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The Schrödinger Equation for Quantum Fields with Nonlinear Nonlocal Scattering*

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Abstract. This paper considers perturbations $H = H_0 + \varepsilon V$ of the Hamiltonian operator H_0 of a free scalar Boson field. V is a polynomial in the annihilation creation operators. Terms of any order are allowed in V, but point interactions, such as $\int : \theta(x)^4 : dx$, are not considered. Unnormalized solutions for the Schrödinger equation are found. For $\varepsilon \to 0$, these solutions have a partial asymptotic expansion in powers of ε. The set of all possible pertubation terms V forms a Lie algebra. General properties of this Lie algebra are investigated.

§ 1. Introduction

We consider Hamiltonian operators of the form

$$H = H_0 + V \tag{1.1}$$

where H_0 is the Hamiltonian for a free field and V is a polynomial in the creation annihilation operators $A^{\pm}$. By this we mean that V is a finite sum of monomials V_{lm} of the form

$$V_{lm} = \int A^+(k_1) \ldots A^+(k_l) v_{lm}(k, k') A^-(k_1') \ldots A^-(k_m') dk\, dk' . \tag{1.2}$$

We require the kernel v_{lm} to be smooth, for example to be in a Schwartz space $\mathscr{S}$. This paper is partly directed toward studying the Lie algebra formed by such H, and it is partly directed toward solving the Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi = H \Psi . \tag{1.3}$$

We solve (1.3) for quite general V of the above form. (See Theorems 7.3 and 9.1.) We find in § 7 a preliminary operator T which intertwines H and H_0,

$$H T = T H_0 . \tag{1.4}$$

Then

$$T \exp(-it H_0) \Phi(0) = \Psi(t)$$

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is a solution of (1.3) with Cauchy data $\Psi(0) = T\Phi(0)$. We have not been able to identify the formal expression for T^{-1} with an operator on any reasonable function space, and so we cannot specify "arbitrary" Cauchy data in (1.3). More serious defects in this T are the following: (a) It seems that the associated scattering is trivial. (b) In perturbation theory (where V is replaced by εV), T contains powers of ε^{-1} as well as powers of ε. In fact, T is a sum of terms $T_{lm} = T_{lm}(\varepsilon)$ such as (1.2) and each

$$T_{lm}(\varepsilon) = \sum\nolimits_{j \geqq J(l,m)} \varepsilon^j T_{lm}^{(j)}$$

has a pole of order $-J(l, m)$ depending on l and m.

Combining this type of T with familiar arguments from perturbation theory, we find in § 9 new T's which have the property that $T_{lm}(\varepsilon)$ is analytic in ε if $l \leqq N$. Here N is a finite number which can be chosen in advance to be as large as desired. In order to do this we must make a finite renormalization of H. The resulting scattering appears to be nontrivial, and could probably be computed, using methods from [2]. This operator T leads to solutions $\Psi = \Psi(t, \varepsilon)$ of the renormalized Schrödinger equation for which

$$\psi_0(t, \varepsilon), \ldots, \psi_N(t, \varepsilon)$$

depend analytically on ε. Here ψ_j is the j particle component of Ψ. The solution Ψ will presumably not be normalized, and will exist as an element of a space larger than the Fock Hilbert space. These results should be compared to Friedrichs' ideas [2]. In his terminology we have considered the case of a totally smooth interaction.

In § 2 and § 3 we realize our operators (1.1) as bounded operators on Frechet spaces, cf. [5]. In § 3 we prove that the Lie algebra of V's in (1.1) has a trivial ideal theory. The only ideal in this Lie algebra is the one dimensional ideal,

$$\mathfrak{z} = \{V : V = V_{00}\}$$

which is its center. The significance for us of this result is primarily negative. Nontrivial ideals would simplify the search for our intertwinting operator T. We will later find T as a product, $T = T_1 T_2$ where in T_1 (respectively T_2) the creation (respectively annihilation) operators dominate. Thus

$$(T_1)_{lm} = 0 \text{ if } l < m \tag{1.5}$$

$$(T_2)_{lm} = 0 \text{ if } l > m\,. \tag{1.6}$$

The operators of the form

$$V = \sum\nolimits_{l > m} V_{lm} \tag{1.7}$$

form a subalgebra which in a generalized sense is nilpotent. In this algebra there are many ideals and the system of equations for the $(T_i)_{lm}$ can be solved successively.

The algebra of all the V's is an algebra which, in a crude sense, is similar to the reductive Lie algebra $\mathfrak{gl}(n, C)$ of $n \times n$ complex matrices; the T_1 and T_2 correspond to the triangular matrices.

Since the Lie algebras we consider are infinite dimensional, it is not clear which topologies should be placed on them. In § 4 we have some results which show that one natural choice for a topology does not seem to be suitable. In § 5 we choose a better topology. For a V of the form (1.7) the exponential map converges to an element of the Lie algebra. Thus we can identify the corresponding infinite dimensional Lie group (Theorem 5.6). This group acts by inner automorphisms on the full Lie algebra, and one of the group elements, namely T_1, puts H in "triangular form". These considerations lead to our first T in § 7. The results of § 2–7 do not depend very strongly on the class of function spaces we have considered. We could replace spaces of type $\mathcal{S}$ by spaces of rapidly decreasing continuous functions.

Sections 8 and 9 are devoted to the process of removing the poles from $T = T(\varepsilon)$, as mentioned above.

§ 2. The operators V and their domains

Let k_i be a vector in Euclidean three space. (We never refer to the components of k_i.) Let μ be a positive number and define

$$\omega(k_i) = (|k_i|^2 + \mu^2)^{1/2},$$

$$H_0 = \int A^+(k)\,\omega(k)\,A^-(k)\,dk .$$

Here A^+ and A^- are the standard creation and annihilation operators for a scalar Boson field. This choice of statistics does not appear to be essential.

Let $\mathfrak{H}$ be the set of sequences

$$\Phi = \{\varphi_0, \varphi_1, \ldots\}$$

where φ_0 is a complex number and φ_n, for $n \geqq 1$, is a symmetric function of the variables $k_1, \ldots, k_n$ and where

$$\varphi_n \in \mathcal{S} .$$

We give $\mathfrak{H}$ the product topology. Then a net $\Phi^{(n)}$ converges if there is convergence $\varphi_j^{(n)} \to \varphi_j^{(\infty)}$ in each term. Let $\mathfrak{H}_0$ be the subset of $\mathfrak{H}$ consisting of those sequences Φ for which $\varphi_n = 0$ for all sufficiently large n. Let

$$\mathfrak{H}^n = \{\Phi\colon \varphi_j = 0 \text{ if } j > n\} .$$

Then $\mathfrak{H}^n$ is a finite direct sum of spaces of type $\mathcal{S}$, and this defines a topology in $\mathfrak{H}^n$. Also

$$\mathfrak{H}_0 = \cup_n \mathfrak{H}^n$$

$$\mathfrak{H}^n \subset \mathfrak{H}^{n+1} .$$

We give $\mathfrak{D}_0$ the inductive limit topology. A linear functional L or a linear transformation T defined on $\mathfrak{D}_0$ is continuous if and only if the restriction

$$L \mid \mathfrak{D}^n$$

or

$$T \mid \mathfrak{D}^n$$

is continuous for each n.

Let $\mathfrak{u}$ be set of all formal sums

$$V = \Sigma\, V_{lm}$$

where V_{lm} is given by (1.2), v_{lm} is in $\mathscr{S}$ and v_{lm} is symmetric in the variables

$$k_1, \ldots, k_l = k$$

and in the variables

$$k'_1, \ldots, k'_m = k' .$$

We give $\mathfrak{u}$ the product topology. Let

$$\mathfrak{u}^n = \{V\colon V \in \mathfrak{u},\ V_{lm} = 0 \text{ if } l + m > n\}$$
$$u_0 = \cup_n\, \mathfrak{u}^n .$$

The above equation defines a topology in $\mathfrak{u}_0$ as an inductive limit of the spaces $\mathfrak{u}^n$.

