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Spectra of Generators in Irreducible Unitary Representations of Non-Compact Semi-Simple Groups

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Abstract. Several theorems concerning the spectra of elements of the complexified Lie algebra in unitary representations of non-compact semi-simple groups are proved. The principal theorem gives purely Lie algebraic sufficient conditions for the type of spectrum (point or continuous) of any element of the real Lie algebra. For elements of special "self adjoint" Cartan subalgebras these conditions are rephrased in terms of the basis-dependent information most readily available to the physicist, namely their hermiticity properties and the values of the structure constants: roots, etc.

1. Introduction

It should be useful in physics to have spectral criteria for the generators of non-compact groups represented unitarily in Hilbert space phrased entirely in terms of their Lie algebraic properties, e.g., in terms of the Killing form norm. Since one also frequently considers the complexified generator algebra (one must, for example, in using a Weyl "canonical" basis $\{h_i, e_\alpha\}$ for semi-simple groups), one would like to be able to develop spectral criteria for at least some, if not all, elements z of the complexified algebra phrased in terms of the most available (basis-dependent) information, namely, the values of the roots, the hermiticity properties of z, etc.

In this paper we prove several theorems about spectra of this type. They fall short of the ideal in that only sufficient conditions can be proved. But without a doubt they can be sharpened to "if and only if"'s if higher Lie algebra invariants beyond the Killing form are introduced. On the other hand it seems remarkable that such "Lie-algebraic" spectral theorems can be proved in general, that is, without any reference to the particular unitary representation. Indeed, this is known to be impossible for the spectral properties of the enveloping algebra.

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2. Assumptions on the Representation

We start with an irreducible unitary representation U of a non-compact semi-simple Lie group G[1] on a complex Hilbert space $\mathscr{H}$. Denote the Lie algebra of G by $\mathbf{G}$[1]. This means in more detail,

(a) $U(G)$ and $U(\mathbf{G})$ are representations of G and $\mathbf{G}$ by unitary and skew adjoint operators, respectively, defined on a common dense invariant domain $\mathscr{D} \subset \mathscr{H}$: $\mathscr{D}$ is dense in $\mathscr{H}$ and $U(G)\,\mathscr{D} \subset \mathscr{D}$, $U(\mathbf{G})\,\mathscr{D} \subset \mathscr{D}$[2].

It follows that the complexified Lie algebra $\mathbf{G}_c$ is also represented on $\mathscr{D}$, and $\mathscr{D}$ is invariant under it. We shall need these detailed properties in the proof of the second half of Theorem 1.

The only assumption we make limiting the type of irreducible unitary representation considered is that $U(\mathbf{G})$ is faithful:

(b) Non-triviality assumption. There is no non-zero element of $U(\mathbf{G})$ which annihilates $\mathscr{D}$.

It follows easily that no non-zero element of $\mathrm{U}(\mathbf{G}_c)$ annihilates $\mathscr{D}$.

Since G is semi-simple by hypothesis, $\mathbf{G}$ is the direct sum of simple Lie algebras $\mathbf{G}_i$, $i = 1, 2, \ldots, n$; and since every representation of a simple algebra except the trivial one is faithful, this restriction is the very light one that no $U(\mathbf{G}_i)$ is the trivial representation: $U(\mathbf{G}_i) = 0$ on $\mathscr{H}$. In particular, if G itself is simple, (b) requires only that $U(\mathbf{G})$ is not the trivial representation.

Note that although the *Lie algebra* $\mathbf{G}$ is represented faithfully on $\mathscr{H}$, this does not imply that the associative *enveloping algebra* need be represented faithfully, and in fact this is what happens in the various degenerate representations of G.

3. Completeness of Weight Vectors

We shall need the primary decomposition of $\mathbf{G}_c$ with respect to the adjoint map of any of its elements:

Definition. Given $z \in \mathbf{G}_c$, its *roots* $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the characteristic roots of $\mathrm{ad}z$ on $\mathbf{G}_c$, i.e., the roots of $\det(\mathrm{ad}z - \lambda) = 0$. $\mathbf{G}_c$ admits a decomposition (the primary decomposition) as a direct sum of *root spaces*

[1] Our notational conventions are as follows. Capital letters like G, K, etc. represent groups and $\mathbf{G}$, $\mathbf{K}$, etc., the corresponding (real) Lie algebras. N. B., for convenience, in referring to the mathematical literature we have used the mathematician's definition of the generator algebra $\mathbf{G}$ of a group G; physicists call $-i\,\mathbf{G}$ the generator algebra. $\mathbf{G}_c = \mathbf{G} \oplus i\,\mathbf{G}$ is the complexification of $\mathbf{G}$. Typical elements of real and complex Lie algebras are denoted x, y and w, z respectively; the operators representing them on the Hilbert space $\mathscr{H}$ are denoted by the corresponding capital letters: X, Y and W, Z, respectively. For the definition of the Hilbert space adjoint* and of a self-adjoint (or skew adjoint) operator, see [1].

