Commun. math. Phys. 8, 147—163 (1968)

# On the Borel Structure of $C^*$-Algebras

(With an Appendix by R. V. KADISON)

E. B. DAVIES
Mathematical Institute, University of Oxford

Received August 20, 1967

**Abstract.** We provide a method of embedding a $C^*$-algebra $\mathscr{A}$ in a $C^*$-algebra $\mathscr{A}^\sim$ called its $\sigma$-envelope. $\mathscr{A}^\sim$ is contained in the enveloping algebra of $\mathscr{A}$ but is generally much smaller, and if $\mathscr{A}$ is commutative with identity then $\mathscr{A}^\sim$ can be identified with the algebra of bounded Baire functions on the spectrum of $\mathscr{A}$. The main result is to completely determine the structure of $\mathscr{A}^\sim$ for all separable G. C. R. algebras $\mathscr{A}$. This provides a good basis for a non-commutative theory of probability.

## 1. Introduction

We obtain a canonical procedure for embedding a $C^*$-algebra $\mathscr{A}$ in a $C^*$-algebra $\mathscr{A}^\sim$ which has the property that every self-adjoint element of $\mathscr{A}^\sim$ has a spectral decomposition in $\mathscr{A}^\sim$. The algebra $\mathscr{A}^\sim$ is a subalgebra of the enveloping algebra $\mathscr{A}^{**}$ and in the case where $\mathscr{A}$ is a commutative $C^*$-algebra with identity, $\mathscr{A}^\sim$ can be identified with the $C^*$-algebra of all bounded Baire functions on the spectrum of $\mathscr{A}$. In the general case our work can be regarded as providing a basis for a non-commutative version of measure theory.

We undertake a close analysis of the structure of the algebra $\mathscr{A}^\sim$ and show that it is closely related to the Borel structures of the spectrum $\hat{\mathscr{A}}$ of $\mathscr{A}$. In the case where $\mathscr{A}$ is a separable G.C.R. algebra we can explicitly write down the structure of $\mathscr{A}^\sim$ (Theorem 4.5). This provides us with a non-commutative generalization of the idea of a standard Borel space [9]. As a particular application we analyse the space of finite positive traces on a separable G.C.R. algebra.

If $\mathscr{A}$ is a separable G.C.R. algebra, the set $\mathscr{P}$ of projections in $\mathscr{A}^\sim$ forms a $\sigma$-complete orthocomplemented lattice. In a further paper we shall show how this observation allows us to relate our theory to Mackey's formulation of quantum mechanics [10], by letting $\mathscr{P}$ be the partially ordered set of questions in some quantum mechanical system. Slightly different work along these lines is being done by R. J. PLYMEN [12].

We should like to thank J. T. LEWIS, G. W. MACKEY and R. J. PLYMEN, who have given us considerable encouragement and insight into the quantum mechanical relevance of the ideas developed here.

We are grateful to Professor R. V. KADISON who in an appendix to this paper obtains certain sufficient conditions for the $\sigma$-closure of a $C^*$-algebra of operators to be a von Neumann algebra.

## 2. On $\Sigma^*$ algebras

For the general theory and notation concerning $C^*$-algebras we shall make systematic use of Dixmier's book [1].

A set $S$ of bounded operators on the Hilbert space $\mathscr{H}$ shall be called *$\sigma$-closed* if given any *sequence* $x_n \in S$ which converges to $x \in \mathscr{L}(\mathscr{H})$ in the weak operator topology, we then have that $x \in S$. Given any set $S$ there is a smallest $\sigma$-closed set containing it, which we call its *$\sigma$-closure* and denote by $\sigma(S)$.

**Lemma 2.1.** *If $\mathscr{A}$ is a $C^*$-subalgebra of the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on the Hilbert space $\mathscr{H}$ then $\sigma(\mathscr{A})$ is a $C^*$-subalgebra such that every increasing sequence in $\sigma(\mathscr{A})$ which is norm bounded has a least upper bound in $\sigma(\mathscr{A})$. If $\mathscr{A}$ is separable then $\sigma(\mathscr{A})$ has an identity element.*

*Proof.* If $\alpha$, $\beta$ are complex numbers and $a \in \mathscr{A}$ then the family of all $x \in \mathscr{L}(\mathscr{H})$ such that $(\alpha a + \beta x) \in \sigma(\mathscr{A})$ and $x^* \in \sigma(\mathscr{A})$ and $ax \in \sigma(\mathscr{A})$ is $\sigma$-closed and contains $\mathscr{A}$, and so contains $\sigma(\mathscr{A})$. Now if $\alpha$, $\beta$ are complex numbers and $b \in \sigma(\mathscr{A})$ then the family of all $x \in \mathscr{L}(\mathscr{H})$ such that $(\alpha x + \beta b) \in \sigma(\mathscr{A})$ and $xb \in \sigma(\mathscr{A})$ is $\sigma$-closed and contains $\mathscr{A}$, and so contains $\sigma(\mathscr{A})$. As a uniformly convergent sequence is convergent in the weak operator topology so we can see that $\sigma(\mathscr{A})$ is a $C^*$-algebra of $\mathscr{L}(\mathscr{H})$. If $x_n \in \sigma(\mathscr{A})$ is a norm bounded sequence such that $x_n \leqq x_{n+1}$ for all $n$ then $x_n$ converges in the weak operator topology; the limit, which is in $\sigma(\mathscr{A})$, is the least upper bound of the sequence $x_n$ in $\mathscr{L}(\mathscr{H})$. If $\mathscr{A}$ is separable let $e_n \in \mathscr{A}$ be a countable increasing approximate identity for $\mathscr{A}$ constructed as in [1, p. 15]. If $e \in \sigma(\mathscr{A})$ is the least upper bound then the set of $x \in \mathscr{L}(\mathscr{H})$ such that $ex = xe = x$ is $\sigma$-closed and contains $\mathscr{A}$, and so contains $\sigma(\mathscr{A})$. That is $e$ is an identity element for $\sigma(\mathscr{A})$.

Now let $\mathscr{A}$ be a $C^*$-algebra and denote by $\mathscr{F}$ the set of all ordered pairs $\{x_n, x\}$ consisting of a sequence $x_n \in \mathscr{A}$ and a point $x \in \mathscr{A}$. If $\mathscr{G} \subseteqq \mathscr{F}$ we denote by $\mathscr{G}^\sigma$ the set of all states $\phi$ in $\mathscr{A}$ such that for all $\{x_n, x\} \in \mathscr{G}$ we have

$$(\phi, x_n) \to (\phi, x)\,.$$

If $\mathscr{H}$ is a set of states on $\mathscr{A}$ we denote by ${}^\sigma\mathscr{H} \subseteqq \mathscr{F}$ the set of all $\{x_n, x\} \in \mathscr{F}$ such that for all $\phi \in \mathscr{H}$ we have

$$(\phi, x_n) \to (\phi, x)\,.$$

It is easy to verify that ${}^\sigma(\mathscr{G}^\sigma) \supseteqq \mathscr{G}$, that $({}^\sigma\mathscr{H})^\sigma \supseteqq \mathscr{H}$ and that

$$({}^\sigma(\mathscr{G}^\sigma))^\sigma = \mathscr{G}^\sigma\,; \quad {}^\sigma(({}^\sigma\mathscr{H})^\sigma) = {}^\sigma\mathscr{H}\,.$$

We now define a *$\Sigma^*$-algebra* $\mathscr{A}$ as a $C^*$-algebra together with a subset $\mathscr{G} \subseteq \mathscr{F}$, called the set of $\sigma$-convergent sequences in $\mathscr{A}$ and denoted $x_n \to x$, such that the following properties hold:

(i) if $x_n \to x$ then there is a constant $K$ such that for all $n$ we have $\|x_n\| \leqq K < \infty$;

(ii) if $x_n \to x$ and $y \in \mathscr{A}$ then $x_n y \to xy$;

(iii) if $x_n \in \mathscr{A}$ is a sequence such that $(\phi, x_n)$ converges for all $\phi \in \mathscr{G}^\sigma$ then there is some $x \in \mathscr{A}$ such that $x_n \to x$.