Let $L(X, Y)$ be the set of continuous linear transformations from a vector space X to a vector space Y.

Theorem 2.1. We have the following inclusions:

$$\mathfrak{u}_0 \subset L(\mathfrak{D}_0, \mathfrak{D}_0) \tag{2.1}$$
$$\mathfrak{u}_0 \subset L(\mathfrak{D}, \mathfrak{D}) \tag{2.2}$$
$$\mathfrak{u} \subset L(\mathfrak{D}_0, \mathfrak{D}). \tag{2.3}$$

In each case the topology in $\mathfrak{u}_0$ or in $\mathfrak{u}$ is stronger than the corresponding topology of uniform convergence on bounded sets.

Proof. A bounded set B is a set with the property that for any neighborhood U of zero there corresponds a $\lambda > 0$ with

$$\lambda B \subset U .$$

The bounded sets in $\mathfrak{D}_0$ are the sets which are bounded subsets of $\mathfrak{D}^n$, for some n. In view of this and the definition of the topology in $\mathfrak{D}_0$, we can replace the first and last inclusions to be proved by

$$\mathfrak{u}_0 \subset L(\mathfrak{D}^n, \mathfrak{D}_0) \tag{2.4}$$
$$\mathfrak{u} \subset L(\mathfrak{D}^n, \mathfrak{D}) , \tag{2.5}$$

$n = 1, 2, \ldots$ Given a Φ in $\mathfrak{D}^n$, an integer $j \geqq 0$ and a V in $\mathfrak{u}_0$ or in $\mathfrak{u}$,

$$(V\Phi)_j \tag{2.6}$$

depends continuously on $\varphi_0, \ldots, \varphi_n$ and if $V \in \mathfrak{u}^k$ for some given k, then

$$V\Phi \in \mathfrak{D}^{n+k}.$$

Thus the set theoretic inclusions (2.4) and (2.5) hold. The term (2.6) depends only on the V_{lm} with $j - n \leqq l \leqq j$, $m \leqq n$. Let a bounded set B in $\mathfrak{D}^n$ be given. (2.6) can be made small uniformly for Φ in B by requiring that these V_{lm} be sufficiently small. This proves (2.5). Let U_j be a neighborhood of zero in the Schwartz space for the variables $k_1, \ldots, k_j$. The sets of the form

$$U = \{\Phi : \varphi_j \in U_j, j = 0, 1, \ldots\}$$

form a fundamental system of neighborhoods for $\mathfrak{D}_0$. Given such a sequence U_j we can find a sequence U_{lm} such that

$$V_{lm} \in U_{lm}$$

and

$$\Phi \in B$$

imply

$$(V\Phi)_j \in U_j, j = 1, 2, \ldots$$

This proves (2.4).

Let V be given in $\mathfrak{u}^k$ for some k. Then (2.6) depends on φ_n only when $n \leqq j + k$. This proves the set theoretic inclusion (2.2). As a bounded set B in $\mathfrak{D}$ we can take a set of the form

$$B = \{\Phi : \varphi_j \in B_j\}$$

where each B_j is a bounded set in $\mathscr{S}$. Given a j and a neighborhood U_j of zero, we find a neighborhood U_{lm} of zero such that

$$\varphi_k \in B_k, \; V_{lm} \in U_{lm}$$

imply

$$(V\Phi)_j \in U_j .$$

This is possible since $(V\Phi)_j$ depends on a product

$$V_{lm}\varphi_k$$

only for $l \leqq j$ and $k - j \leqq m \leqq k$. This proves (2.2).

§ 3. The structure of $\mathfrak{u}_0$ and $\mathfrak{u}$

Theorem 3.1. Let $P, Q \in \mathfrak{u}_0$ and let $R \in \mathfrak{u}$. Then

$$PQ \in \mathfrak{u}_0$$

$$PR, RP \in \mathfrak{u} .$$

It follows from this theorem that $\mathfrak{u}_0$ is an algebra and also a Lie algebra with the bracket

$$[P, Q] = PQ - QP .$$

Furthermore, for $R \in \mathfrak{u}$,

$$\operatorname{ad} P(R) = [P, R] = PR - RP$$

is defined.

Definition. $\mathfrak{m}$ is a closed ideal in $\mathfrak{u}$ if $\operatorname{ad} A\,\mathfrak{m} \subset \mathfrak{m}$ for each A in $\mathfrak{u}_0$ and if $\mathfrak{m}$ is a closed subset of $\mathfrak{u}$.

Let $\mathfrak{z} = \{V : V \in \mathfrak{u}_0,\ V = V_{00}\}$. One can see that $\mathfrak{z}$ is a closed ideal in $\mathfrak{u}$, that $\mathfrak{z}$ is the center of $\mathfrak{u}_0$ and that the elements of $\mathfrak{z}$ act as multiples of the identity operator I on the domains $\mathfrak{D}_0$ and $\mathfrak{D}$.

Theorem 3.2. $\mathfrak{z}$ is the only nontrivial closed ideal in $\mathfrak{u}$.

Proof of Theorem 3.1. A product $P_{jk}Q_{lm}$ does not have the right form to be in $\mathfrak{u}_0$ since k annihilators from P_{jk} precede l creators from Q_{lm}. However, by use of the commutator identity

$$[A^-(k), A^+(k')] = \delta(k - k')\,, \tag{3.1}$$

we can interchange the order of an A^- and an A^+. Each such interchange leads to a new term with the A^+ and A^- replaced by a δ function. If we perform the integration corresponding to the variables of the δ function the result is an operator with a smooth kernel. Thus $PQ \in \mathfrak{u}_0$, and more precisely we have proved

Lemma 3.3. For some choice of S and T in $\mathfrak{u}_0$ we have

$$P_{l_1 m_1} Q_{l_2 m_2} = \sum\nolimits_{0 \leq r \leq \min\{m_1, l_2\}} S_{l_1 + l_2 - r,\, m_1 + m_2 - r}$$

$$[P_{l_1 m_1}, Q_{l_2 m_2}] = \sum\nolimits_{1 \leq r \leq J} T_{l_1 + l_2 - r,\, m_1 + m_2 - r}$$

where

$$J = \max\{\min\{m_1, l_2\}, \min\{m_2, l_1\}\}\,.$$

Let $P \in \mathfrak{u}^n$ for some n and let a j and a k be given. It follows from Lemma 3.3 that $(PR_{lm})_{jk} = 0 = (R_{lm}P)_{jk}$ if $l + m > j + k + n$. Thus $(PR)_{jk}$ and $(RP)_{jk}$ are finite sums of elements of $\mathfrak{u}_0$ and so are in $\mathfrak{u}_0$. Hence PR and RP are in $\mathfrak{u}$.

Proof of Theorem 3.2. First we show that the closed ideal generated by any

$$V_{01} = \int v(k) A^-(k)\, dk \qquad (v \neq 0)$$

is all of $\mathfrak{u}$. If $\varphi \in \mathscr{S}$ then one can compute

$$[\int A^+(k)\bar{v}(k)\,\varphi(k') A^-(k)\, dk\, dk',\ vA^- dk] = -\|v\|^2 \int \varphi A^- dk\,.$$

Thus $\int \varphi A^- dk$ is in the ideal generated by $\int v A^- dk$. Now let g_{lm} be a kernel with finite rank and let

$$g_{l+1,m} = Sym_k\, \varphi(k_1)\, g(k_2, \ldots, k_{l+1}, k')\,.$$

If φ is suitably chosen then

$$[G_{l+1,m}, \int \varphi A^- dk] = -\|\varphi\|^2 G_{lm},$$

where the G's are the operators with kernels g. Thus the ideal contains all operators in $\mathfrak{u}_0$ whose kernels have finite rank and since this is dense in $\mathfrak{u}$, the ideal is all of $\mathfrak{u}$.

Similarly one proves that the closed ideal generated by $\int vA^+dk$ is $\mathfrak{u}$.

