(k)} , \tag{5.9}$$

$$\Sigma_{AB} = -i [\gamma_A, \gamma_B] \tag{5.10}$$

$$= -(i/2) \sum_{k=1}^{2\lambda} (\gamma_A^{(k)} \gamma_B^{(k)} - \delta_{AB}) \tag{5.10}$$

afford a generally reducible representation of $SO(n+1)$ with a maximal eigenvalue λ. Σ_{AB} are also reducible representations of $SO(n)$. Multiplying (5.6) by $\gamma_A^{(k)}$, and summing over all $k \leqq 2\lambda$, we arrive at Eqs. (5.1). The irreducible representations of the latter equations are obtained by the full reduction of (5.10) under $SO(n)$.

In order to put (5.7) in the usual Bargmann-Wigner [19] form, take $p_A = 0$ for $A > 5$, and $p_5 = m$, and denote

$$\gamma_\mu^{(k)} = S_{\mu 5}^{(k)} = -(i/\lambda)\, \Sigma_{\mu 5}^{(k)} .$$

It is easily verified that the complete set of Eqs. (5.7) are obtained by multiplying the Bargmann-Wigner equations

$$(\gamma_\mu^{(k)} p_\mu + m) \psi = 0$$

by the different elements $\Sigma_{AB}^{(k)}$. The original Bargmann-Wigner equations were derived for the case $n = 5$, where $\gamma_\mu^{(k)}$ are four-dimensional Dirac matrices.

6. Observables and Auxiliary Equations

For convenience, we define $S_{AB} = -(i/\lambda)\, \Sigma_{AB}$ such that the field equations become

$$P_A \psi \equiv S_{AB} p_B \psi = p_A \psi , \tag{6.1}$$

where P_A is redefined from now on as $P_A = S_{AB}\, p_B$. Relations (4.5) and (4.7) are not satisfied automatically; they hold only when applied to the state vector ψ:

$$[P_A, P_B]\,\psi = 0\,, \tag{6.2}$$

$$[P_B, \Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}]\,\psi = 0\,. \tag{6.3}$$

The definition of an observable O should be modified. As usual, if ψ is a solution of (6.1), then also $O\psi$ is a solution of the same equations. O needs not commute with $p_A - P_A$; it suffices that

$$[p_A - P_A, O]\,\psi = 0\,. \tag{6.4}$$

Hence P_A and $\Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}$ are observables of the theory. Also the total angular momentum tensor J_{AB} is observable. In fact,

$$i\,[J_{AB}, p_C - P_C] = \delta_{BC}(p_A - P_A) - \delta_{AC}(p_B - P_B)\,,$$

such that

$$[J_{AB}, p_C - P_C]\,\psi = 0\,. \tag{6.5}$$

We derive now auxiliary equations satisfied by the observables $\Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}$, which are helpful in solving the field equations. Contracting (2.4) with p_B and using (6.1) we get

$$\begin{aligned}(-1)^n\, i(\lambda + k)\,(k+1)\,\Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}\,\psi \\ = \sum_{\{A\}} (-1)^P\, p_{A_1}\, \Gamma^{(k+1)}_{A_2 A_3 \cdots A_{n-2k-1}}\,\psi\,.\end{aligned} \tag{6.6}$$

These equations hold for $1 \leqq k \leqq N-1$ if $n = 2N+1$, and for $1 \leqq k \leqq N-2$ if $n = 2N$. Further, contracting (2.5) with p_C, using (6.1) we get

$$-iN(\lambda + N - 1)\,\Pi^{(N-1)}_A\,\psi = p_A\,\Gamma^{(N)}\,\psi \tag{6.7}$$

for $n = 2N$. Contracting (6.6) and (6.7) with the corresponding $\Pi^{(k)}_{A_1 A_2 \cdots A_{n-2k-1}}$ we arrive at

$$C^{(k)}\psi = 0, \quad 1 \leqq k \leqq N-1\,. \tag{6.8}$$

This is true for n odd or even. Further, contracting (2.6) with p_B, and using (6.1), we get for $n = 2N+1$

$$\Pi^{(N)}\,\psi = C^{(N)}\,\psi = 0\,. \tag{6.9}$$

Finally, for $n = 2N$, contracting (6.6) for $k = N-2$ with $P_{A_1 A_2 A_3}$, we obtain the similar equation (6.9). Hence, for the finite-dimensional representations with $\Delta = 0$, all invariants $C^{(k)} = 0$.