[2] The assumption of the unitary group representation guarantees these domain and range properties, see [2].

(weight spaces of $\mathrm{ad}z$ on $\mathfrak{G}_c$), namely,

$$\mathfrak{G}_c = \mathfrak{G}_{\alpha_1} \oplus \mathfrak{G}_{\alpha_2} \oplus \cdots \oplus \mathfrak{G}_{\alpha} \tag{3.1}$$

where for any non-zero element $e_\alpha \in \mathfrak{G}_\alpha$

$$(\mathrm{ad}z - \alpha)^k e_\alpha = 0\,, \quad \text{some integer} \quad k \geqq 1\,. \tag{3.2}$$

If $d_\alpha \equiv \dim \mathfrak{G}_\alpha$, the least such integer $k \leqq d_\alpha$. For further properties of the primary decomposition see, say, HELGASON [3]. The terminology used here follows JACOBSON [4], see p. 61 ff.

Now we can prove a useful lemma which, for generality, we state about any element of the complexified Lie algebra.

Lemma 1. *Let $z \in \mathfrak{G}_c$. If Z has a weight vector $\in \mathscr{H}$, then its weight vectors span $\mathscr{H}$. If $z = x \in \mathfrak{G}$, the weight vectors are all eigenvectors and the weights are pure imaginary.*

Proof. Assume then that f is a weight vector of Z to weight m, that is $(Z - m)^n f = 0$, $f \in \mathscr{D}$, for some integer $n \geqq 1$ and some number m. We first prove that if e_α is an element of the root space $\mathfrak{G}_\alpha$, then $E_\alpha f$ is also a weight vector, to weight $m + \alpha$[3]. First note that for any two operators A, B and number λ

$$(A - \lambda)^N B = \sum \binom{N}{i} [(\mathrm{ad}A - \lambda)^{N-i} B] A^i\,.$$

Here $(\mathrm{ad}A)\, B \equiv [A, B]$, the commutator. Prove by inspection or induction. Now let us take $A = Z - m$, $B = E_\alpha$, and $\lambda = \alpha$, and apply this identity to f, noting that $\mathrm{ad}(Z - m) = \mathrm{ad}Z$:

$$[Z - (m + \alpha)]^N E_\alpha f = \sum \binom{N}{i} [(\mathrm{ad}Z - \alpha)^{N-i} E_\alpha]\, (Z - m)^i f\,. \tag{3.3}$$

Therefore, taking $N \geqq d_\alpha + n - 1$, and comparing with (3.2) and $(Z - m)^n f = 0$, we find that one of the exponents in each term is always big enough to annihilate that factor. Hence $[Z - (\alpha + m)]^N E_\alpha f = 0$, so that $E_\alpha f$ is a weight vector to weight $m + \alpha$, QED.

Now choosing bases in each of the root spaces we get a basis of the complex Lie algebra $\mathfrak{G}_c$ by (3.1). Applying all monomials in these basis elements an arbitrary number of times on f, we get an infinite set of weight vectors $\subset \mathscr{D}$, by what has just been proved above. The closure of the finite complex linear span $\mathscr{M}_f$ of this set is a sub-Hilbert space $\mathscr{H}_f = \overline{\mathscr{M}}_f$ which is a non-trivial[4] closed invariant subspace under $U(G)$. Because $U(G)$ is irreducible, $\mathscr{H}_f = \mathscr{H}$; in other words, the weight vectors of Z span $\mathscr{H}$, QED.

Consider the case that $z = x \in \mathfrak{G}$. Then since every weight is also an eigenvalue, all the weights of the skew adjoint operator X are pure

[3] This is implicit in JACOBSON [4]; Proposition 5, p. 64.