Now let $\mathfrak{m}$ be a closed ideal in $\mathfrak{u}$ which is not contained in $\mathfrak{z}$. We will prove that $\mathfrak{m}$ contains an element of the form $\int vA^{\pm}dk$ with $v \neq 0$. This will prove the theorem. Let M be an element of $\mathfrak{m}$ which is not in $\mathfrak{z}$. Let

$$N = \int A^+(k)\delta(k-k')A^-(k')dk\,dk' .$$

By a limiting procedure one proves that $[N, \mathfrak{m}] \subset \mathfrak{m}$. Furthermore for any polynomial p,

$$p(\mathrm{ad}N)\mathfrak{m} \subset \mathfrak{m} .$$

One computes that

$$[N, M_{lm}] = (l-m)M_{lm} .$$

If we choose $p = p_{n,r}$ so that

$$p(r) = 1$$

$$p(k) = 0 \text{ if } k \in Z, k \neq r, |k| \leqq n ,$$

then

$$\lim_n p_{nr}(\mathrm{ad}N)M = \textstyle\sum_{l-m=r} M_{lm} \in \mathfrak{m} .$$

Thus we can suppose that M has the form $\sum_{l-m=r} M_{lm}$ for some r. Each operator $\mathrm{ad}\int vA^{\pm}dk$ applied to M removes an $A^{\mp}(k)$ from each term of M and replaces it with a $\mp v(k)$. If products of the operators $\mathrm{ad}\int vA^{\pm}dk$ are applied to M, we can successively remove annihilation-creation operators from M and for suitably chosen v's we achieve the following result. Either there is an $M = M_{10} \neq 0$ in $\mathfrak{m}$ (and so the proof is finished) or there is an M in $\mathfrak{m}$ of the form

$$M = \textstyle\sum_l M_{l,l+1};\ M_{01} \neq 0 .$$

Let φ be an element of $\mathscr{S}$ with unit norm and let v_{lm} be a kernel. We can write v_{lm} uniquely as a sum

$$v_{lm} = \textstyle\sum_{\alpha,\beta=0}^{l,m} \mathrm{Sym}_k \mathrm{Sym}_{k'} \varphi^{-\alpha\otimes} v_{l-\alpha,m-\beta}\varphi^{\beta\otimes} , \tag{3.2}$$

where $v_{l-\alpha,m-\beta}$ is a kernel orthogonal to φ in the following sense:

$$\begin{aligned} 0 &= \int \varphi(k_1)v_{l-\alpha,m-\beta}(k_1,\ldots,k')dk_1 \\ &= \int \varphi(k_1')v_{l-\alpha,m-\beta}(k_1,\ldots,k'_{m-\beta})dk_1', \end{aligned}$$

and where $\varphi^{\alpha\otimes} = \varphi\otimes\cdots\otimes\varphi$ is an α-fold tensor product. Let $v_{lm}^{\alpha\beta}$ be the α, β term in this sum and let $V_{lm}^{\alpha\beta}$ be the corresponding operator. Let

$$P(\varphi) = \int A^+\bar{\varphi}\otimes\varphi A^- dk\,dk' .$$

One can compute

$$[P(\varphi), V_{lm}^{\alpha\beta}] = (\alpha-\beta)V_{lm}^{\alpha\beta} .$$

We choose φ proportional to m_0, the kernel of M_{01}. We use the $p_{n,-1}$ defined above. We have

$$\lim_n p_{n,-1}(\operatorname{ad} P(\varphi))M = \sum M_{l,l+1}^{\alpha,\alpha+1} \in \mathfrak{m} .$$

Since $M_{01} = M_{01}^{01}$, we can suppose that M has the form

$$M = \sum M_{l,l+1}^{\alpha,\alpha+1} ; \quad M_{01} \neq 0 .$$

We return to a general φ and we let

$$B = \operatorname{ad} \int \varphi \otimes \varphi A^- A^- dk' \operatorname{ad} \int A^+ A^+ \bar{\varphi} \otimes \bar{\varphi} dk$$

$$C = \operatorname{ad} \int \varphi A^- dk' \operatorname{ad} \int A^+ \bar{\varphi} dk .$$

One can compute

$$[B, V^{\alpha\beta}] = 4\{-(\alpha+1)\beta V^{\alpha\beta} + \alpha\beta(\beta-\alpha-2) W^{\alpha-1,\beta-1} + \\ + \alpha(\alpha-1)\beta(\beta-1) X^{\alpha-2,\beta-2}\} .$$

Here $V = V_{lm}$ has its kernel given by (3.2) and $W^{\alpha-1,\beta-1}$ and $X^{\alpha-2,\beta-2}$ have kernels

$$w^{\alpha-1,\beta-1} = \operatorname{Sym}_{k,k'} \varphi^{(\alpha-1)\otimes} v_{l-\alpha,m-\beta} \varphi^{(\beta-1)\otimes}$$

$$x^{\alpha-2,\beta-2} = \operatorname{Sym}_{k,k'} \varphi^{(\alpha-2)\otimes} v_{l-\alpha,m-\beta} \varphi^{(\beta-2)\otimes} .$$

Also

$$[C, V^{\alpha\beta}] = \alpha\beta W^{\alpha-1,\beta-1}$$

$$[C, [C, V^{\alpha\beta}]] = \alpha(\alpha-1)\beta(\beta-1) X^{\alpha-2,\beta-2}$$

It follows that for some linear combination D of $\operatorname{ad} B$, $\operatorname{ad} C$ and $(\operatorname{ad} C)^2$, we have

$$DM = \sum (\alpha+1)^2 M_{l,l+1}^{\alpha,\alpha+1} \in \mathfrak{m} .$$

$$\lim_n p_{n,1}(D)M = \sum M_{l,l+1}^{01} \in \mathfrak{m} .$$

We write $m_{l,l+1}^{01} = \operatorname{Sym}_{k'} q_{l,l} \otimes m_{01}$, for some kernels $q_{l,l}$ which are orthogonal to m_{01}. We choose an orthonormal base $\varphi_0, \varphi_1, \ldots$ of $\mathscr{S}$ consisting of Hermite functions. Without loss of generality $m_{01} = \varphi_0$. The kernels q_{ll} can be written as an infinite series of tensor products of the φ_j with convergence in $\mathscr{S}$, see [5, § 2] for example. Thus if we choose an integer $K = K(L)$ sufficiently large, we will have

$$q_{ll} = r_{ll} + s_{ll}, \text{ all } l \leqq L$$

where s_{ll} is small in $\mathscr{S}$, each term of r_{ll} contains at least one factor from the set

$$\{\varphi_1, \ldots, \varphi_K\} ,$$

and where s_{ll} is independent of $\varphi_1, \ldots, \varphi_K$ (and of φ_0 also).

The expansion of the kernels leads to a corresponding expansion of the $M^{01}_{l,l+1}$. Each such term is an eigenvector for $\operatorname{ad} P(\varphi_j)$ with eigenvalue

$$\{\text{number of times } \varphi_j \text{ occurs in the first } l \text{ variables}\} -$$
$$- \{\text{number of times } \varphi_j \text{ occurs in the last } l+1 \text{ variables}\} .$$

Since $\operatorname{ad} P(\varphi_j) M_{01} = 0$, we can argue as above and eliminate those terms for which the eigenvalue is not zero. We do this for $1 \leqq j \leqq K$.

We set $\varphi = \varphi_j$ in our definition of $B = B_j$ and $C = C_j$. For a new linear combination E_j of $\operatorname{ad} B_j$, $\operatorname{ad} C_j$ and $(\operatorname{ad} C_j)^2$ we have each term in the expansion of the $M^{01}_{l,l+1}$ an eigenvector of E_j with eigenvalue

$$\begin{aligned} &\geqq 1 \text{ if } \varphi_j \text{ occurs} \\ &= 0 \text{ if } \varphi_j \text{ does not occur} . \end{aligned}$$

As above

$$[\Pi_{j=1}^{K} \lim_n p_{n,0}(E_j)] M^{01}_{ll+1} = s_{ll} \otimes \varphi_0$$

for $1 \leqq l \leqq L$. Also $p_{n,0}(E_j) M_{01} = M_{01}$. It follows that $M_{01} \in \mathfrak{m}$ and the proof is complete.