7. Generalized Stepanovskii Equations

In the case of the Poincaré group $SO(3, 1)$, Eqs. (6.1) become

$$P_\mu \psi \equiv S_{\mu\nu} p_\nu \psi = p_\mu \psi, \quad 1 \leqq \mu \leqq 4\,, \tag{7.1}$$

such that $p_\mu p_\mu \psi = 0$. These are the minimal linear field equations for massless particles. These equations hold for any irreducible representation $D(s_1, s_2)$ of the generators $S_{\mu\nu}$ of the homogeneous Lorentz group. In this case, there is only one auxiliary equation (6.7), which may be written in the form

$$\tilde{S}_{\mu\nu} p_\nu \psi = \gamma_5 p_\mu \psi \,. \tag{7.2}$$

Here $\tilde{S}_{\mu\nu} = (1/2)\, \varepsilon_{\mu\nu\alpha\beta} S_{\alpha\beta}$ is the dual of $S_{\mu\nu}$, and

$$\gamma_5 = \Sigma_{\mu\nu} \tilde{\Sigma}_{\mu\nu} / 4\lambda(\lambda + 1) \tag{7.3}$$

is the chirality operator. As is well-known [20], in the finite-dimensional representation $D(s_1, s_2)$, the spin s is restricted by the condition

$$|s_1 - s_2| \leqq s \leqq s_1 + s_2 \,. \tag{7.4}$$

λ is the maximal value of s in this representation. Thus

$$\lambda = s_1 + s_2 \,. \tag{7.5}$$

The two invariants of $D(s_1, s_2)$ are

$$\begin{aligned} \Sigma_{\mu\nu}\Sigma_{\mu\nu} &= 4\,[s_1(s_1 + 1) + s_2(s_2 + 1)] \,, \\ \Sigma_{\mu\nu}\tilde{\Sigma}_{\mu\nu} &= 4\,(s_1 - s_2)\,(s_1 + s_2 + 1) \,, \end{aligned} \tag{7.6}$$

where $\Sigma_{\mu\nu} = i\lambda S_{\mu\nu}$. Hence

$$\gamma_5 = (s_1 - s_2)/(s_1 + s_2) \,. \tag{7.7}$$

$2s_1$ and $2s_2$ are two nonnegative integers.

Denoting

$$\begin{aligned} S^{(s)}_{\mu\nu} &= \frac{1}{2}\,(S_{\mu\nu} + \tilde{S}_{\mu\nu}) = \tilde{S}^{(s)}_{\mu\nu} \,, \\ S^{(a)}_{\mu\nu} &= \frac{1}{2}\,(S_{\mu\nu} - \tilde{S}_{\mu\nu}) = -\tilde{S}^{(a)}_{\mu\nu} \,, \end{aligned} \tag{7.8}$$

Eqs. (7.1), (7.2) and (7.7) lead to

$$S^{(s)}_{\mu\nu} p_\nu \psi = \frac{s_1}{s_1 + s_2}\, p_\mu \psi \,, \tag{7.9}$$

$$S^{(a)}_{\mu\nu} p_\nu \psi = \frac{s_2}{s_1 + s_2}\, p_\mu \psi \,. \tag{7.10}$$

Each set of these equations is covariant under the proper Lorentz group. Under spatial inversion, the two sets transform into each other. For $s_1 = s$ and $s_2 = 0$, Eqs. (7.9) become just STEPANOVSKII's [13] equations. DIRAC [20] had derived the same equations from his spinor theory rather intuitively. Our Eqs. (7.9) and (7.10) are more general, and apply to all realizations of the Poincaré group with $m = 0$. The corresponding realizations for arbitrary m were studied by SHAW [5].

The decomposition [21]

$$\Sigma_{\mu\nu} = \Sigma^{(s)}_{\mu\nu} + \Sigma^{(a)}_{\mu\nu} \tag{7.11}$$

corresponds to the decomposition $0(4) = 0(3) \times 0(3)$. In fact, $[\Sigma^{(s)}_{\mu\nu}, \Sigma^{(a)}_{\alpha\beta}] = 0$. Let

$$\begin{aligned} \Sigma^{(s)}_{kl} &= -i s_1 \varepsilon_{k\,l\,n} \sigma^{(1)}_n, \qquad \Sigma^{(s)}_{4k} = i s_1 \sigma^{(1)}_k, \\ \Sigma^{(a)}_{kl} &= -i s_2 \varepsilon_{k\,l\,n} \sigma^{(2)}_n, \qquad \Sigma^{(a)}_{4\,k} = -i s_2 \sigma^{(2)}_k, \end{aligned} \tag{7.12}$$

where $k, l, n = 1, 2, 3$ (spatial components).