[4] Because it contains $f \neq 0$.

imaginary. Take any weight vector ϕ and corresponding pure imaginary weight m: $(X - m)^n = 0$. Then n vectors

$$\phi, A\phi, A^2\phi, \ldots, A^{n-1}\phi$$

span a finite-dimensional subspace V_ϕ of $\mathscr{H}$ invariant under the self-adjoint operator $A \equiv i(X - m)$. But A restricted to V_ϕ is both self-adjoint and nilpotent, thus vanishes, i.e., $A\phi = 0 \Leftrightarrow \phi$ is an eigenvector of X, QED[5]. This completes the proof of Lemma 1.

Note how we need the domain and range properties detailed in (a) of Sec. 2 to be able to carry through this proof.

Remarks

(a) Lemma 1 generalizes the familiar theorem that in any finite-dimensional irreducible representation of a complex semi-simple Lie algebra, the eigenvectors of any element of any Cartan subalgebra span the representation space V [5]. The generalization is in two directions: first, z need not belong to any Cartan subalgebra and, second, the representation may be infinite dimensional.

Besides the specialization to $z = x \in \mathfrak{G}$ mentioned in the Lemma, note the following specialization. If $z \in \mathfrak{C} \equiv$ some Cartan subalgebra of $\mathfrak{G}_c$, then ad$z$ is semi-simple[6], so that one can specialize to $d_\alpha = 1$ in the proof. Then it turns out that Z is semi-simple too, i.e., if it has a single weight vector, then its *eigenvectors* span $\mathscr{H}$.

(b) Note that Lemma 1 is actually a pure algebra theorem, does not need the strong hypothesis of a unitary group representation. Stated in its full generality, the theorem proved reads[7] as follows:

Theorem. *Given an irreducible representation R of any complex Lie algebra $\mathscr{L}_c$ (N.B., not necessarily semi-simple) on a Hilbert space such that $R(\mathscr{L}_c)$ is defined on a common dense domain $\mathscr{D} \subset \mathscr{H}$ invariant under R: $R(\mathscr{L}_c)\mathscr{D} \subset \mathscr{D}$. Let $z \in \mathscr{L}_c$, then if Z has a weight vector $\in \mathscr{H}$, its weight vectors span $\mathscr{H}$.*

(c) Remark on spectra. Since any generator X is a skew adjoint operator on $\mathscr{H}$ by Stone's theorem, its spectrum consists only of points of the point spectrum ("eigenvalues") or of the continuous spectrum [6]. Hence whenever we can show that X has no eigenvalue, the conclusion is that it has a pure continuous spectrum. Now Lemma 1 says that if X has any eigenvalue at all, then it has a pure point spectrum. Hence the alternatives for the spectrum of any generator are cut down to merely these two: pure point or pure continuous spectrum.

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[5] This argument that the weight vectors of X are all eigenvectors is due to N. LIMIĆ.

[6] See HELGASON [3]; the definition p. 137.

[7] In slightly different language this theorem asserts that if for any element $z \in \mathscr{L}_c$, Z has a weight vector $f \in \mathscr{H}$, then the representation is cyclic, with cyclic vector f. I thank H. DOEBNER for a conversation on this point.

4. Spectra of Elements of the Lie Algebra G

Let $B(x, y) = \mathrm{Tr}(\mathrm{ad}x\,\mathrm{ad}y)$ be the Killing form on G. The basic theorem is:

Theorem 1. *Given $x \in$ G, X has a pure point spectrum if x belongs to some maximal compact subalgebra* K; *if $B(x, x) \geqq 0$, X has a pure continuous spectrum.*

Proof. Given $x \in$ K, X generates a unitary representation $\exp tX$, $-\infty < t < \infty$, of a one-parameter subgroup $k(t)$ of $K \equiv$ connected subgroup of G generated by K. Since K is compact[8], $\exp tX$ is a unitary representation of a compact (abelian) group. Hence it is reducible into a *direct sum* of finite-dimensional irreducible unitary representations [7], [8], which must then be one-dimensional; in other words $\mathscr{H}$ = direct sum of eigenvectors of X to pure imaginary eigenvalues, QED.

Remark. Here the assumed existence of a representation of the *group* G, not just of the algebra G, was essential, so that we could bring into play the powerful theorems on the reduction of *group* representations.

Now given $B(x, x) \geqq 0$, we assume that X has an eigenvector and prove a contradiction.

First, we can easily get the expression

$$B(x, x) = \sum_{i=1}^{p} \alpha_i^2 d_{\alpha_i} \tag{4.1}$$

where the α_i are the roots of x and d_{α_i}, the dimensions of the corresponding root spaces G_{α_i}.