(iv) if $0 \neq x \in \mathscr{A}$ then there is some $\phi \in \mathscr{G}^\sigma$ such that $(\phi, x) \neq 0$.

It is clear from the definition that $\mathscr{G} = {}^\sigma(\mathscr{G}^\sigma)$ so that the $\Sigma^*$-algebra may be alternatively specified in terms of $\mathscr{G}^\sigma$, called the set of *$\sigma$-states* of the $\Sigma^*$-algebra $\mathscr{A}$. We note the following elementary properties.

(v) If $x_n \to x$ then $x_n^* \to x^*$;

(vi) if $x_n \to x$ and $y_n \to y$ then $(x_n + y_n) \to (x + y)$;

(vii) if $x_n \to x$ and $\alpha_n$ is a sequence of complex numbers converging to $\alpha$ then $\alpha_n x_n \to \alpha x$;

(viii) if $x_n \to x$ and $y \in \mathscr{A}$ then $y x_n \to yx$;

(ix) the set $\mathscr{G}^\sigma$ is a norm-closed convex set in $\mathscr{A}^*$.

If $\mathscr{H}$ is a Hilbert space and $\mathscr{A}$ is a $C^*$-subalgebra of $\mathscr{L}(\mathscr{H})$ such that $\mathscr{A}$ is a $\sigma$-closed set, then $\mathscr{A}$ becomes a $\Sigma^*$-algebra if we define the $\sigma$-convergent sequences to be the sequences of operators $x_n \in \mathscr{A}$ which are convergent in the weak operator topology. We call such algebras *$\Sigma^*$-subalgebras* of $\mathscr{L}(\mathscr{H})$; clearly $\mathscr{L}(\mathscr{H})$ itself is a $\Sigma^*$-algebra. By a *$\sigma$-representation* $\pi$ of the $\Sigma^*$-algebra $\mathscr{A}$ on the Hilbert space $\mathscr{H}$ we shall mean a representation such that if $x_n \to x$ then $\pi x_n \to \pi x$. By a *faithful $\sigma$-representation* we shall mean a faithful representation such that $\pi\mathscr{A}$ is $\sigma$-closed and $x_n \to x$ if and only if $\pi x_n \to \pi x$.

**Lemma 2.2.** *Every $\Sigma^*$-algebra $\mathscr{A}$ has a faithful $\sigma$-representation as a $\Sigma^*$-subalgebra of the algebra of operators on a Hilbert space.*

*Proof.* The algebra $\mathscr{A}_1$ obtained from $\mathscr{A}$ by adjoining an identity $e$ becomes a $\Sigma^*$-algebra if we say that $x_n \oplus \lambda_n e \to x \oplus \lambda e$ if and only if $x_n \to x$ and $\lambda_n \to \lambda$. If $\phi$ is a $\sigma$-state on $\mathscr{A}$ its extension to a state on $\mathscr{A}_1$ is also a $\sigma$-state. It is easy to check that the representation $\pi_\phi$ on $\mathscr{A}$ induced by $\phi$ is a $\sigma$-representation, on a Hilbert space $\mathscr{H}_\phi$. If

$$\mathscr{H} = \sum_{\phi \in \mathscr{G}^\sigma} \oplus \mathscr{H}_\phi$$

and $\pi$ is the direct sum representation then $\pi$ is also a $\sigma$-representation and is faithful. Now let $x_n \in \mathscr{A}$ and let $\pi x_n$ converge to $y \in \mathscr{L}(\mathscr{H})$ in the weak operator topology. For each $\phi \in \mathscr{G}^\sigma$ there is a vector $\xi_\phi \in \mathscr{H}$ such that for all $x \in \mathscr{A}$ we have

$$\phi(x) = \langle (\pi x)\, \xi_\phi, \xi_\phi \rangle$$

so we see that $\phi(x_n)$ converges for each $\phi \in \mathscr{G}^\sigma$. By condition (iii) there exists $x \in \mathscr{A}$ such that $x_n \to x$. It follows that $\pi x = y$ which proves that $\pi$ is a faithful $\sigma$-representation.

Now let $X$ be a space with a given $\sigma$-field of subsets. The space $\mathscr{B}\{X\}$ of all bounded measurable functions on $X$ is a commutative $C^*$-algebra in an obvious sense. We say that a sequence $f_n$ in $\mathscr{B}\{X\}$ is $\sigma$-convergent to $f$ in $\mathscr{B}\{X\}$ if and only if $\|f_n\| \leqq K$ for some $K$ and all $n$, and $f_n$ also converges pointwise to $f$. It is easy to verify that the family of $\sigma$-states is exactly the set of probability measures on $X$. Now let $f_n \in \mathscr{B}\{X\}$ be a sequence such that $\phi(f_n)$ converges for all $\phi \in \mathscr{G}^\sigma$. Regarding the $f_n$ as continuous linear functionals on the Banach space of all bounded signed measures on $X$, [4], we see by the uniform boundedness theorem that there is a constant $K$ such that $\|f_n\| \leqq K$ for all $n$. The functions $f_n$ converge pointwise and the limit must be in $\mathscr{B}\{X\}$. It follows that $\mathscr{B}\{X\}$ is a $\Sigma^*$-algebra.

**Lemma 2.3.** *Let $\mathscr{A}$ be a $\Sigma^*$-subalgebra of the algebra of bounded operators on the Hilbert space $\mathscr{H}$. Let $\pi : C(\Omega) \to \mathscr{A}$ be a representation of the $C^*$-algebra of continuous functions on the compact Hausdorff space $\Omega$ into $\mathscr{A}$. Then $\pi$ has a unique extension to a $\sigma$-representation of the $\Sigma^*$-algebra $\mathscr{B}\{\Omega\}$ of bounded Baire functions on $\Omega$ into $\mathscr{A}$.*

*Proof.* The uniqueness of such a representation is clear. Conversely it is shown in [8] that there is a natural extension of $\pi : C(\Omega) \to \mathscr{A}$ to a representation $\pi^\sim : \mathscr{B}\{\Omega\} \to \mathscr{L}(\mathscr{H})$ such that for each vector $\xi \in \mathscr{H}$ there is a Baire measure $\mu_\xi$ on $\Omega$ such that for all $f \in B\{\Omega\}$ we have

$$\langle(\pi^\sim f)\,\xi, \xi\rangle = \int f d\mu_\xi\,.$$

From this formula and the Lebesgue dominated convergence theorem we see that $\pi^\sim$ is a $\sigma$-representation and hence that its range is contained in $\mathscr{A}$.