Our Hamiltonians $H = H_0 + V$ and H_0 are not in $\mathfrak{u}$ since the kernel of H_0 is singular. However the set

$$\{\lambda H_0\} + \mathfrak{u} = \mathfrak{w} \tag{3.3}$$

is a Lie algebra since the bracket $[H_0, V]$ is defined and is in $\mathfrak{u}$; $\mathfrak{u}$ is thus an ideal in $\mathfrak{w}$. It can be seen that the algebra $\mathfrak{a}$ of operators of the form

$$\lambda H_0 + V_{00} \tag{3.4}$$

is a maximal abelian subalgebra of $\mathfrak{w}$. We remark that Theorem 3.2 and its proof remain valid if we replace the requirement that the kernels belong to $\mathscr{S}$ by the requirement that they belong to L_2.

§ 4. Canonical transformations of $\mathfrak{u}$

The constructions of automorphisms and derivations which we shall consider lead to operators in $\mathfrak{u}$ which are not in $\mathfrak{u}_0$, even if our perturbation V is in $\mathfrak{u}_0$. Thus $\mathfrak{u}_0$ is too small to provide a satisfactory framework for the theory. On the other hand $\mathfrak{u}$ is not a Lie algebra and we will show in this section that an attempt to study derivations and automorphisms of $\mathfrak{u}$ leads to certain pathological phenomena.

We consider the following explicitly soluble problem. Let $V = \sum_{m>0} V_{0m}$ consist entirely of annihilation operators. Let $\Gamma V = \sum \Gamma V_{0m}$, where ΓV_{0m} has the kernel

$$-(\Sigma \omega_j)^{-1} v_{0m} .$$

Then $\Gamma V \in \mathfrak{u}$ and ΓV is a solution of the equation

$$[H_0, \Gamma V] = V ;$$

also V and ΓV commute. Using this, it can be seen that

$$\begin{aligned}
&\exp(\Gamma V)\,(H_0 + V)\exp(-\Gamma V)\\
&= \exp(\Gamma V) H_0 \exp(-\Gamma V) + V\\
&= H_0 + [\Gamma V, H_0] + V\\
&= H_0 .
\end{aligned}$$

We let

$$\begin{aligned}
\mathfrak{a}_V &= \{\lambda H\} + V_{00}\\
&= \exp(-\Gamma V)\,\mathfrak{a}\exp(\Gamma V) .
\end{aligned}$$

Since $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{w}$, one might expect that $\mathfrak{a}_V$ would be maximal abelian also. However we will see that this is not the case. We let

$$\begin{aligned}
B^+(k) &= \exp(\Gamma V) A^+(k)\exp(-\Gamma V)\\
&= A^+(k) + [\Gamma V, A^+(k)]\\
B^-(k) &= \exp(\Gamma V) A^-(k)\exp(-\Gamma V) = A^-(k) .
\end{aligned}$$

One might hope that the map

$$\begin{aligned}
G &= G(A^{\pm}) \to G(B^{\pm})\\
&= G(A^+ + [\Gamma V, A^+], A^-)
\end{aligned}$$

obtained by substituting $B^{\pm}$ for $A^{\pm}$ would be an automorphism of $\mathfrak{u}$. Although $\exp(\pm\Gamma V)$ and $B^{\pm}$ are defined (since $V = \sum_m V_{0m}$), $G(B^{\pm})$ is not in general defined. By this we mean that formally

$$G(B^{\pm}) = F = \textstyle\sum_{lm} F_{lm}$$

and that each kernel f_{lm} is expressed as an infinite sum of terms depending linearly on the kernels $g_{l'm'}$. Since there is no restriction in the rate of growth of the $g_{l'm'}$ as $l', m' \to \infty$, such a sum will not in general converge.

Definition. Let $g = g(k_1, \ldots, k_p)$ be a distribution of class $\mathcal{S}^*$ which is symmetric in the variables $k_1, \ldots, k_p$. If $v_{l+p,m} = v_{l+p,m}(k_1, \ldots, k_{l+p}, k_1', \ldots, k_m')$ is a kernel of the class we have considered (§ 2), let

$$\begin{aligned}
&\langle g, v_{l+p,m}\rangle (k_{p+1}, \ldots, k_{l+p}, k')\\
&\qquad = \int \bar{g}(k_1, \ldots, k_p) v_{l+p,m}(k_1, \ldots, k_{l+p}, k')\, dk_1 \ldots dk_p .
\end{aligned}$$

Theorem 4.1. Let g be given as above. If $g \neq 0$, then the range of the map

$$v_{l+p,m} \to \langle g, v_{l+p,m}\rangle \tag{4.1}$$

is the set of all our kernels in the variables $k_1, \ldots, k_l, k_1', \ldots, k_m'$.

Proof. It is clear that $\langle g, v_{l+p,m}\rangle$ is a kernel of the right class. Let φ be a function of one variable k_1, $\varphi \in \mathscr{S}$, such that

$$0 \neq \int \bar{g}(k_1, \ldots, k_p)\varphi(k_1) \ldots \varphi(k_p) dk_1 \ldots dk_p .$$

Let a kernel v_{lm} be given. There are kernels $u_{0m}, \ldots, u_{lm}$ for which

$$v_{lm} = \sum_{j=0}^{l} \varphi^{(l-j)\otimes} \otimes u_{jm} .$$

Here $\varphi^{k\otimes}$ is the k-fold tensor product of φ (and $\langle \varphi, u_{jm}\rangle = 0$). All tensor products are taken to be symmetric in the first l variables. Any kernel of the form

$$\varphi^{l\otimes} \otimes u_{0m} = \text{const.}\ \langle g, \varphi^{(l+p)\otimes} \otimes u_{0m}\rangle$$

is in the range of (4.1). We suppose by induction on J that for $j < J$ we have kernels of the form $\varphi^{(l-j)\otimes} \otimes u_{jm}$ in the range of (4.1). Then

$$\langle g, \varphi^{(l-J+p)\otimes} \otimes u_{Jm}\rangle = \langle g, \varphi^{p\otimes}\rangle\, \varphi^{(l-J)\otimes} \otimes u_{Jm} + (\text{terms in the range of } (4.1))$$

and so $\varphi^{(l-J)\otimes} \otimes u_{Jm}$ is in the range of (4.1). This completes the induction and the proof.

Suppose that φ has L_2 norm equal to one. The u_{jm} are given explicitly by a formula

$$\text{const.}\ u_{jm} = \langle \varphi^{(l-j)\otimes}, v_{lm}\rangle - \sum_{i=0}^{j-1} \text{const.}\ \varphi^{(j-i)\otimes} \otimes u_{im}$$

where the constants are ratios of the factorials. It follows that the u_{jm} depend in a linear continuous fashion on v_{lm}. Thus $v_{l+p,m}$ can be chosen to be a continuous linear function of v_{lm}.

If g and v_{lm} are in the class $L^{\wedge}$ of rapidly decreasing continuous functions introduced in § 8, then φ can be chosen in this class also. The u_{jm} and v_{l+p} belong to this class (this requires the use of formula (8.10)), and v_{l+p} can again be chosen to be a linear continuous function of v_{lm}. We need these facts in § 9.