Proof. Take the basis of G_c such that $\mathrm{ad}x$ restricted to any root space G_α has the $d_\alpha \times d_\alpha$ matrix

$$\begin{pmatrix} \alpha & & & * \\ & \alpha & & \\ & & \alpha & \\ & 0 & & \ddots \\ & & & & \alpha \end{pmatrix}$$

where * denotes possibly non-zero elements. Then $\mathrm{Tr}(\mathrm{ad}x\,\mathrm{ad}x)$ restricted to $\mathrm{G}_\alpha = \alpha^2 d_\alpha$, QED.

Since $B(x, x) \geqq 0$ by hypothesis, if all the roots do not vanish then by (4.1) there is at least one root, say α, which is not pure imaginary. Now assume that X has an eigenvector $\in \mathscr{H}$. Then by Lemma 1 $\mathscr{H}$ is spanned by X's weight vectors and all weights are pure imaginary. Now there exists a weight vector f such that $E_\alpha f \neq 0$ for any non-zero $e_\alpha \in G_\alpha$, otherwise one would have $E_\alpha \mathscr{D} = 0$, contrary to the non-triviality assumption of Sec. 2. But if the (pure imaginary) weight of f is m, the weight of the

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[8] Cf. Helgason [3]; Theorem 1.1, p. 214. We are assuming that our semi-simple group G is connected and has a finite centre.

non-zero vector $E_\alpha f$ is $m + \alpha \neq$ pure imaginary, contradiction. The conclusion is that X has no eigenvector, therefore no point spectrum, therefore a pure continuous spectrum by remark (c) of Sec. 3, QED.

It remains to treat the case $B(x, x) = 0$ and all roots vanish, i.e., $\mathrm{ad}\, x$ is nilpotent. Assuming that X has an eigenvector, to weight m say, it follows just as before that $\mathscr{H}$ is spanned by X's weight vectors. But now, since all the roots are zero, all the weights $= m$, i.e., the operator $A \equiv X - m$ is skew adjoint and nilpotent on $\mathscr{H}$ in the sense that there is a basis ϕ_i, $i = 1, 2, \ldots$, of $\mathscr{H}$ such that $A^{n_i}\phi_i = 0$ for some integer n_i, $i = 1, 2, \ldots$. Just as in the proof of Lemma 1, we can then infer that $A\phi_i = 0$, i.e., $X =$ constant operator m on $\mathscr{H}$.

Now $\mathrm{ad}\, x \neq 0$ since G is semi-simple, therefore there exist non-zero elements $y, v \in$ G such that $(\mathrm{ad}\, x) v = y$. Represented on $\mathscr{H}$ this reads

$$(\mathrm{ad}\, X)\, V = [X, V] = Y .$$

But since $X =$ const on $\mathscr{H}$, $[X, V] = Y = 0$, contrary to the non-triviality assumption of Sec. 2, contradiction. Hence X has no eigenvector, thus a pure continuous spectrum in this case also. This completes the proof of Theorem 1, QED.

Remarks on Theorem 1

(a) Unfortunately Theorem 1 cannot be sharpened to an "if and only if" in terms of the Killing form alone. For although $B(x, x) < 0$ is necessary that X have a pure point spectrum, it is not sufficient. As a counter-example, consider the element

$$x = \mu i M_{12} + \lambda i M_{34}, \quad \mu, \lambda \neq 0 \text{ real}, \quad |\mu| > |\lambda| \tag{4.2}$$

of the Lie algebra **L** of the Lorentz group $L = SO(3, 1)$ in the usual (physical) basis $M_{\mu\nu} = -M_{\nu\mu}$, $\mu, \nu = 1, \ldots, 4$. One has the Killing form norms of iM_{12} and iM_{34} equal to -4 and $+4$ respectively, and they are orthogonal with respect to B. Hence $B(x, x) = -4(\mu^2 - \lambda^2) < 0$. In spite of the fact that $B(x, x) < 0$ it can be directly proved (Appendix) that x is *not compact* $\equiv$ belongs to no maximal compact subalgebra K of **L** or, equivalently, given any such K, there is no inner automorphism[9] of G which transforms x into K. Therefore one is not *guaranteed* via Theorem 1 that X has a pure point spectrum. On the other hand, one can directly prove that it has a pure continuous spectrum.