**Lemma 2.4.** *Let $x$ be a self-adjoint element of the $\Sigma^*$-subalgebra $\mathscr{A}$ of the algebra of bounded operators on the Hilbert space $\mathscr{H}$. Then the range projection of $x$ is in $\mathscr{A}$.*

*Proof.* Let $\Omega$ be the spectrum of $x$ and let $\pi : C(\Omega) \to \mathscr{L}(\mathscr{H})$ be the faithful representation such that $\pi(1) = 1$ and $\pi(f) = x$, where $f$ is the function $f(z) = z$. Let $\pi^\sim$ be the $\sigma$-representation induced on $\mathscr{B}\{\Omega\}$. The set of functions $g$ in $\mathscr{B}\{\Omega\}$ such that $\pi^\sim(g)$ is in $\mathscr{A}$ is $\sigma$-closed and contains all continuous functions vanishing at the origin. Therefore if $h \in \mathscr{B}\{\Omega\}$ is the function given by $h(0) = 0$ and $h(z) = 1$ if $z \neq 0$ we see that $p = \pi^\sim(h)$ is a projection in $\mathscr{A}$ such that $px = xp = x$. Moreover if $q$ is a projection in $\mathscr{L}(\mathscr{H})$ such that $qx = xq = x$ then the set of all functions $g$ in $\mathscr{B}\{\Omega\}$ such that

$$\pi^\sim(g)\,q = q\pi^\sim(g) = \pi^\sim(g)$$

is $\sigma$-closed and contains all polynomials with zero constant coefficient. Therefore $h$ is such a function and $pq = qp = p$. This implies that $p$ is the range projection of $x$.

## 3. The $\sigma$-envelope of a $C^*$-algebra

In defining this we make systematic use of the enveloping algebra of a $C^*$-algebra as defined in [1, 5]. We summarize the facts in the form we shall need them.

Let $\mathscr{A}$ be a $C^*$-algebra. Every positive linear functional $\phi$ on $\mathscr{A}$ defines a cyclic representation $\pi_\phi$ on $\mathscr{A}$ and we call the direct sum $\pi$ of these representations $\pi_\phi$, one each for positive linear functional, the *universal representation* of $\mathscr{A}$. It is a faithful representation and if $\mathscr{H}$ is the Hilbert space on which it acts, we denote the weak operator closure of $\pi\mathscr{A}$ by $\overline{\pi\mathscr{A}}$. Now define $S(\mathscr{A})$ by

$$S(\mathscr{A}) = \{\phi \in \mathscr{A}^* : 0 \leqq \phi \text{ and } \|\phi\| \leqq 1\}$$

so that $S(\mathscr{A})$ is a compact convex set in the weak* topology of $\mathscr{A}^*$. Each vector $\xi \in \mathscr{H}$ with $\|\xi\| \leqq 1$ defines a functional $\phi_\xi$ in $S(\mathscr{A})$ by the equation

$$(x, \phi_\xi) = \langle(\pi x)\, \xi, \xi\rangle$$

and $\phi$ is then a map from the unit ball of $\mathscr{H}$ onto $S(\mathscr{A})$.

The $C^*$-algebra $\mathscr{A}$ can be identified as a Banach space with the space $A_0(S(\mathscr{A}))$ of all continuous complex valued linear functionals on $S(\mathscr{A})$ and under this identification the self-adjoint elements of $\mathscr{A}$ correspond to the real linear functionals and the positive elements of $\mathscr{A}$ correspond to the positive linear functionals. As shown in [1, 5] the map $\phi$ allows us to extend $\pi$ to an identification $\pi^{**}$ of the Banach space $\mathscr{A}^{**}$, or equivalently of the space of all bounded linear functionals on $S(\mathscr{A})$, with the von Neumann algebra $\overline{\pi(\mathscr{A})}$ in such a way that for $x \in \mathscr{A}^{**}$ and $\xi \in \mathscr{H}$ with $\|\xi\| \leqq 1$ we still have

$$(x, \phi_\xi) = \langle(\pi^{**} x)\, \xi, \xi\rangle .$$

The map $\pi^{**}$ identifies real elements of $\mathscr{A}^{**}$ with self-adjoint elements of $\overline{\pi\mathscr{A}}$, positive elements of $\mathscr{A}^{**}$ with positive operators in $\overline{\pi\mathscr{A}}$, and identifies the weak* topology of $\mathscr{A}^{**}$, or equivalently the topology of pointwise convergence on $\mathscr{A}^{**}$ regarded as the space of bounded linear functionals on $S(\mathscr{A})$, with the weak operator topology on $\overline{\pi\mathscr{A}}$.

We now define the *$\sigma$-envelope* $\mathscr{A}^\sim$ of $\mathscr{A}$ as the smallest $\sigma$-closed family of bounded linear functionals on $S(\mathscr{A})$ containing $\mathscr{A}$. As in Lemma 2.1 we see that $\mathscr{A}^\sim$ is a closed linear subspace of $\mathscr{A}^{**}$ and it is clear that the functions in $\mathscr{A}^\sim$ are bounded Baire functions on $S(\mathscr{A})$.

**Theorem 3.1.** *If $\mathscr{A}$ is a $C^*$-algebra then the $\sigma$-envelope $\mathscr{A}^\sim$ is a $\Sigma^*$-subalgebra of the enveloping algebra $\mathscr{A}^{**}$. If $\mathscr{A}$ is separable then $\mathscr{A}^\sim$ has an identity element. If $\mathscr{H}$ is a Hilbert space and $\lambda$ is a representation of $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$, then there is a unique extension to a $\sigma$-representation $\lambda^\sim$ of $\mathscr{A}^\sim$ into $\mathscr{L}(\mathscr{H})$; moreover every $\sigma$-representation of $\mathscr{A}^\sim$ arises in this way. If $\phi$ is a state on $\mathscr{A}$ then it has a unique extension to a $\sigma$-state on $\mathscr{A}^\sim$ and every $\sigma$-state on $\mathscr{A}^\sim$ arises in this way.*

*Proof.* We define the $\sigma$-convergent sequences in $\mathscr{A}^\sim$ as being those sequences of functions in $\mathscr{A}^\sim$ which are pointwise convergent on $S(\mathscr{A})$ with limits in $\mathscr{A}^\sim$. The first two statements of the theorem now follow immediately from Lemma 2.1 and the properties of $\pi^{**}$.

It is shown in [1] that $\lambda:\mathscr{A} \to \mathscr{L}(\mathscr{H})$ has a unique extension to a representation $\lambda^{**}:\mathscr{A}^{**} \to \mathscr{L}(\mathscr{H})$ such that $\lambda^{**}$ is continuous with respect to the weak* topology of $\mathscr{A}^{**}$ and the weak operator topology of $\mathscr{L}(\mathscr{H})$. Defining $\lambda^\sim$ as the restriction of $\lambda^{**}$ to $\mathscr{A}^\sim$ it is clear that $\lambda^\sim$ is a $\sigma$-representation of $\mathscr{A}^\sim$ which extends $\lambda$. Uniqueness follows immediately from the fact that $\mathscr{A}^\sim$ is the $\sigma$-closure of $\mathscr{A}$. As every $\sigma$-representation of $\mathscr{A}^\sim$ must coincide with the $\sigma$-extension of its restriction to $\mathscr{A}$ so every $\sigma$-representation arises in the above way.

Every state $\phi$ of $\mathscr{A}$ defines a point in $S(\mathscr{A})$ and so by pointwise evaluation a $\sigma$-state $\phi^\sim$ in $\mathscr{A}^\sim$. As above it is clear that $\phi^\sim$ is unique and that every $\sigma$-state on $\mathscr{A}^\sim$ arises in this way.

Following [1], we now define the *spectrum* $\hat{\mathscr{A}}$ of a $C^*$-algebra $\mathscr{A}$ as the set of unitary equivalence classes of irreducible representations of $\mathscr{A}$. The *reduced atomic representation* [3, 6] of $\mathscr{A}$ is defined as the direct sum of the irreducible representations of $\mathscr{A}$ taking one from each unitary equivalence class. The following theorem provides one very important respect in which the $\sigma$-envelope $\mathscr{A}^\sim$ is better behaved than the enveloping algebra $\mathscr{A}^{**}$.