Theorem 4.2. Let

$$H = H_0 + \sum_{l-m>-r} V_{lm} + V_{0r}$$

for some r with $r \geqq 2$. Suppose that $V_{0r} \neq 0$. There is a G in $\mathfrak{u}$ for which

$$[H, G] = 0 ,$$
$$G \neq G_{00} .$$

Proof. We write $G = \sum_{k>0} G_k$, where $G_k = \sum_{l-m=k} G_{lm}$. The kernel of $[H_0, G_{lm}]$ is $(\sum \omega) g_{lm}$, where

$$\sum \omega = \sum_{i=1}^{l} \omega(k_i) - \sum_{j=1}^{m} \omega(k'_j) .$$

The equation for G can be written as

$$[V_{0r}, G_k] = \text{terms depending on } G_1, \ldots, G_{k-1} \tag{4.3}$$

for $k = 1, 2, \ldots$ We solve this by induction on k. For fixed k, (4.3) is equivalent to

$$[V_{0r}, G_k]_{j+k-r,j} = \text{known operator}$$

or to

$$\begin{aligned}[V_{0r}, G_{j+k,j}]_{j+k-r,j} = {} & \text{known operator} \\ & + \text{terms depending on } G_{k,0}, \ldots, G_{j-1+k,j-1},\end{aligned} \tag{4.4}$$

for $j \geqq 0$, $j \geqq r - k$. We solve this by induction on j. Then (4.4) is equivalent to

$$\langle \bar{v}_{0r}, g_{j+k,j} \rangle = \text{known kernel}\,.$$

This can be solved by Theorem 4.1. Finally we observe that G_{10}, for example, can be chosen arbitrarily. Thus we can have $G \neq G_{00}$.

§ 5. A Lie algebra between $\mathfrak{u}_0$ and $\mathfrak{u}$

Let $\mathfrak{u}_{1r}$ be the set of Q in $\mathfrak{u}$ which have the form

$$Q = \sum\nolimits_{l-m \geqq r} Q_{lm}\,.$$

$\mathfrak{u}_{1r}$ is a closed subset of $\mathfrak{u}$; we give it the relative topology from $\mathfrak{u}$. Let

$$\mathfrak{u}_1 = \cup_r \mathfrak{u}_{1r}\,,$$

and let $\mathfrak{u}_1$ have the inductive limit topology.

Theorem 5.1. We have the following inclusion:

$$\mathfrak{u}_1 \subset L(\mathfrak{D}, \mathfrak{D})\,.$$

The topology of $\mathfrak{u}_1$ is stronger than the topology of uniform convergence on bounded sets.

Theorem 5.2. $\mathfrak{u}_1$ is a subalgebra of $L(\mathfrak{D}, \mathfrak{D})$ and the product PQ of two elements of $\mathfrak{u}_1$ is jointly continuous in its two factors P and Q provided one of the factors P or Q is required to remain in a bounded set.

The algebra $\mathfrak{u}_1$ gives an unsymmetrical preference to creation operators. If we formed an analogous algebra but gave the preference to annihilation operators instead, we could obtain the set of formal adjoints $\mathfrak{u}_1^*$ to $\mathfrak{u}_1$. Here we have

$$\mathfrak{u}_1^* \subset L(\mathfrak{D}_0, \mathfrak{D}_0)\,.$$

Proof of Theorem 5.1. Let B be a bounded set in D. We can suppose that B has the form

$$B = \{\Phi : \varphi_j \in B_j\}$$

where each B_j is a bounded set in an $\mathscr{S}$ space. For any given k and for Q in $\mathfrak{u}_{1r}$, the component $(Q\Phi)_k$ depends only on

$$\varphi_0, \ldots, \varphi_{k+|r|}$$

and on Q_{lm} for $l \leqq k$, $m \leqq k + |r|$. Thus if these terms of Q are sufficiently small in $\mathscr{S}$ and if $\Phi \in B$, then $(Q\Phi)_k$ can be made as small as desired. In view of the definition of the topology in $\mathfrak{u}_1$, this proves the theorem.

Proof of Theorem 5.2. It is sufficient to prove separate continuity and also joint continuity at zero. To prove separate continuity, it is sufficient to consider the case $P \in \mathfrak{u}_{1r}$, for some fixed r. Then a term $(PQ)_{lm}$ depends only on

$$\{P_{jk}: j \leqq l,\, k \leqq l + |r|\}$$

and on

$$\{Q_{jk}: j \leqq 2l + |r|,\, k \leqq m\}$$

by Lemma 3.3. Thus $PQ \in \mathfrak{u}$ and the product is separately continuous. By Lemma 3.3,

$$\mathfrak{u}_{1r}\, \mathfrak{u}_{1s} \subset \mathfrak{u}_{1r+s}$$

and so $\mathfrak{u}_1$ is a subalgebra of $L(\mathfrak{D}, \mathfrak{D})$. The bounded sets in $\mathfrak{u}_1$ are just the sets which are bounded subsets of $\mathfrak{u}_{1r}$ for some r. Thus if P is in some bounded set, the above argument shows the joint continuity at zero.

Now let B be a bounded set in some $\mathfrak{u}_{1r}$ and suppose that $Q \in B$. We can suppose that B has the form

$$\{Q : Q \in \mathfrak{u}_{1r},\, Q_{lm} \in B_{lm}\}$$

where the B_{lm} are bounded subsets of $\mathscr{S}$ spaces. To give a neighborhood U of zero in $\mathfrak{u}_1$ we choose an integer k_0. For each $k \geqq k_0$ we choose an integer $l(k) \geqq l(k-1)$ and we choose neighborhoods U_{lm}. We define U to be the set

$$\{R : R \in \mathfrak{u}_1,\, R_{l,\,l+k} \in U_{l,\,l+k} \quad \text{for} \quad k \geqq k_0 \quad \text{and} \quad l \leqq l(k)\}\,.$$

These U form a basis for neighborhoods of zero in $\mathfrak{u}_1$. A term $P_{\alpha,\,\alpha+\beta}$ contributes to a term $(PQ)_{l,\,l+k}$ only when $k \leqq \beta + r$ and $\alpha \leqq l$. The formulas

$$(P_{\alpha,\,\alpha+\beta}\, Q)_{l,\,l+k} \in 2^{-\alpha-|\beta|}\, U_{l,\,l+k}\,, \quad l \leqq l(k)$$

will be satisfied for $k \geqq k_0$ provided each of the terms

$$\{P_{\alpha,\,\alpha+\beta} : \beta \geqq k_0 - r,\, \alpha \leqq l(\beta + r)\}$$

is sufficiently small. These terms will be small if P is a suitable neighborhood V of zero and this completes the proof of the theorem.

Let $\mathfrak{w}_1$ be the set of operators of the form $H = \lambda H_0 + V$, $V \in \mathfrak{u}_1$. $\mathfrak{w}_1$ is also a Lie algebra. Theorem 3.2 and its proof are valid for $\mathfrak{u}_1$; consequently $\mathfrak{u}_1/\mathfrak{z}$ is simple. Each subspace $\mathfrak{u}_{1r}$, $0 \leqq r$, of $\mathfrak{u}_1$ is a subalgebra and a Lie subalgebra of $\mathfrak{u}_1$. This is not the case for $r < 0$. If $0 \leqq r_1 \leqq r_2$, then $\mathfrak{u}_{1r_2}$ is an ideal in $\mathfrak{u}_{1r_1}$. The derived subalgebra of $\mathfrak{u}_{1r}$ is contained in $\mathfrak{u}_{12r}$ for $0 \leqq r$.

Theorem 5.3. If $Q \in \mathfrak{u}_{1r}$ and if $0 \leqq r$, then the power series $\exp Q$ converges in the topology of $\mathfrak{u}_1$ to an element of $I + \mathfrak{u}_{1r}$. The set

$$\{\textstyle\sum_{n \leqq N} (n!)^{-1}\, Q^n : N = 0, 1, \ldots\}$$

of partial sums is a bounded set.

Theorem 5.4. If $Q \in \mathfrak{u}_{1r}$, if $1 \leqq r$ and if $\{a_0, a_1, \ldots\}$ is any sequence of numbers, then the power series

$$\sum a_i Q^i$$

converges.

Proof of Theorem 5.4. Let l and m be given and let Q and r be as in the theorem. Then $(Q^n)_{l\,m} = 0$ if $n > l - m$.