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[9] Incidentally, the Theorem 5—3 given in R. Hermann [9] is wrong. The last part of it states that if $\boldsymbol{K}'$ is a subalgebra of a semi-simple $\boldsymbol{G}$ such that B restricted to $\boldsymbol{K}'$ is negative definite, then there is an inner automorphism of $\boldsymbol{G}$ transforming $\boldsymbol{K}'$ into any maximal compact subalgebra $\boldsymbol{K}$. A counter example is given in the Appendix. Moreover, it is not clear how to fix up this theorem: for example, adding the hypothesis that $\boldsymbol{K}'$ is semi-simple does not seem to help. The further hypotheses must guarantee that the corresponding connected subgroup K' is compact, so that the analogue of Theorem 5—3 on the group level (Helgason [3]; Theorem 2.1, p. 218) can be applied.

Proof. iM_{12} belongs to a maximal compact subalgebra, while $B(iM_{34}, iM_{34}) = +4 > 0$, therefore by Theorem 1 they have respectively pure point and pure continuous spectra. Moreover, $[M_{12}, M_{34}] = 0$. Assume X has an eigenvector; then by Lemma 1 $\mathscr{H}$ is spanned by its eigenvectors, and since $[X, iM_{12}] = 0$ we can find a basis ϕ_i, $i = 1, 2, \ldots$, of $\mathscr{H}$ composed of common eigenvectors of X and iM_{12}. But then the ϕ_i are eigenvectors of $\lambda i M_{34} = X - \mu i M_{12}$, contradiction. Therefore X has a pure continuous spectrum, QED.

(b) We conjecture that Theorem 1 can be sharpened as follows: "X has a pure point spectrum if *and only if* $x \in$ some maximal compact subalgebra." This can be proved for the Lorentz group, for example, by means of a classification of all its one-parameter subgroups up to conjugacy under inner automorphism [10]. Representatives of all the noncompact subgroups with $B(x, x) < 0$ can be directly examined, and Theorem 1 gives that they all have pure continuous spectra. Any equivalent x' then also has a pure continuous spectrum since on the Hilbert space an inner automorphism of G corresponds to a unitary transformation, which preserves spectra (see Sec. 6).

For the general semi-simple group we expect that further Lie algebra invariants — beyond $B(x, x)$ — analogous to the higher Casimir invariants can be introduced by the method of WINTERNITZ et al., so that the compactness of x can be *characterized* by their values. Assuming that the above conjecture were true, the discreteness of the spectrum of X would then be characterized by certain values of these Lie algebra invariants. I thank P. WINTERNITZ for fruitful discussions on this point.

(c) The spectral conditions given in Theorem 1 (and conjectured in point (b)) should be useful in physics, because there one usually works with the Lie algebra in a definite basis x_i, $i = 1, 2, \ldots N$, with definite structure constants $c_{ij}{}^k$. The Killing form metric is then $g_{ij} = \sum_{l,k} c_{ik}{}^l c_{jl}{}^k$.

If the element in question is $x = \sum a^i x_i$, one has

$$B(x, x) = \sum a^i a^j g_{ij} .$$

Another form of Theorem 1 follows immediately from the above:

Corollary 1. *Given $x \in$ G, X has a pure continuous spectrum unless its roots are all pure imaginary.*

Proof. If the roots are either all zero or not all pure imaginary, then by the last part of the proof of Theorem 1 one sees that X cannot have an eigenvector, QED.

5. Spectra of „Self-Adjoint" Cartan Subalgebras of G_c

It is convenient to introduce the notion of a "*-map" of a complex Lie algebra $\mathscr{L}_c$, with the same properties which would arise if one had a skew-adjoint representation of one of its real forms on some Hilbert space. Thus we define: a $*$-map of the complex Lie algebra $\mathscr{L}_c$ is an antilinear map which is involutory and reverses the Lie bracket. I.e.,

$$(aw + bz)^* = \bar{a}w^* + \bar{b}z^* \qquad a, b \text{ complex numbers}; \; w, z \in \mathscr{L}_c \qquad (5.1)$$
$$(z^*)^* = z, \quad [wz]^* = [z^* w^*] \,.$$

It can be shown that $*$-maps are associated $1-1$ with real forms $\mathscr{L}$ of $\mathscr{L}_c$, that is $* \leftrightarrow \mathscr{L}$ if and only if $x^* = -x$ for all $x \in \mathscr{L}$[10]. Now in any unitary representation of our group G, the Hilbert space adjoint acting on the complexified generator algebra $\mathfrak{G}_c$ represented on $\mathscr{H}$ has the same properties as (5.1). Hence the unitary representation determines uniquely[11] a $*$-map of $\mathfrak{G}_c$ and we can use the same symbol $*$ for both this $*$-map and the Hilbert space adjoint on the representing operators. This provides a rigorous justification for neglecting the difference between an element z of $\mathfrak{G}_c$ and its image Z in $U(\mathfrak{G}_c)$ in the following theorems.