**Theorem 3.2.** *Let $\lambda:\mathscr{A} \to \mathscr{L}(\mathscr{H})$ be the reduced atomic representation of a $C^*$-algebra $\mathscr{A}$. Then the induced $\sigma$-representation $\lambda^\sim$ is a faithful $\sigma$-representation of $\mathscr{A}^\sim$ onto the $\Sigma^*$-subalgebra $\sigma(\lambda\mathscr{A})$ of $\mathscr{L}(\mathscr{H})$.*

*Proof.* We need to make use of Choquet boundary theory, and use [11] as the basic reference for terminology.

The extreme boundary $\partial S$ of the compact convex set $S(\mathscr{A})$ consists of the set of pure states $P(\mathscr{A})$ together with the origin. As each pure state on $\mathscr{A}$ induces an irreducible representation so we can identify $P(\mathscr{A})$ with a certain subset of the unit sphere of $\mathscr{H}$. Now let $\mu$ be a probability measure on $S(\mathscr{A})$ with barycentre $s \in S(\mathscr{A})$. The set of bounded linear Baire functions $f$ on $S(\mathscr{A})$ such that

$$f(s) = \int f\,d\mu$$

is a $\sigma$-closed subset of $\mathscr{A}^{**}$ because of the uniform boundedness theorem and as this set contains $\mathscr{A}$ so it contains $\mathscr{A}^\sim$. Now suppose that $\mu$ is a maximal representing measure for $s$, that $f \in \mathscr{A}^\sim$ satisfies $\lambda^\sim f = 0$ and that

$$E = \{t \in S(\mathscr{A}) : f(t) \neq 0\} .$$

Then $E$ is a Baire subset of $S(\mathscr{A})$ not meeting $\partial S$ so that by [11, p. 30] $\mu(E) = 0$. It follows that $f(s) = 0$ so that $E = \emptyset$. Therefore $\lambda^\sim$ is a faithful representation.

To prove that $\lambda^\sim$ is a faithful $\sigma$-representation we have to prove more. Let $f_n \in \mathscr{A}^\sim$ and let $\lambda^\sim f_n$ converge in the weak operator topology to $g \in \mathscr{L}(\mathscr{H})$. Then by the uniform boundedness theorem the $f_n$ are uniformly bounded. Using the identification of $P(\mathscr{A})$ with a subset of the unit sphere of $\mathscr{H}$ we see that $f_n$ converges on the set $\partial S$. Then as in the proof of Rainwater's theorem [11, p. 33] we see that $f_n$ converges on $S(\mathscr{A})$ to a limit $f$ which must be in $\mathscr{A}^\sim$. It follows that $\lambda^\sim f = g$ and that $\lambda^\sim$ is a faithful $\sigma$-representation.

**Corollary 3.3** *If $\Omega$ is a compact Hausdorff space and $\mathscr{A}$ is the $C^*$-algebra $C(\Omega)$ of all continuous functions on $\Omega$ then $\mathscr{A}^\sim$ can be identified with the $\Sigma^*$-algebra $\mathscr{B}\{\Omega\}$ of all bounded Baire functions on $\Omega$.*

*Proof.* We have already shown that $\mathscr{B}\{\Omega\}$ is a $\Sigma^*$-algebra. The result now follows from the fact that it is the $\sigma$-closure of $C(\Omega)$ for the reduced atomic representation.

**Corollary 3.4.** *If $\mathscr{A}$ is the $C^*$-algebra of compact operators on a separable Hilbert space $\mathscr{H}$ then $\mathscr{A}^\sim$ can be identified with $\mathscr{L}(\mathscr{H})$.*

*Proof.* This follows from the fact that the reduced atomic representation is the identity representation.

## 4. On separable G. C. R. algebras

We now start on a more detailed analysis of $\mathscr{A}^\sim$ using the sets $\mathrm{Irr}(\mathscr{A})$, $P(\mathscr{A})$, $\hat{\mathscr{A}}$ and $\mathrm{Prim}(\mathscr{A})$ as defined in [1]. Throughout this section we suppose $\mathscr{A}$ is a separable $C^*$-algebra.

The set $\mathrm{Irr}_n(\mathscr{A})$ is the set of irreducible representations of $\mathscr{A}$ in a fixed $n$-dimensional Hilbert space $\mathscr{H}_n$, and is a standard Borel space in a natural way. Choosing a fixed unit vector $\xi_n \in \mathscr{H}_n$ we get an induced Borel map $\lambda_n : \mathrm{Irr}_n(\mathscr{A}) \to P(\mathscr{A})$ such that if $x \in \mathscr{A}$ and $\pi \in \mathrm{Irr}_n(\mathscr{A})$ then

$$(x, \lambda_n \pi) = \langle (\pi x)\, \xi_n, \xi_n \rangle .$$

If we still denote by $\pi$ the extension to a $\sigma$-representation of $\mathscr{A}^\sim$ then we see that the above formula still holds for all $x \in \mathscr{A}^\sim$. Defining $\lambda : \mathrm{Irr}(\mathscr{A}) \to P(\mathscr{A})$ as the union of the maps $\lambda_n$ we know that $\lambda$ is a Borel map of the standard Borel space $\mathrm{Irr}(\mathscr{A})$ onto the standard Borel space $P(\mathscr{A})$. If $\mu : \mathrm{Irr}(\mathscr{A}) \to \hat{\mathscr{A}}$ and $\nu : P(\mathscr{A}) \to \hat{\mathscr{A}}$ are the natural Borel maps defined in [1] then $\nu\lambda = \mu$ and we can show, using the theory of standard

Borel spaces [7, 10], that the Mackey Borel structure of $\hat{\mathscr{A}}$ may be characterised either as the quotient Borel structure of $\hat{\mathscr{A}}$ from $\operatorname{Irr}(\mathscr{A})$ under $\mu$ or as the quotient Borel structure of $\hat{\mathscr{A}}$ from $P(\mathscr{A})$ under $\nu$; we shall use the second characterisation.

**Theorem 4.1.** *If $\mathscr{A}$ is a separable $C^*$-algebra then the centre of its $\sigma$-envelope $\mathscr{A}^\sim$ can be canonically identified with the $\Sigma^*$-algebra of all bounded measurable functions on $\hat{\mathscr{A}}$ with respect to a certain $\sigma$-field of subsets of $\hat{\mathscr{A}}$. This $\sigma$-field is larger than the topological Borel structure and smaller than the Mackey Borel structure and so for a separable* G.C.R. *algebra coincides with both.*

*Proof.* An element $x \in \mathscr{A}^\sim$ is central if and only if for each $\pi \in \operatorname{Irr}(\mathscr{A})$ we know that $\pi x$ is a multiple of the identity. This happens if and only if for any two unit vectors $\xi_1, \xi_2$ of the representation space of $\pi$ we know that

$$\langle (\pi x)\, \xi_1, \xi_1 \rangle = \langle (\pi x)\, \xi_2, \xi_2 \rangle$$

or equivalently if and only if for any two $\phi_1, \phi_2 \in P(\mathscr{A})$ with $\nu \phi_1 = \nu \phi_2$ we have

$$(x, \phi_1) = (x, \phi_2)\,.$$

If $x_1, x_2 \in$ centre $(\mathscr{A}^\sim)$ and $\phi \in P(\mathscr{A})$ then it is easy to verify that

$$(\alpha x_1 + \beta x_2, \phi) = \alpha (x_1, \phi) + \beta (x_2, \phi)$$
$$(x_1^*, \phi) = \overline{(x_1, \phi)}$$
$$(x_1 x_2, \phi) = (x_1, \phi)\,(x_2, \phi)\,.$$

Moreover as $\mathscr{A}^\sim$ can be identified by Theorem 3.2 with a vector space of bounded Borel functions on $P(\mathscr{A})$, so centre $(\mathscr{A}^\sim)$ is equal to the $\Sigma^*$-algebra of all bounded functions $f$ on $\hat{\mathscr{A}}$ such that $\nu f : P(\mathscr{A}) \to C$ is in $\mathscr{A}^\sim$. Under this identification pointwise convergence on $\hat{\mathscr{A}}$ corresponds to $\sigma$-convergence in centre $(\mathscr{A}^\sim)$ so we see that there is a $\sigma$-field $G$ of sets in $\hat{\mathscr{A}}$ such that centre $(\mathscr{A}^\sim)$ can be identified with the set of all $G$-measurable bounded functions on $\hat{\mathscr{A}}$.