Proof of Theorem 5.3. A product

$$Q_{l_1 m_1} Q_{l_2 m_2}$$

has a nonzero $l\,m$ term only if

$$l_1 \leqq l$$

$$l_2 \leqq l_2 + (l_1 - m_1) \leqq l\,.$$

Hence

$$(Q^n)_{lm} = ((\textstyle\sum_{m_1 \leqq l_1 \leqq l} Q_{l_1 m_1})^n)^{l\,m}$$

and so there are at most $(2\,l)^n$ terms of the form

$$(Q_{l_1 m_1} \ldots Q_{l_n m_n})_{l\,m}$$

which contribute to $(Q^n)_{l\,m}$. Each of these terms is a sum

$$\textstyle\sum_{0 \leqq j_1, \ldots, j_n \leqq l} (\ldots (Q_{l_1 m_1} \underset{j_1}{-\circ-} Q_{l_2 m_2}) \underset{j_2}{-\circ-} \ldots) \underset{j_n}{-\circ-} Q_{l_n m_n})_{l\,m}\,. \tag{5.1}$$

Here $\underset{j}{-\circ-}$ is defined by the formula

$$Q_{l_1 m_1} \underset{j}{-\circ-} Q_{l_2 m_2} = (Q_{l_1 m_1} Q_{l_2 m_2})_{l_1 + l_2 - j,\, m_1 + m_2 - j}\,,$$

which differs slightly from the definition in [2]. There are l^n summands in (5.1) and so there are $(2\,l^2)^n$ terms in $(Q^n)_{l\,m}$ which have the form of a summand in (5.1). Hence for any seminorm $\| \cdot \|$ defined on the operators $Q_{l\,m}$ we can find a constant K such that

$$\|(Q^n)_{l\,m}\| \leqq K^n\,.$$

The convergence of $\exp Q$ and the boundedness of the partial sums follow from this.

Let $H \in \mathfrak{w}_1$ and let $Q \in \mathfrak{u}_{1,0}$. The map

$$H \to \exp(Q)\, H \exp(-Q) \tag{5.2}$$

is an automorphism of $\mathfrak{w}_1$. This map is determined by its effect on the $A^{\pm}$. Let

$$B^{\pm} = \exp(Q)\, A^{\pm} \exp(-Q)\,. \tag{5.3}$$

Then

$$\exp(Q)\, V \exp(-Q) = \textstyle\sum_{l\,m} \int B^+(k_1) \ldots B^+(k_l)\, v_{l\,m}(k, k')\, B^-(k_1') \ldots B^-(k_m')\, dk\, dk'\,. \tag{5.4}$$

The conventional relation between automorphisms and derivations holds for (5.2).

Theorem 5.5. Let H be in $\mathfrak{w}_1$ and let Q be in $\mathfrak{u}_{1,0}$. Then

$$\exp(Q)\,H\exp(-Q) = \sum_{n=1}^{\infty} (n!)^{-1}(\operatorname{ad} Q)^n H\,.$$

The series converges in $\mathfrak{w}_1$.

Proof. We have

$$\exp(Q)\,H\exp(-Q) = \lim_{J,K\to\infty} \textstyle\sum_{j\leqq J,\,k\leqq K} (j!\,k!)^{-1}\,Q^j\,H(-Q)^k\,.$$

Because the partial sums $\sum_{j\leqq J}(j!)^{-1}(\pm Q)^k$ are bounded, we can write this limit as

$$\lim_{N\to\infty} \textstyle\sum_{n\leqq N}(n!)^{-1}\sum_{j+k=n}\binom{n}{j}\,Q^j\,H(-Q)^k$$
$$= \lim_{N\to\infty}\textstyle\sum_{n\leqq N}(n!)^{-1}(\operatorname{ad} Q)^n H\,.$$

This proves the theorem.

In a similar fashion one proves

$$\exp(sQ)\exp(t\,Q) = \exp((s+t)\,Q)\,.$$

The group $I + \mathfrak{u}_{1,1}$ is the (infinite dimensional Lie) group corresponding the to Lie algebra $\mathfrak{u}_{1,1}$. This group and its Lie algebra are in some sense nilpotent, although they are not nilpotent if one uses the conventional definitions [3]. The next theorem gives a property which they have in common with nilpotent Lie groups and algebras.

Theorem 5.6. The mapping exp is a one-one map of $\mathfrak{u}_{1,1}$ onto $I + \mathfrak{u}_{1,1}$.

Proof. Let V be in $\mathfrak{u}_{1,1}$ and let

$$V_r = \textstyle\sum_{l-m=r} V_{lm}\,.$$

We must find a solution $Q \in \mathfrak{u}_{1,1}$ of the equation

$$\exp Q = I + V\,. \tag{5.5}$$

This is equivalent to solving the sequence of equations

$$Q_r = V_r - \textstyle\sum_{n\geqq 2} n!^{-1}(Q^n)_r \tag{5.6}$$

for $r = 1, 2, \ldots$. For a Q in $\mathfrak{u}_{1,1}$ the operator

$$(Q^n)_r$$

depends only on $Q_1, \ldots, Q_{r-n+1}$. Thus the equations (5.6) can be solved by induction, and the solution is unique. This proves the theorem.

Let an r be given. There is a polynomial p_r such that

$$\textstyle\sum_{j\leqq r} Q_j = \sum_{j\leqq r} p_r(V)_j\,.$$

This can be proved directly by induction on r. It also follows from the Baker-Hausdorff formula [3]. This formula exhibits p_r explicitly and in fact gives Q as a power series in V.

Let H be an element of $\mathfrak{w}_1$ and let

$$\mathfrak{c}_H = \{Q : Q \in \mathfrak{u}_{1,1}, [Q, H] = 0\} .$$

Theorem 5.7. $\mathfrak{c}_H$ is a subalgebra of $\mathfrak{u}_{1,1}$ and $I + \mathfrak{c}_H$ is a subgroup of $I + \mathfrak{u}_{1,1}$. Also

$$\begin{aligned} I + \mathfrak{c}_H &= \exp \mathfrak{c}_H \\ &= \{Q : Q \in I + \mathfrak{u}_{1,1}, [Q, H] = 0\} . \end{aligned}$$

Proof. If $Q \in \mathfrak{c}_H$ then $[Q^n, H] = 0$ and $Q^n \in \mathfrak{c}_H$. Thus $\mathfrak{c}_H$ is an algebra, $I + \mathfrak{c}_H$ is a group and

$$\exp \mathfrak{c}_H \subset I + \mathfrak{c}_H .$$

The reverse inclusion follows from the fact that the Q in (5.5) is a limit of polynomials (without constant terms) in V. The last equality is obvious.

We remark that $\mathfrak{c}_H$ and $I + \mathfrak{c}_H$ are closed in $\mathfrak{u}$.

§ 6. Canonical forms for certain elements of $\mathfrak{w}_1$

In this section we consider elements H of $\mathfrak{w}_1$ of the form

$$H = \lambda H_0 + \textstyle\sum_{l-m > -r} V_{lm} + V_{0r} \tag{6.1}$$

$$r > 0 , \quad V_{0r} \neq 0 . \tag{6.2}$$

The main result is

Theorem 6.1. If H is given by (6.1), (6.2) then there is a Q in $\mathfrak{u}_{1,1}$ for which

$$\exp(Q)\, H \exp(-Q) = \lambda H_0 + V_{0r} .$$

It can be seen that the equations

$$\begin{aligned} &\exp(Q)\, (\lambda H_0 + S_{0r}) \exp(-Q) \\ &\quad = \mu H_0 + T_{0r} , \\ &\quad Q \in \mathfrak{u}_{1,1} \end{aligned}$$

imply $\mu = \lambda$ and $S_{0r} = T_{0r}$. Thus we regard

$$H_1 = \lambda H_0 + V_{0r} \tag{6.3}$$

as a canonical form for H with respect to the inner automorphism group $\exp(\mathfrak{u}_{1,1})$. In proving this theorem we do not assume $\lambda \neq 0$, and we regard V_{0r} rather than H_0 as the dominant term in H.