Consider a Cartan subalgebra $\mathfrak{C} \subset \mathfrak{G}_c$ which is fixed under the $*$-map given by U, i.e., a *self-adjoint* Cartan subalgebra:

$$\text{Def. } \mathfrak{C}^* = \mathfrak{C} \Leftrightarrow h \in \mathfrak{C} \Rightarrow h^* \in \mathfrak{C} \,. \qquad (5.2)$$

It can easily be proved that $\mathfrak{C}^* = \mathfrak{C}$ if and only if one can find a basis of $\mathfrak{C}$ which belongs to $\mathfrak{G}$, i.e., if $\mathfrak{C}$ is just the complexification of a Cartan subalgebra $\mathfrak{C}_G$ of $\mathfrak{G}$[12]. This also rests on Lemma 2, which we now give.

Lemma 2. *If $z \in \mathfrak{G}_c$, $Z^* = -Z$ if and only if $z \in \mathfrak{G}$.*

Proof. The proof is an immediate consequence of the non-triviality assumption that the representation of the Lie algebra $\mathfrak{G}$ is faithful ($\equiv$ the kernel is zero).

Let $\alpha_1, \alpha_2, \ldots \alpha_r$ be the non-zero roots and e_{α_i}, elements of the root spaces $\mathfrak{G}_{\alpha_i}$. Thus

$$[h e_{\alpha_i}] = \alpha_j(h) e_{\alpha_i}, \quad h \in \mathfrak{C} \qquad (5.3)$$

for $j = 1, 2, \ldots r$. Then we have

Theorem 2. *Given a self-adjoint Cartan subalgebra $\mathfrak{C} \subset \mathfrak{G}_c : \mathfrak{C}^* = \mathfrak{C}$. If $h \in \mathfrak{C}$ and its roots $\alpha_j(h)$ are real, then H has a pure continuous spectrum unless $h^* = h$.*

Proof. We introduce a basis of $\mathfrak{C}: h_i$, $i = 1, \ldots l$, such that $i h_i \in \mathfrak{G}$, as we may by the remark above. Let

$$h = \sum a_i h_i, \quad \text{thus} \quad \alpha_j(h) = \sum_{i=1}^{l} a_i \alpha_j(h_i) \quad j = 1, 2, \ldots r \,. \qquad (5.4)$$

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[10] Thus there are an infinite number of *different* $*$-maps of a given $\mathscr{L}_c$, because not only do two non-isomorphic real forms give different $*$'s but also the infinite number of isomorphic real forms of any one type.

[11] This reduces to showing that given $Z \in U(G_c)$, then $Z^* = -Z$ if and only if $Z \in U(G)$. This is a consequence of Lemma 2 below.

[12] *Any* Cartan subalgebra C of G_c is conjugate to a self-adjoint one by an inner automorphism of G_c, however, in general $C^* \neq C$. See further remarks in Sec. 6.

Assume that H has an eigenvector $f \in \mathscr{D}$ and eigenvalue. But since the H_i are (strongly) commuting self-adjoint operators, this will be so if and only if all the H_i appearing in first sum (5.4) have this eigenvector f and corresponding points m_i of their point spectra, thus by Lemma 1 if and only if they all have pure point spectra. So let the sums go from 1 to $l' \leqq l$, where all the a_i are now non-zero; therefore, by the above, all the $\alpha_j(h_i)$, $i = 1, \ldots l'$, appearing in the second sum are real for $j = 1, 2 \ldots r$. But since the $\alpha_j(h)$ are real, this forces all the a_i to be real.

Proof. One gets immediately

$$\sum_{i=1}^{l'} (\operatorname{Im} a_i)\, \alpha_j(h_i) = 0, \quad j = 1, \ldots r .$$

Since the simple roots α_j, $j = 1, \ldots, l \geqq l'$, are linearly independent[13], this has only the trivial solution $\operatorname{Im} a_i = 0$, q.e.d. But then $h^* = \sum \bar{a}_i h_i^* = \sum a_i h_i = h$, QED. This completes the proof of Theorem 2.