The characteristic function $\chi(E)$ of any set $E$ of $G$ is in the centre of $\mathscr{A}^\sim$ and every element of $\mathscr{A}^\sim$ is given by a Borel function on $P(\mathscr{A})$. Therefore $\nu^{-1}(E)$ is a Borel subset of $P(\mathscr{A})$ so that the Mackey Borel structure on $\hat{\mathscr{A}}$ is larger than $G$.

On the other hand let $U \subseteq \hat{\mathscr{A}}$ be an open set where $\hat{\mathscr{A}}$ has the topology of [1, p. 60]. Then the closed set $\hat{\mathscr{A}} - U$ corresponds to a closed ideal $I$ of $\mathscr{A}$. Let $e_n \in I$ be a countable increasing approximate identity for $I$ constructed as in [1, p. 15] and let $e \in \mathscr{A}^\sim$ be the least upper bound in $\mathscr{A}^\sim$ of the sequence $e_n$. Then $e$ is a central projection in $\mathscr{A}^\sim$ and is the characteristic function of the set $\nu^{-1}(U)$. It follows that $U$ is in $G$ so that $G$ contains the topological Borel structure of $\hat{\mathscr{A}}$.

For separable G.C.R. algebras we are able to go very much further because of the existence of a Borel cross-section for the map $\mu:\mathrm{Irr}(\mathscr{A}) \to \hat{\mathscr{A}}$. We first make some further definitions.

Let $X$ be a space with a given $\sigma$-field of subsets, and let $\mathscr{H}$ be a separable Hilbert space. We say that a function $f:X \to \mathscr{H}$ is measurable if for each $\xi \in \mathscr{H}$ the function $\langle f(x), \xi\rangle$ is a measurable function on $X$; equivalently we may suppose that $\xi$ is an arbitrary element of a given fixed orthonormal basis of $\mathscr{H}$. We denote the vector space of all norm-bounded measurable functions $f:X \to \mathscr{H}$ with the obvious operations by $\mathscr{B}\{X, \mathscr{H}\}$. Similarly we say a function $f:X \to \mathscr{L}(\mathscr{H})$ is measurable if for each $\xi_1, \xi_2 \in \mathscr{H}$ the function $\langle f(x)\,\xi_1, \xi_2\rangle$ is a measurable function on $X$. The space $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ of all norm-bounded measurable functions $f:X \to \mathscr{L}(\mathscr{H})$ is a $C^*$-algebra in an obvious way. If $K$ is the Hilbert space of all functions $f:X \to \mathscr{H}$ of countable support such that

$$\sum_{x \in X} \|f(x)\|^2 < \infty$$

then $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ is naturally identified with a $\Sigma^*$-subalgebra of $\mathscr{L}(\mathscr{K})$. If $f, f_n \in \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ then $f_n$ is $\sigma$-convergent to $f$ if and only if for some $k$, all $n$ and all $x \in X$ we have

$$\|f_n(x)\| \leqq k < \infty$$

and for all $x \in X$ the sequence $f_n(x)$ converges to $f(x)$ in the weak operator topology. The centre of the $\Sigma^*$-algebra $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ may be identified with $\mathscr{B}\{X\}$, the $\Sigma^*$-algebra of bounded complex-valued measurable functions on $X$. We denote the characteristic function of a set $E \subseteqq X$ by $\chi(E)$.

The following three lemmas will be needed in the proof of Theorem 4.5.

**Lemma 4.2.** *Let $p$ be a projection in the $\Sigma^*$-algebra $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ such that* trace $(p)$ *and* trace $(1 - p)$ *are constant on $X$. Then there is a unitary operator $u \in \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ such that $u^*\,p\,u$ is constant on $X$.*

*Proof.* Let $e_n$ be a complete orthonormal basis in $\mathscr{H}$ so that for each $x \in X$ the vectors $(p\,e_n)(x)$ span the range of $p(x)$. Now define the sequence of vectors $y_n^1 \in \mathscr{B}\{X, \mathscr{H}\}$ inductively as follows:

$$y_1^1(x) = \begin{cases} 0 \text{ when } p\,e_1(x) = 0 \\ \|p\,e_1(x)\|^{-1} p\,e_1(x) \quad \text{otherwise} \end{cases}$$

so that $\|y_1^1(x)\|$ is equal to zero or one. Given $y_1^1, \ldots, y_{n-1}^1$ define

$$z_n(x) = p\,e_n(x) - \sum_{n=1}^{n-1} \langle p\,e_n(x), y_r^1(x)\rangle\, y_r^1(x)$$

and then

$$y_n^1(x) = \begin{cases} \text{o when } z_n(x) = 0 \\ \|z_n(x)\|^{-1} z_n(x) \quad \text{otherwise}\,. \end{cases}$$

Then for each $x \in X$ the vectors $y_n^1(x)$ are orthogonal, span $p(x)\mathscr{H}$, and have norm equal to zero or one. We observe that $y_n^1 \in \mathscr{B}\{X, \mathscr{H}\}$ and let $E_n$ be the measurable set on which $y_n^1(x)$ is non-zero.

We now define a new sequence of vectors $u_n \in \mathscr{B}\{X, \mathscr{H}\}$, inductively. Suppose that vectors $y_n^m \in \mathscr{B}(X, \mathscr{H})$ are defined for $n = 1, 2, \ldots$ so that $y_n^m = 0$ for $n \leqq (m-1)$, and suppose $E_n^m$ is the set on which $y_n^m(x)$ is non-zero. Define

$$u_m = \sum_{n=1}^{\infty} \chi\left(E_n - \bigcup_{r<n} E_r\right) y_n^m$$

observing that for each $x \in X$ the sum has only one non-zero term. We now define the new subsidiary sequence $y_n^{m+1} \in \mathscr{B}(X, \mathscr{H})$ by

$$y_n^{m+1} = \chi\left(\bigcup_{r<n} E_r^m\right) y_n^m$$

so that $y_n^{m+1} = 0$ for $n \leqq m$.

If trace $\{p(x)\} = \infty$ then $u_n(x)$ is non-zero for each $n$ and all $x \in X$; if trace $\{p(x)\} = N < \infty$ then $u_n(x)$ is zero for each $n > N$ and $x \in X$. Considering only the non-zero $u_n$ we see that $u_n \in \mathscr{B}\{X, \mathscr{H}\}$ and that for each $x \in X$ the $u_n(x)$ form an orthonormal basis for $p(x)\mathscr{H}$. Carrying out the same procedure for the projection $(1-p)$ it is now elementary to construct a unitary operator $u \in \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ with the required properties.

**Lemma 4.3.** *Let $X$ be a space with a $\sigma$-field of subsets and let $\mathscr{H}$ be a Hilbert space of dimension $n < \infty$. Let $B$ be a $\Sigma^*$-subalgebra of $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ such that the centre of $B$ contains $\mathscr{B}\{X\}$ and for each $x \in X, B(x)$ is equal to $\mathscr{L}(\mathscr{H})$. If $B$ is the $\sigma$-envelope of some countable subset then $B$ is equal to $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$.*

*Proof.* The lemma is trivial for $n = 1$ and we assume that it has been proved for all values of $n < m$.