The total spin is

$$\boldsymbol{\Sigma} = s_1 \boldsymbol{\sigma}^{(1)} + s_2 \boldsymbol{\sigma}^{(2)}\,, \tag{7.13}$$

and

$$\begin{aligned} \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(1)} &= (s_1 + 1)/s_1\,, \\ \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{\sigma}^{(2)} &= (s_2 + 1)/s_2\,. \end{aligned} \tag{7.14}$$

With this notation, Eqs. (7.9) become

$$\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}\,\psi = p_0 \psi\,, \tag{7.15a}$$

$$(\boldsymbol{\sigma}^{(1)} p_0 + i\,\boldsymbol{\sigma}^{(1)} \wedge \mathbf{p})\,\psi = \mathbf{p}\,\psi\,, \tag{7.15b}$$

and

$$-\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}\,\psi = p_0 \psi\,, \tag{7.16a}$$

$$(-\boldsymbol{\sigma}^{(2)} p_0 + i\,\boldsymbol{\sigma}^{(2)} \wedge \mathbf{p})\,\psi = \mathbf{p}\,\psi\,. \tag{7.16b}$$

Eqs. (7.15a) and (7.16a) are HAMMER-GOOD's equations [16] with opposite chiralities. Eqs. (7.15b) and (7.16b) give the additional supplementary conditions, which ensure that $p_\mu p_\mu \psi = 0$. Actually, the Hammer-Good theory corresponds to the choice $s_1 = s$ and $s_2 = 0$. The general theory with $s_1, s_2 \neq 0$ is of rather theoretical significance. In fact, the two known massless particles, the neutrino and the photon correspond to $s_1 = s = 1/2$, 1 and $s_2 = 0$, or $s_1 = 0$ and $s_2 = s = 1/2$, 1. The two conjugate representations (under spatial inversion) $D(s, 0)$ and $D(0, s)$ may be used to define particles with opposite chiralities. In this case the original equations (6.1) should be used. For the neutrino this affords a possible unified description of the two neutrinos associated with the electron and the muon [22]. The similar possibility of two photons with opposite chiralities has been discussed recently [23].

8. One Mass-component Theories

If $p_5 = m$ and $p_A = 0$ for all $A > 5$, the dynamical symmetry group reduces to the inhomogeneous de Sitter group. This gives rise to an internal symmetry subgroup $SO(q_1 - 4, q_2 - 1)$, which commutes with the de Sitter group. Denoting

$$\begin{aligned} \gamma_\mu = S_{\mu 5}, \qquad \beta^{(k)}_\mu = S_{\mu,\,k+5}\,, \\ \beta^{(k)}_5 = S_{5,\,k+5}, \qquad 1 \leqq \mu \leqq 4, \qquad 1 \leqq k \leqq n-5\,, \end{aligned} \tag{8.1}$$

Eqs. (6.1) reduce to

$$P_\mu \psi \equiv (S_{\mu\nu} p_\nu + m\,\gamma_\mu)\psi = p_\mu \psi\,, \tag{8.2}$$

$$(\gamma_\mu p_\mu + m)\psi = 0\,, \tag{8.3}$$

$$(\beta_\mu^{(k)} p_\mu + m\,\beta_5^{(k)})\psi = 0\,. \tag{8.4}$$

Since P_{ABC}, given by (2.11), are observables of the dynamical equations, we see that

$$P_{\mu\nu\alpha} = i\,\varepsilon_{\mu\nu\alpha\beta} W_\beta \tag{8.5}$$

and

$$P_{5,k+5,l+5} = m\Sigma_{k+5,l+5}\,, \qquad (1 \leqq k, l \leqq n-5)\,, \tag{8.6}$$

are observables.

$$W_\mu = -(i/2)\,\varepsilon_{\mu\nu\alpha\beta} p_\nu \Sigma_{\alpha\beta} \tag{8.7}$$

is the Pauli-Lubanski pseudo-vector. The Lorentz invariant

$$W_\mu W_\mu = m^2 s(s+1) \tag{8.8}$$

characterizes the spin s of the particle [3]. $\Sigma_{k+5,l+5}$ are the generators of the internal symmetry subgroup $SO(q_1-4,\ q_2-1)$. Thus the spin and the internal symmetries are observables.