A sort of a converse follows immediately:

Corollary 2. *Let* $\mathfrak{C}$ *be any Cartan subalgebra of* $\mathfrak{G}_c$ *and* α_j, $j = 1, 2, \ldots r$, *the corresponding non-zero roots. If* $h \in \mathfrak{C}$ *and* $h^* = h$, *then* H *has a pure continuous spectrum unless the* $\alpha_j(h)$ *are all real.*

Proof. Then $ih \in \mathfrak{G}$ by Lemma 2, and use Cor. 1, QED.

6. Spectra of Other Elements of $\mathfrak{G}_c$

In practice one is interested only in the spectra of Cartan subalgebras of $\mathfrak{G}_c$, so we shall confine ourselves to those elements. So given a general Cartan subalgebra $\mathfrak{C} \subset \mathfrak{G}_c$ what can we say about the spectra of its elements ? For $h \in \mathfrak{C}$, $H = J + iL$ where J and L are self-adjoint operators on $\mathscr{H}$, in general unbounded and non-commuting. Not much is known about the spectra of such operators. Even though we have algebraic criteria above which may decide whether J and L have pure point or pure continuous spectra, we can say little about the type of spectrum of H.

Let us try another way. One knows that $\mathfrak{C}$ is conjugate to any other Cartan subalgebra of $\mathfrak{G}_c$ by an inner automorphism[14] of $\mathfrak{G}_c$, thus in particular to any self-adjoint Cartan subalgebra, for whose spectra we have the criteria above. Let then

$$\mathfrak{C} = (\exp \operatorname{ad} z)\, \mathfrak{C}', \quad \mathfrak{C}'^* = \mathfrak{C}', \ z \in \mathfrak{G}_c \tag{6.1}$$

for example.

Now notice that if any two elements x', $x \in \mathfrak{G}$ are connected by an inner automorphism of $\mathfrak{G}$, they have the same type of spectrum.

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[13] Jacobson [4]; chapter IV, paragraph 3.

[14] Jacobson [4]; Theorem 3, p. 273.

Proof. It is sufficient to treat the case

$$x' = (\exp \operatorname{ad} y)\, x, \quad y \in \mathfrak{G}\,. \tag{6.2}$$

In proof realize that on $\mathscr{H}$ (6.2) reads[15]

$$X' = e^Y X e^{-Y} \tag{6.3}$$

where $\exp(\pm Y)$ are unitary operators, whence the skew adjoint X' and X have the same spectra [11], q.e.d.

But one cannot infer that H and H' such that h and h' are connected by (6.1) have the same spectra. We know this because it is easy to find (even self-adjoint) Cartan subalgebras which have different type spectra. The failure of the inner automorphism (6.1) of $\mathfrak{G}_c$ to preserve the spectrum may be due to either of the following possibilities: (1) $\exp \operatorname{ad} z$ may not be "implementable" to a relation like (6.3) on $\mathscr{H}$, that is, $\exp(\pm Z)$ may not "exist" as an operator on $\mathscr{H}$, more precisely, may not have the dense domain $\mathscr{D}$[16]. (2) $\exp(\pm Z)$ may "exist" in this sense but, being in general non-unitary and even unbounded, it may not preserve the spectrum of a self-adjoint operator.

The upshot is that the connection (6.1) of our general Cartan subalgebra $\mathfrak{C}$ with a self-adjoint one can teach us nothing about the spectra of elements of $\mathfrak{C}$ on the basis of the theorems presented here if $\mathfrak{C}^* \neq \mathfrak{C}$.

7. Application

For a simple example, take the conformal space-time group $\cong SO(4,2)$ and let $M_{ab} = -M_{ba}$ $(a, b = 0, 1, \ldots, 5)$ be the usual (in physics) generators relative to co-ordinates ξ^a such that the fundamental quadratic form is $\boldsymbol{\xi}^2 - (\xi^4)^2 - 2\xi^0\xi^5$. One can introduce the self-adjoint Cartan subalgebra $\mathfrak{C} =$ complex span $\{h_1 \equiv M_{12}, h_2 \equiv iM_{34}, h_3 \equiv iM_{05}\}$, for which the non-zero roots $\alpha_j(h_i)$ are

$$\alpha_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad \alpha_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \alpha_4 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \alpha_5 = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \alpha_6 = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

with the matrix convention

$$\alpha = \begin{pmatrix}\alpha(h_1)\\ \alpha(h_2)\\ \alpha(h_3)\end{pmatrix} \tag{7.1}$$

and their negatives. Thus all roots are real. Since $h_1^* = h_1$, $h_2^* = -h_2$, $h_3^* = -h_3$, one can immediately infer from Theorems 1 and 2 what sort of spectra the h_i have in any non-trivial irreducible unitary representa-

[15] JACOBSON [4]; top p. 282.