The self-adjoint part of $B$ is the $\sigma$-envelope of a countable subset and so by using Lemma 2.3 we can find a countable set $\{p_r\}_{r=1}^{\infty}$ of projections in $B$ such that $B$ is the $\sigma$-envelope of the linear subspace spanned by the $p_r$. For each $p_r$ the function trace $\{p_r(x)\}$ is a measurable function on $X$ taking integer values between zero and $m$. Define the Borel set $X_r \subseteq X$ for $r = 1, 2, \ldots$ as the set of all $x \in X$ such that $r$ is the smallest integer for which

$$0 < \text{trace}\,\{p_r(x)\} < m$$

and then for $1 < s < m$ define $X_{r,s}$ as the Borel set

$$X_{r,s} = X_r \cap \{x \in X : \text{trace}\,\{p_r(x)\} = s\},$$

so that $X_{r,s}$ partition $X$ into a countable number of disjoint Borel sets. Let $e_{r,s}$ be the central projection $\chi(X_{r,s})$ and let $u_{r,s}$ be a unitary operator in $\mathscr{B}\{X_{r,s}, \mathscr{L}(\mathscr{H})\}$ constructed as in Lemma 4.2 such that

$u_{r,s}^* p_r u_{r,s}$ is a constant proper projection on $X_{r,s}$. Also we see that the $\Sigma^*$-algebra

$$u_{r,s}^* e_{r,s} B e_{r,s} u_{r,s}$$

satisfies the conditions of this lemma with respect to $\mathscr{B}\{X_{r,s}, \mathscr{L}(\mathscr{H})\}$, so it is now clearly only necessary to establish this lemma in the case where $B$ contains a constant proper projection $p$.

If this is the case then $pBp$ satisfies the conditions of this lemma in $\mathscr{B}\{X, \mathscr{L}(p\mathscr{H})\}$ and so by our inductive hypothesis

$$B \supseteq pBp = \mathscr{B}\{X, \mathscr{L}(p\mathscr{H})\}$$

and similarly

$$B \supseteq (1-p)\, B(1-p) = \mathscr{B}\{X, \mathscr{L}((1-p)\, \mathscr{H})\}\,.$$

Now for each $x \in X$ we know that $B(x) = \mathscr{L}(\mathscr{H})$. Let $e_1, \ldots, e_s$ be an orthonormal basis for $p\mathscr{H}$ and $e_{s+1}, \ldots, e_m$ an orthonormal basis for $(1-p)\mathscr{H}$. It follows by considering the countable set of operators $\{p_r\}_{r=1}^{\infty}$ that $X$ can be partitioned into a countable number of disjoint Borel sets $Y_n$ such that for each $n = 1, 2, \ldots$ there is some $q \in B$ and integers $a \leqq s$ and $b > s$ such that for all $x \in Y_n$,

$$\langle q(x)\, e_a, e_b \rangle \neq 0\,.$$

Now defining

$$Y_{n,l} = Y_n \cap \left\{x \in X : |\langle q(x)\, e_a, e_b\rangle| \geqq \frac{1}{l}\right\}$$

and using the fact that the function

$$\chi(Y_{n,l})\, \langle q(x)\, e_a, e_b\rangle^{-1}$$

is in the centre of $B$, we see that the operator

$$\chi(Y_{n,l})\, e_a \otimes \overline{e_b}$$

is in $B$ for all integers $a$, $b$. Again using the fact that the centre of $B$ is equal to $\mathscr{B}\{X\}$ we see that

$$B \supseteq \chi(Y_{n,l})\, \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$$

so that

$$B = \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}\,.$$

**Lemma 4.4.** *Let $X$ be a space with a $\sigma$-field of subsets and let $\mathscr{H}$ be a separable Hilbert space of infinite dimension. Let $B$ be a $\Sigma^*$-subalgebra of $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ such that the centre of $B$ contains $\mathscr{B}\{X\}$, and for each $x \in X$, $B(x)$ is dense in $\mathscr{L}(\mathscr{H})$ for the weak operator topology, and $B$ is the $\sigma$-envelope of a countable subset. Suppose there is a countable increasing family of projections $\{q_r\}_{r=1}^{\infty}$ in $B$ whose weak limit is the identity operator and such that for all $n = 1, 2, \ldots$ and $x \in X$,* trace $\{q_r(x)\}$ *is finite. Then*

$$B = \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}\,.$$

*Proof*. Choose any particular $q_r$ and partition $X$ into a countable number of Borel sets $X_n$ by defining

$$X_n = \{x \in X : \text{trace}\,\{q_r(x)\} = n\}\,.$$

By Lemma 4.2 there is a unitary operator $u_n$ in $\mathscr{B}\{X_n, \mathscr{L}(\mathscr{H})\}$ such that $u_n^* q_r u_n$ is a constant finite dimensional projection on $X_n$. Now applying Lemma 4.3 to the $\Sigma^*$-subalgebra

$$\chi(X_n)\,(u_n^* q_r u_n)\,(u_n^* B u_n)\,(u_n^* q_r u_n)\,\chi(X_n)$$

of

$$\mathscr{B}\{X_n, \mathscr{L}((u_r^* q_r u_n)\,\mathscr{H})\}$$

we see that they are equal so that

$$q_r B q_r = q_r \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}\, q_r\,.$$

Now for any operator $b \in B$ we know that $q_r b q_r$ converges in the weak operator topology to $b$ from which we conclude that

$$B = \mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}\,.$$

**Theorem 4.5.** *Let $\mathscr{A}$ be a separable G. C. R. algebra. Then each $n = \infty, 1, 2, \ldots$ defines a central projection $e_n$ in the $\sigma$-envelope $\mathscr{A}^\sim$ and so a $\sigma$-ideal $\mathscr{A}_n^\sim = e_n \mathscr{A}^\sim e_n$ such that*

$$\mathscr{A}^\sim = \sum_{n=1}^{n=\infty} \oplus\, \mathscr{A}_n^\sim\,.$$

*Each $\Sigma^*$-algebra $\mathscr{A}_n^\sim$ has a faithful $\sigma$-representation as $\mathscr{B}\{\hat{\mathscr{A}}_n, \mathscr{L}(\mathscr{H}_n)\}$, the $\Sigma^*$-algebra of all bounded Borel functions from $\hat{\mathscr{A}}_n$ to $\mathscr{L}(\mathscr{H}_n)$, where $\mathscr{H}_n$ is an n-dimensional Hilbert space, separable for $n = \infty$.*

*Remark*. This theorem may be regarded as completely determining the *Borel* structure of all separable G.C.R. algebras. Some similar but more complicated results on the *topological* structure of a very special subclass of the G.C.R. algebras have been obtained in [2, 13].