The field equations (8.2)—(8.4) reduce actually to (8.2) of the de Sitter group. In fact [10], (8.3) follows automatically from (8.2). We prove now that (8.4) is satisfied automatically by the irreducible solutions of (8.2). Decompose ψ into the direct sum of realizations of $SO(4, 1)$:

$$\psi = \sum_{\lambda_1,\lambda_2} \oplus\ \psi(\lambda_1, \lambda_2)\,. \tag{8.9}$$

$\lambda_1 \geqq \lambda_2 \geqq 0$ characterize the irreducible representation $R_5(\lambda_1, \lambda_2)$ of $SO(5)$. For the finite-dimensional representations, λ_1 and λ_2 are both integers (for bosons) or half-integers (for fermions) [24]. Each "de Sitter" component $\psi(\lambda_1, \lambda_2)$ satisfies (8.2). As is well-known from the theory of de Sitter-symmetric field equations [10], we should have $\lambda_1 = \lambda$. Further, $\psi(\lambda_1, \lambda_2)$ satisfies

$$W_\mu W_\mu \psi(\lambda_1, \lambda_2) = m^2 \lambda_2(\lambda_2 + 1)\,\psi(\lambda_1, \lambda_2)\,. \tag{8.10}$$

The spin of the particle is $s = \lambda_2$. The theory admits different (total) spin states $s \leqq \lambda$. For a definite spin s,

$$\psi(\lambda_1, \lambda_2) = \delta_{\lambda_1\lambda}\delta_{\lambda_2 s}\psi(\lambda, s)\,. \tag{8.11}$$

We are left with one nonvanishing de Sitter component for each spin s.

To prove that this solution satisfies (8.4) automatically, we turn to the rest-system $p_4 = i\varepsilon m$, $p_1 = p_2 = p_3 = 0$. Then (8.3) and (8.4) become, for any $k' = k + 5$

$$\Sigma_{54}\psi = \varepsilon\lambda\psi\,, \tag{8.12}$$

$$(\Sigma_{4k'} + i\varepsilon\Sigma_{k'5})\psi = 0\,. \tag{8.13}$$

As argued in Section 5, (8.13) follows from (8.12) automatically. Hence (8.4) is satisfied by the solution (8.11).

Since the reduced dynamical symmetry is $SO(4, 1)$, we can introduce a mass-formula consistent with the dynamical equations of the form

$$M = A + BD^{(1)} + CD^{(2)} . \tag{8.14}$$

Here A, B and C are functions of the internal symmetry subgroup, and $D^{(1)}$ and $D^{(2)}$ are the two invariants of $SO(4, 1)$ (compare (2.7)). In the finite-dimensional irreducible representation $R_5(\lambda_1, \lambda_2)$, we have [25]

$$D^{(1)} = (1/2)\Sigma_{ab}\Sigma_{ab} = \lambda_1(\lambda_1 + 3) + \lambda_2(\lambda_2 + 1) , \tag{8.15}$$

and [26]

$$D^{(2)} = \Gamma_a^{(2)}\Gamma_a^{(2)} = 4\lambda_2(\lambda_2 + 1)(\lambda_1 + 1)(\lambda_1 + 2) , \tag{8.16}$$

where the summation extends over five dimensions. Since for the solution (8.11), $\lambda_1 = \lambda$ is a characteristic of the representations of the whole group $SO(q_1, q_2)$, and $\lambda_2 = s$ is the spin, we get

$$M\psi = \{A' + B's(s + 1)\}\psi . \tag{8.17}$$

Here A' and B' depend only on the internal symmetries and the invariants of the whole group. Whether $M = m$ or m^2 remains arbitrary. It is interesting that the dynamical symmetry leads unambiguously to the spin dependence $s(s + 1)$, irrespective of the type of the internal symmetries. This is just the spin dependence of the $SU(6)$-symmetry [27].

We note that for the group $SO(4, 4)$, the maximal internal symmetry subgroup is $SO(3)$. It can be related to the isospin group $SU(2)$. However, its generators have the same eigenvalues as the dynamical spin Σ_{ab}. This would imply that bosons are isobosons and fermions are isofermions. This is obviously an unphysical restriction.

The $SU(3)$ internal symmetry may be embedded into $SO(6)$, as demonstrated by Halbwachs [12] for $SO(4, 2)$. In fact, $SU(3) \subset SU(4)$ and $SO(6) \sim SU(4)/Z_2$. The group $SO(6)$ can be taken now unambiguously as an internal symmetry subgroup of an inhomogeneous $SO(q_1, q_2)$ group. We stress that this combined symmetry scheme is different from that proposed by Halbwachs [12]. In fact, Halbwachs defines the hyperchange and the isospin in terms of the dynamical spin, while our $SO(6)$ commutes with the whole dynamical group. The smallest group of this type is $SO(10, 1)$. However, in order to establish the connection with the $SU(6)$-symmetry, it is more tempting to try $SO(10, 2)$ or $SO(11, 1)$. The dynamical subgroup may be taken as $SO(4, 2)$ or $SO(5, 1)$. The reduction of this dynamical symmetry into the de Sitter symmetry would follow if we assume a mass formula of the form (8.14). Similarly, the $SU(4)$ internal symmetry may be broken in an $SU(3)$ symmetric way.

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