Proof of Theorem 6.1. We write

$$Q = \textstyle\sum_{s=0}^{\infty} Q_s \tag{6.4}$$

where

$$Q_s = \textstyle\sum_{l-m=s} Q_{lm} , \quad s > 0$$

and

$$Q_0 = 0$$

and we let

$$Q^{(t)} = \sum_{s \leqq t} Q_s .$$

Let a $t \geqq 0$ be given. We suppose inductively that $Q_0, Q_1, \ldots, Q_t$ have been chosen and that

$$\sum_{n=0}^{\infty} (n!)^{-1} (\mathrm{ad}\, Q^{(t)})^n H \equiv H_1 \quad (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+1}) .$$

The inductive hypothesis is true for $t = 0$; we use the hypothesis and define Q_{t+1}. Let

$$\sum_{n=0}^{\infty} (n!)^{-1} (\mathrm{ad}\, Q^{(t)})^n H \equiv H_1 + \sum_{l-m=-r+t+1} V_{lm}(t) \ (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+2}).$$

We choose Q_{t+1} as a solution of the equation

$$[Q_{t+1}, V_{0r}] = - \sum_{l-m=-r+t+1} V_{lm}(t) .$$

This equation is equivalent to the system of equations

$$\sum_{j=1}^{r} V_{0r} \underset{j}{-\!\circ\!-} Q_{m+j-r+t+1, m+j} = V_{m-r+t+1, m}(t) . \qquad (6.5)$$

We solve (6.5) by induction on m. The equation for $Q_{m+t+1,m}$ then has the form

$$V_{0r} \underset{r}{-\!\circ\!-} Q_{m+t+1,m} = \text{known function} .$$

This equation has solutions, by Theorem 4.1. The induction starts with $m = \max\{0, r - t - 1\}$, and for $0 \leqq m < r - t - 1$, $Q_{m+t+1,m}$ can be chosen arbitrarily.

However

$$(\mathrm{ad}\, Q^{(t)})^n H \equiv (\mathrm{ad}\, Q^{(t+1)})^n H \quad (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+2})$$

for $n > 1$. Thus

$$\sum_{n=0}^{\infty} (\mathrm{ad}\, Q^{(t+1)})^n H \equiv H_1 \quad (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+2}) ,$$

and the induction on t is complete. We define Q by (6.4). As above

$$(\mathrm{ad}\, Q^{(t)})^n H \equiv (\mathrm{ad}\, Q)^n H \quad (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+1}) ,$$

and

$$\sum_{n=0}^{\infty} (n!)^{-1} (\mathrm{ad}\, Q)^n H \equiv H_1 \quad (\mathrm{mod}\, \mathfrak{u}_{1,-r+t+1}) .$$

This is true for $t = 1, 2, \ldots$ and this completes the proof.

§ 7. The transformation and solution of the Schrödinger equation

In this section we consider time dependent operators. We set

$$A^{\pm}(k, t) = e^{\mp i \omega t} A^{\pm}(k, 0) , \qquad (7.1)$$

where the $A^{\pm}(k, 0)$ are the standard annihilation creation operators introduced in § 2. We consider operators of the form

$$V_{lm} = \int A^{+}(k_1, t) \ldots A^{+}(k_l, t) v_{lm}(k, k') A^{-}(k'_1, t) \ldots A^{-}(k'_m, t) \, dk \, dk' , \quad (7.2)$$

and of the form

$$V = \sum V_{lm} \,. \tag{7.3}$$

Such operators are said not to depend explicitly on t. We see that the $A^{\pm}$ satisfy the differential equation

$$i \frac{\partial}{\partial t} A^{\pm}(k, t) = [H_0, A^{\pm}(k, t)] \,. \tag{7.4}$$

Let

$$\Phi(t) = \exp(-itH_0)\, \Phi(0)$$

be a solution of the free field Schrödinger equation

$$i \frac{\partial}{\partial t} \Phi = H_0 \Phi \,. \tag{7.5}$$

If $\Phi(0) \in \mathfrak{D}_0$ then $\Phi(t) \in \mathfrak{D}_0$ for all t and $(\partial/\partial t)\, \Phi$ exists in the topology of $\mathfrak{D}_0$. A similar statement holds if $\Phi(0) \in \mathfrak{D}$. In either case

$$\psi(t) = \int A^{\pm}(k, t)\, v(k)\, dk\, \Phi(t)$$

is a solution of (7.5). [This follows formally from (7.4).] Thus $A^{\pm}(k, t)$ is an operator on the Cauchy data of Φ for the time t which creates or annihilates a particle of momentum k at the time $t = 0$. Similarly $A^{\pm}(k, t - s)$ acts on the same Cauchy data (at time t) and creates or annihilates a particle at the time $t = s$.

We use the symbol $\mathfrak{w}_1$, etc. to denote the class of operators (7.3) which for each fixed t belong to the class $\mathfrak{w}_1$, etc. considered earlier.

Lemma 7.1. If $H \in \mathfrak{w}$ (resp. $\mathfrak{w}_0$, $\mathfrak{w}_1$) then $(\partial/\partial t)\, H$ exists in the topology of $\mathfrak{w}$ (resp. $\mathfrak{w}_0$, $\mathfrak{w}_1$) and

$$i \frac{\partial}{\partial t} H = [H_0, H] \,. \tag{7.6}$$

If $\Phi(0) \in \mathfrak{D}_0$ (resp. $\mathfrak{D}_0$, $\mathfrak{D}$) then $H\, \Phi(t)$ is a solution of (7.5) in $\mathfrak{D}$ (resp. $\mathfrak{D}_0$, $\mathfrak{D}$).

Proof. It is clear that the derivative exists and that it defines a derivation of $\mathfrak{w}_0 = \{\lambda H_0\} + \mathfrak{u}_0$ which is continuous in the $\mathfrak{w}$ topology. The right member of (7.6) defines a derivation of $\mathfrak{w}_0$ with the same properties. The exponential factors cancel in

$$H_0(t) = \int A^{+}(k, t)\, \omega(k)\, A^{-}(k, t)\, dk \,,$$

so $H_0(t) = H_0(0)$ and (7.6) holds for $H = H_0$. For $H = V_{01}$ or $H = V_{10}$, (7.6) is equivalent to (7.4). For a general H, (7.6) follows from these two cases and the derivation property. The last statement follows from (7.6).

Let Q be in $\mathfrak{u}_{1,0}$ and define $B^{\pm}$ by (5.3). We regard the $A^{\pm}$ as independent variables and we regard an H in $\mathfrak{w}_1$ as a function of these variables,

$$H = H(A^{\pm}) \,.$$

We use (5.4) to define the operation of "substituting" $B^\pm$ for $A^\pm$, and thus we define

$$H(B^\pm) = \exp(Q)\, H(A^\pm) \exp(-Q) \,. \tag{7.7}$$

If the multiplications in the right member of (7.7) are performed, we obtain a new function of the $A^\pm$

$$\exp(Q)\, H(A^\pm) \exp(-Q) = \tilde{H}(A^\pm) \,,$$

and so

$$H(B^\pm) = \tilde{H}(A^\pm) \,.$$

These considerations and Lemma 7.1 provide the proof for the following theorem.

Theorem 7.2. Let Q be in $\mathfrak{u}_{1,0}$ and let

$$H = \exp(-Q)\, H_0 \exp(Q)$$

$$B^\pm = \exp(Q)\, A^\pm \exp(-Q) \,.$$

Then

$$i \frac{\partial}{\partial t} B^\pm(k, t) = [H(B^\pm), B^\pm(k, t)] \,.$$

Since $H(B^\pm) = H_0(A^\pm) = H_0$, we see that (7.5) is equivalent to the Schrödinger equation for a problem with interaction:

$$i \frac{\partial}{\partial t} \Phi(t) = H(B^\pm(0))\, \Phi(t) \,. \tag{7.8}$$

Let $\Phi \in \mathfrak{H}$ be a solution of (7.8) and let $V \in \mathfrak{u}_1$. Then $V(B^\pm) = \tilde{V}(A^\pm)$ with $\tilde{V} \in \mathfrak{u}_1$. Thus

$$V(B^\pm(t))\, \Phi(t)$$

is a solution of (7.5) and (7.8).