*Proof*. As $\mathscr{A}$ is a G.C.R. algebra so by [1, p. 95] the natural maps $\lambda_n : \mathrm{Irr}_n(\mathscr{A}) \to \hat{\mathscr{A}}_n$ have Borel cross-sections. We now identify $\hat{\mathscr{A}}_n$ with its Borel cross-section in $\mathrm{Irr}_n(\mathscr{A})$. For each $x \in \mathscr{A}^\sim$ and $\xi_1, \xi_2 \in \mathscr{H}_n$ we noted that

$$\langle(\pi x)\,\xi_1, \xi_2\rangle$$

is a Borel function on $\mathrm{Irr}_n(\mathscr{A})$ so that the direct sum of the $\pi \in \mathrm{Irr}_n(\mathscr{A})$ is a $\sigma$-representation

$$\phi_n : \mathscr{A}^\sim \to \mathscr{B}\{\hat{\mathscr{A}}_n, \mathscr{L}(\mathscr{H})\}\,.$$

The direct sum of the $\phi_n$ is the induced $\sigma$-representation of the reduced atomic representation of $\mathscr{A}$, and is a faithful $\sigma$-representation by Theorem 3.2. By Theorem 4.1 the characteristic function of $\hat{\mathscr{A}}_n$ is a central projection $e_n$ in $\mathscr{A}^\sim$ and if we define $\mathscr{A}_n^\sim = e_n \mathscr{A}^\sim e_n$ then it is clear that

$$\mathscr{A}^\sim = \sum_{n=1}^{n=\infty} \oplus \mathscr{A}_n^\sim$$

and that each $\mathscr{A}_n^\sim$ is a $\Sigma^*$-subalgebra of $\mathscr{B}\{\hat{\mathscr{A}}_n, \mathscr{L}(\mathscr{H}_n)\}$. If $\{x_m\}_{m=1}^\infty$ is. a countable dense set in $\mathscr{A}$ then $\mathscr{A}_n^\sim$ is the $\sigma$-envelope of $\{e_n x_m e_n\}_{m=1}^\infty$. Each $\pi \in \hat{\mathscr{A}}_n$ is an irreducible representation so $\mathscr{A}_n^\sim(\pi)$ is weakly dense in $\mathscr{L}(\mathscr{H}_n)$ for each $\pi \in \hat{\mathscr{A}}_n$. It is now immediate from Theorem 4.1 and Lemma 4.3 that $\mathscr{A}_n^\sim$ is equal to $\mathscr{B}\{\hat{\mathscr{A}}_n, \mathscr{L}(\mathscr{H}_n)\}$ for finite $n$.

Now consider the case $n = \infty$. Let $I_\varrho$, $\varrho \in R$ be a composition series of closed ideals in $\mathscr{A}$ such that $I_{\varrho+1}/I_\varrho$ is a non-trivial C.C.R. algebra for each $\varrho \in R$; then $R$ is a countable set. The sets $X_\varrho \subseteq \mathscr{A}_\infty$ defined by

$$X_\varrho = \{\pi \in \hat{\mathscr{A}}_\infty : \pi | I_\varrho = 0 \quad \text{but} \quad \pi | I_{\varrho+1} \neq 0\}$$

form a partition of $\hat{\mathscr{A}}_\infty$ into a countable number of disjoint Borel sets. Each $\pi \in X_\varrho$ maps $I_{\varrho+1}$ onto the algebra of all compact operators on $\mathscr{H}_\infty$.

By using the spectral decomposition in $\mathscr{A}_\infty^\sim$ of a countable dense subset of the self-adjoint part of $I_{\varrho+1}$ we can find a countable set of projections $p_n$ in $\mathscr{A}_\infty^\sim$ such that $p_n(x)$ are finite-dimensional for all $x \in X_\varrho$ and the vector space spanned by $p_n(x)\,\mathscr{H}_\infty$ for $n = 1, 2, \ldots$ is dense in $\mathscr{H}_\infty$ for all $x \in X_\varrho$. Now let $q_n \in \mathscr{A}_\infty^\sim$ be the range projection of

$$p_1 + \cdots + p_n$$

and observe that $q_n$ is an increasing sequence of projections in $\mathscr{A}_\infty^\sim$ such that for each $x \in X_\varrho$, $q_n(x)$ is finite-dimensional and converges weakly to the identity operator. By Lemma 4.4 we see that

$$\chi(X_\varrho)\, \mathscr{A}_\infty^\sim \chi(X_\varrho) = \mathscr{B}\{X_\varrho, \mathscr{L}(\mathscr{H}_\infty)\}$$

so that it becomes trivial

$$\mathscr{A}_\infty^\sim = \mathscr{B}\{\hat{\mathscr{A}}_\infty, \mathscr{L}(\mathscr{H}_\infty)\}\,.$$

## 5. The finite traces on a $C^*$-algebra

A (finite) *trace* $\phi$ on a $C^*$-algebra $\mathscr{A}$ is defined as a positive functional such that for all $x, y \in \mathscr{A}$ we have

$$\phi(xy) = \phi(yx)\,.$$

Suppose that $\mathscr{A}$ is a separable $C^*$-algebra. Then $\phi$ has a unique natural extension to a weak* continuous trace on $\mathscr{A}^{**}$, and restricting to $\mathscr{A}^\sim$ we see that every trace on $\mathscr{A}$ has a unique extension to a $\sigma$-trace on $\mathscr{A}^\sim$ and all $\sigma$-traces on $\mathscr{A}^\sim$ are obtained in this way.

**Theorem 5.1.** *Let $\mathscr{A}$ be a separable* G. C. R. *algebra, so that*

$$\mathscr{A}^{\sim} \cong \sum_{n=1}^{n=\infty} \oplus \mathscr{B}\{\hat{\mathscr{A}}_n, \mathscr{L}(\mathscr{H}_n)\}.$$

*Then there is a one-one correspondence between the set of finite traces $\phi$ on $\mathscr{A}$, the set of finite $\sigma$-traces $\phi^{\sim}$ on $\mathscr{A}^{\sim}$, and the set of finite measures $\mu$ on $\hat{\mathscr{A}}$ such that*

$$\sum_{n=1}^{\infty} n\, \mu(\hat{\mathscr{A}}_n) < \infty .$$

*This correspondence is defined by the equation*

$$(x, \phi) = \int_{\hat{\mathscr{A}}} \operatorname{trace}\{\pi x\}\, \mu(d\pi)$$

*for all $x \in \mathscr{A}^{\sim}$.*

*Proof.* Clearly all that we have to do is characterize the finite $\sigma$-traces on the $\Sigma^*$-algebra $\mathscr{B}\{X, \mathscr{L}(\mathscr{H})\}$ for $n = \infty, 1, 2, \ldots$ If $n = \infty$ then a $\sigma$-trace $\phi$ is a $\sigma$-trace on the algebra of constant elements and so must vanish.

Now suppose that $n$ is finite. Every finite measure $\mu$ on $X$ defines a finite $\sigma$-trace on $B = \mathscr{B}\{X, \mathscr{L}(\mathscr{H}_n)\}$ by the equation

$$(b, \phi_\mu) = \int_X \operatorname{trace}\{b(x)\}\, \mu(dx)$$

and by considering the restriction of $\phi_\mu$ to the centre of $B$ we see that $\phi_\mu$ determines $\mu$. Conversely let $\phi$ be a finite $\sigma$-trace on $B$ and let $\mu$ be the measure defined by restriction to $\mathscr{B}\{X\}$, the centre of $B$. Let $G$ be the compact group of all constant unitary operators in $B$, with Haar measure $dg$. Then for all $b \in B$ and $g \in G$ we have

$$(b, \phi) = (g^* b g, \phi)$$

so that

$$\begin{aligned}(b, \phi) &= \Big(\int_G (g^{-1} b g)\, dg, \phi\Big) \\ &= (\operatorname{trace}(b), \phi) \\ &= \int_X \operatorname{trace}\{b(x)\}\, \mu(dx) .\end{aligned}$$

Therefore $\phi = \phi_\mu$ and the theorem is proved.

## References


1. Dixmier, J.: Les $C^*$-algèbres et leurs representations. Cahiers scientifiques, fasc. XXIX, Paris: Gauthier-Villars 1964.
2. Fell, J. M. G.: The structure of algebras of operator fields. Acta Math. **106**, 233—280 (1961).
3. Glimm, J., and R. V. Kadison: Unitary operators in $C^*$-algebras. Pacific J. Math. **10**, 547—556 (1960).