[16] According to H. DOEBNER, it is very probably provable that possibility (1) cannot arise.

tion of $SO(4, 2)$. In particular $h_3 = iM_{05}$, which has the geometrical interpretation of the generator of space-time dilatations, has a pure continuous spectrum in any such representation of the conformal group.

Consider also $iM_{5\mu} = ip_\mu$, which generates translations, i.e., p_μ is the four-momentum. Using the structure constants one computes

$$B(ip_\mu, ip_\mu) = 0, \quad \mu = 1, \ldots, 4;$$

in fact right from the Lie commutation relations one sees that the $\mathrm{ad}(ip_\mu)$ are nilpotent. Thus from Theorem 1 the momenta have pure continuous spectra in any unitary representations of this type.

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Appendix

Since any inner automorphism of $\mathbf{L}$ is equivalent to a Lorentz transformation on the tensor indices of $M_{\mu\nu}$, the question is whether there exists a Lorentz transformation $L^\mu{}_\nu$ such that if $M'^{\mu\nu} \equiv L^\mu{}_\lambda L^\nu{}_\xi M^{\lambda\xi}$, then $\mu M^{12} + \lambda M^{34}$ = real linear combination of the *spatial* components M'^{ij}, $i, j = 1, 2, 3$. This because one knows that *any* maximal compact subalgebra is generated by the spatial components M'^{ij} in some Lorentz frame. But if we could attain this form, by a further spatial rotation we could align this axial three-vector along the z-axis say, and divide out the non-zero real coefficient. Hence it suffices to show that there is no Lorentz transformation $L^\mu{}_\nu$ such that

$$M'^{12} \equiv L^1{}_\mu L^2{}_\nu M^{\mu\nu} = \mu M^{12} + \lambda M^{34}$$
$$\mu, \lambda \text{ real}, \neq 0 . \tag{A.1}$$

Set $L^1{}_\mu \equiv X_\mu$, $L^2{}_\mu \equiv Y_\mu$ for convenience; then (A.1) is equivalent to

$$\begin{aligned} &(a)\ X_1Y_2 - X_2Y_1 = \mu \qquad &&(d)\ X_1Y_4 - X_4Y_1 = 0 \\ &(b)\ X_3Y_4 - X_4Y_3 = \lambda \qquad &&(e)\ X_2Y_3 - X_3Y_2 = 0 \\ &(c)\ X_1Y_3 - X_3Y_1 = 0 \qquad &&(f)\ X_2Y_4 - X_4Y_2 = 0 \end{aligned} \tag{A.2}$$

Proof. Consider (c) and (d). If all quantities there were non-zero, by division of (c) by (d) and cross multiplication we would get $X_3Y_4 - X_4Y_3 = 0$, $\asymp\!\!\!\!\times$ ($\equiv$ contradiction) with (b), since $\lambda \neq 0$. Therefore at least one of X_1, X_3, X_4, Y_1, Y_3, Y_4 must vanish if $L^\mu{}_\nu$ exists.

Case $X_1 = 0$. Then $X_3Y_1 = 0$ from (c). But $Y_1 \neq 0$ from (a) since $\mu \neq 0$, therefore $X_3 = 0$. From (d) $X_4 = 0$. But then $\asymp\!\!\!\!\times$ with (b).

Case $Y_3 = 0$. Then $X_3Y_1 = 0$ from (c). But $X_3 \neq 0$ from (b) since $\lambda \neq 0$, thus $Y_1 = 0$. From (a) $X_1 \neq 0$ since $\mu \neq 0$. Thus from (d) $Y_4 = 0$. But $\asymp\!\!\!\!\times$ with (b).

By similar arguments the other four cases likewise give contradictions. Hence $L^{\mu}{}_{\nu}$ does not exist, QED.

Note that we have proved something actually stronger: the element on the right hand of (A.1) can never be transformed into the single component $\theta M'^{\mu\nu}$, θ real $\neq 0$, any μ, ν, by *any* linear transformation of space-time.

The same conclusion, namely that the element (4.2) generates a non-compact subgroup for any $\mu, \lambda \neq 0$, is a special case of the general results of Ref. 15.

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