4. HALMOS, P. R.: Measure theory, Princeton-London-Toronto: 1st edit. van Nostrand. 1950.
5. KADISON, R. V.: Transformation of states in operator theory and dynamics; Topology Vol. 3, Supp. 2, 177—198 (1964—1965).
6. —, and J. R. RINGROSE: Derivations and Automorphisms of operator algebras; Commun. Math. Phys. 4, 32—63 (1967).
7. KURATOWSKI, C.: Topologie I, espaces metrisables, espaces complets; 1st edit. Warsaw 1933.
8. LOOMIS, L.: An introduction to abstract harmonic analysis, 1st edit. Princeton-London-Toronto: van Nostrand, 1953.
9. MACKEY, G. W.: Borel structure in groups and their duals. Trans. Am. Math. Soc. 85, 134—165 (1957).
10. — The mathematical foundations of quantum mechanics, lecture note vol., New York-Amsterdam: W. A. Benjamin Inc. 1963.
11. PHELPS, R. R.: Lectures on Choquet's theorem, 1 st edit. Princeton-London-Toronto: van Nostrand, 1966.
12. PLYMEN, R. J.: $C^*$-algebras and Mackey's axioms; to appear.
13. VASIL'EV, N. B.: $C^*$-algebras with finite dimensional irreducible representations; Russian Math. Surveys, 21, (1), 137—155 (1966).

Dr. E. B. DAVIES
Mathematical Institute
University of Oxford
24—29, St. Giles
Oxford, Great Britain

## Appendix by R. V. KADISON

**Theorem A.** *If $A$ is a $C^*$-algebra acting on a separable Hilbert space $H$, then $\sigma(A)$ is $A''$ (the von Neumann algebra generated by $A$).*

*Proof.* The unit ball of $A''$ is metrizable in the weak-operator topology and

$$d(a,b) = \sum_n \frac{|([a-b]\,\xi_n, \xi_n)|}{2^n \|\xi_n\|^2}$$

is a metric for this topology (where $\{\xi_n\}$ is a dense denumerable subset of $H$). If $b$ lies in this ball, it is, therefore, a weak *sequential* limit point of any dense subset. From the Kaplansky Density Theorem, the unit ball of $A$ is such a dense subset. Since $\sigma(A)$ is sequentially closed, $b$ lies in $\sigma(A)$; and $\sigma(A) = A''$.

Somewhat more generally:

**Theorem B.** *If $\sigma(A)$ is countably-decomposable (i. e., each family of mutually-orthogonal projections in $\sigma(A)$ is countable — $H$ need not be separable), then $\sigma(A) = A''$.*

*Proof.* This result follows from [2'; Theorem 2, p. 179].

In view of Theorem B, it should be noted that, even if $A$ is norm-separable, $\sigma(A)$ need be neither countably-decomposable nor a von Neumann algebra.

*Example C.* Let $A$ be $C([0, 1])$, the algebra of complex-valued continuous functions on $[0, 1]$, and $\pi_\lambda$ the one-dimensional representation of $A$ defined by: $\pi_\lambda(a) = a(\lambda)$. With $\pi = \sum_\lambda \oplus \pi_\lambda$, $\sigma(\pi(A))$ (acting on $H = \sum_\lambda \oplus H_\lambda$) contains $e_\lambda$, the (one-dimensional) projection of $H$ onto $H_\lambda$ (isomorphic to the complex numbers). To see this, let $f_n$ be 1 at $\lambda$, 0 on $[0. \lambda - 1/n]$ and $[\lambda + 1/n, 1]$, and linear on $[\lambda - 1/n, \lambda]$ and $[\lambda, \lambda + 1/n]$ (with obvious modification if $\lambda$ is either 0 or 1). Then $\pi(f_n)$ is a monotone decreasing sequence of positive operators in $\pi(A)$, each greater than $e_\lambda$ (since $\pi_\lambda(f_n) = 1$). For each $\lambda' \neq \lambda$, there is an $n'$ such that $\pi(f_n) e_{\lambda'} = 0$ if $n \geqq n'$. Thus $\pi(f_n)$ tends strongly to $e_\lambda$, $e_\lambda$ lies in $\sigma(\pi(A))$; and the norm-separable $\pi(A)$ contains the uncountable family $\{e_\lambda : 0 \leqq \lambda \leqq 1\}$ of mutually orthogonal projections.

Each operator $a'$ on $H$ gives rise to a function $a$ on $[0, 1]$ such that $e_\lambda a' e_\lambda = a(\lambda) e_\lambda$ (recall that $e_\lambda$ is one-dimensional). If $a'$ lies in $\sigma(\pi(A))$, $a$ is a Baire function on $[0, 1]$; for if $a'_n$ on $H$ tends weakly to $a'$, $e_\lambda a'_n e_\lambda = a_n(\lambda) e_\lambda$ tends to $e_\lambda a' e_\lambda = a(\lambda) e_\lambda$, i.e., $a_n$ tends pointwise to $a$ on $[0, 1]$ (while $\sigma(\pi(A))$ is obtained from $\pi(A)$ by the process of taking weak sequential limits). With $S$ a non-Baire subset of $[0, 1]$ (say, non-measurable) and $e'$ the projection $\bigvee_\lambda \{e_\lambda : \lambda \text{ in } S\}$, the function $e$ on $[0, 1]$ corresponding to $e'$ is the characteristic function of $S$. Thus $e'$ (in $\pi(A)''$) is not in $\sigma(\pi(A))$. Moreover, a projection $\bar{e}$ in $\sigma(\pi(A))$ which is a least upper bound of $\{e_\lambda : \lambda \text{ in } S\}$ would have to correspond to a Baire set $S_0$ in $[0, 1]$ coinciding with $S$. Thus $\sigma(\pi(A))$ has no faithful representation as a von Neumann algebra.

The "measure-theoretic" (or, "commutative") phenomena noted above represent the only possibility for $\sigma(A)$ to fail to coincide with $A''$ when $A$ is norm-separable.

**Theorem D.** *If $A$ is a norm-separable (equivalently, countably-generated) $C^*$-algebra acting on $H$ and the center of $A''$ is countably decomposable then $\sigma(A) = A''$.*

*Proof.* From [1'; Cor., p. 20] the center $C$ of $A''$ has a separating vector $\xi$. Since $\xi$ is separating for $C$, the smallest projection in $C$ whose range contains $\xi$ is 1; so that $e'$, the (cyclic) projection in $A'$ whose range is the closure of $\{A\xi\}$, has central carrier 1. Note that this range, $e'H$, is separable since $A$ is norm-separable.

It follows from [1'; Prop. 2, p. 19] that the mapping $a \to ae'$ of $A''$ onto $A''e'$ is an isomorphism, and from [1'; Cor. 1, p. 57] that this mapping is ultraweakly bicontinuous. Thus, if $(a_n e')$ is a monotone increasing sequence in $\sigma(A)e'$ with limit $ae'$, then $(a_n)$ is monotone increasing in $\sigma(A)$ with limit $a$. Hence $a \in \sigma(A)$, $ae' \in \sigma(A)e'$, and $\sigma(A)e'$

contains the limit of each bounded monotone increasing sequence of its elements. Since $e'H$ is separable the argument of [2'; Theorem 2] yields that $\sigma(A)\, e' = A'' e'$. As $a \to a e'$ is an isomorphism $\sigma(A) = A''$.

## References

1'. Dixmier, J.: Les algèbres d'opérateurs dans l'espace hilbertien. Paris: Gauthiers-Villars 1957.
2'. Kadison, R. V.: Operator algebras with a faithful weakly-closed representation. Ann. Math. 64, 175—181 (1956).

Prof. R. V. Kadison
Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania 19104
USA