Next we consider the Hamiltonian

$$H_1 = H_0 + V_{0r} \,.$$

As above and as in § 4 we define

$$B^\pm = \exp(\Gamma V_{0r})\, A^\pm \exp(-\Gamma V_{0r}) \,.$$

Then

$$B^- = A^-$$

$$B^+ = A^+ + [\Gamma V, A^+] \,.$$

One can show directly that

$$H_1(B^\pm) = H_0(A^\pm)$$

$$i \frac{\partial}{\partial t} B^\pm(t) = [H_1(B^\pm), B^\pm(t)] \,.$$

Finally we consider the Hamiltonian H given by (6.1), (6.2) with $\lambda = 1$. Let $Q = Q(A^\pm)$ be given by Theorem 6.1. In a formal sense

$$C^\pm(t) = \exp\big(Q(B^\pm(t))\big)\, B^\pm(t) \exp\big(-Q(B^\pm(t))\big) \tag{7.9}$$

is a solution of the equation

$$i \frac{\partial}{\partial t} C^{\pm}(t) = [H(C^{\pm}), C^{\pm}(t)] . \tag{7.10}$$

Although Q and $\exp(V_{0r})$ have meaning as operators on certain Fock type spaces, the space for Q is different than the space for $\exp(V_{0r})$ and the expression

$$Q(B^{\pm}) = \exp(\Gamma V_{0r}) Q \exp(-\Gamma V_{0r})$$

appears not to have a meaning as an operator on the domains we have considered. Consequently we have not found canonical annihilation creation operators $C^{\pm}(t)$ for particles H by the method of Theorem 7.2.

We continue to consider the same Hamiltonian H, but we now do this in the Schrödinger picture. We let $A^{\pm}$ denote time independent annihilation creation operators, as in the previous sections.

Theorem 7.3. Let H and Q be as in Theorem 6.1. Let

$$T = \exp(-Q) \exp(-\Gamma V_{0r}) . \tag{7.11}$$

Then $T \in \mathfrak{u} \subset L(\mathfrak{D}_0, \mathfrak{D})$ and

$$H T = T H_0 . \tag{7.12}$$

If $\Phi = \Phi(t)$ is a solution of the free Schrödinger equation (7.5) and if $\Phi(0) \in \mathfrak{D}_0$ then

$$\psi(t) = T \Phi(t)$$

is a solution of the interacting Schrödinger equation

$$i \frac{\partial}{\partial t} \psi = H \psi . \tag{7.13}$$

The Q of Theorem 6.1 is not unique. Let

$$\exp(Q_i) H \exp(-Q_i) = H_1 , \quad i = 1, 2 .$$

Then

$$\exp(-Q_1) \exp(Q_2) \in I + \mathfrak{c}_H$$

and by Theorem 5.7,

$$\exp(-Q_1) = \exp(R) \exp(-Q_2)$$

where $R \in \mathfrak{c}_H$. Thus the T in (7.11) is determined up to multiplication on the left by an element of $I + \mathfrak{c}_H$. This nonuniqueness of T does not affect the solubility of (7.13).

Proposition 7.4. Let $\psi(t)$ be a differentiable function of t with values in $\mathfrak{D}$ and let P be in $I + \mathfrak{c}_H$. Then ψ is a solution of (7.13) if and only if $P \psi$ is.

Proof of Theorem 7.3. It is evident that the product T in (7.11) exists and defines an element of $\mathfrak{u}$. If $\Phi(0) \in \mathfrak{D}_0$ then $\Phi(t) \in \mathfrak{D}_0$ and $(\partial/\partial t) \Phi(t)$ exists in $\mathfrak{D}_0$. Thus $T \Phi = \psi$ is defined and $(\partial/\partial t) \psi$ exists in $\mathfrak{D}$. Furthermore

$$\begin{aligned} H T &= \exp(-Q) H_1 \exp(-\Gamma V_{0r}) \\ &= T H_0 , \end{aligned}$$

and this implies that ψ is a solution of (7.12).

We do not know what physical significance, if any, these solutions have. In § 9 we find solutions which appear to correspond to the standard formal solutions of perturbation theory.

§ 8. The operation Γ

The operator $[H_0, V_{lm}]$ has a kernel

$$(\sum \omega_i - \sum \omega_j') \, v_{lm}(k, k') \tag{8.1}$$

where

$$\omega_j = \omega(k_j) \,, \quad \omega_j' = \omega(k_j') \,.$$

Thus any solution ΓV_{lm} of the equation

$$[H_0, \Gamma V_{lm}] = V_{lm}$$

will in general have a singular kernel. In this section we derive some properties of such singular operators, following the ideas of [2].

We let Ω_j denote the angular variables $k_j/|k_j|$ and we write

$$d k_j = \omega_j |k_j| \, d\omega_j \, d\Omega_j \,.$$

Let

$$\hat{\varphi}(\lambda, \Omega) = \int e^{-i\lambda \cdot \omega} \varphi(k) \, d\omega$$

$$\hat{v}(\lambda, \lambda', \Omega, \Omega') = \int e^{-i(\lambda \cdot \omega - \lambda' \omega')} v(k, k') \, d\omega \, d\omega' \,.$$

Actually it is not these Fourier transforms which interest us, but rather the transform in the next lemma.

Lemma 8.1. Let $\varphi \in \mathscr{S}$. There is a constant M such that

$$|[\Pi_j (\omega_j |k_j|)^{1/2} \varphi]^\wedge(\lambda)| \leqq M \, \Pi_j (1 + |\lambda_j|)^{-5/4} \,, \tag{8.2}$$

and similarly for v.

Proof. (Cf. [4], p. 124.) It is sufficient to consider the case where $\varphi = \varphi_1$ depends on k_1 only. We write

$$|k| = (\omega^2 - \mu^2)^{1/2}$$
$$= (\omega - \mu)^{1/2} (\omega + \mu)^{1/2} \,.$$

Then (8.2) is given by

$$|\int_\mu^\infty e^{-i\lambda\omega} (\omega - \mu)^{1/4} \chi((\omega - \mu)^{1/2}) \, d\omega| = |2 \int_0^\infty e^{-i\lambda\tau^2} \tau^{3/2} \chi(\tau) \, d\tau| \,, \tag{8.3}$$

where $\chi(\tau) = (\tau^2 + \mu)^{1/2} (\tau^2 + 2\mu)^{1/2} \varphi(\tau(\tau^2 + 2\mu)^{1/2} \Omega)$ is in $\mathscr{S}$. We consider first the case $\chi_1 = a \exp(-\tau^2)$. If we substutite this in (8.3) and deform the contour of integration in the complex plane, we obtain

$$2a \int_0^\infty \exp(-|1 + i\lambda| \tau^2) \, \tau^{3/2} \, d\tau \leqq 2a (1 + |\lambda|)^{-5/4} \int_0^\infty e^{-\tau^2} \tau^{3/2} \, d\tau \,. \tag{8.4}$$

Next we consider the case $\chi_2 = b\,\tau \exp(-\tau^2)$. As before we obtain in this case the bound

$$2b(1+|\lambda|)^{-7/4} \int_0^\infty e^{-\tau^2} \tau^{5/2}\, d\tau \tag{8.5}$$

for (8.3). We set $a = \chi(0)$, $b = \chi'(0)$ and

$$\chi_3 = \chi - \chi_1 - \chi_2 \,.$$

Then

$$(\omega - \mu)^{1/4} \chi_3((\omega-\mu)^{1/2}) = 2^{-1} \chi_3''(0)\,(\omega-\mu)^{5/4} + \cdots$$

has two derivatives in L_1. Thus

$$\left| \int_\mu^\infty e^{-i\lambda\omega} (\omega-\mu)^{1/4} \chi_3((\omega-\mu)^{1/2})\, d\omega \right| \leqq \text{const.}\ (1+|\lambda|)^{-2} \,.$$

The lemma follows from this, (8.4) and (8.5).

We now define

$$F\,\varphi = [\Pi_j (\omega_j\, |k_j|)^{1/2}\, \varphi]^\wedge \,,$$

and similarly for v. Let $\Lambda_j = i\,\partial/\partial\lambda_j$, let $\nu = \nu_1, \ldots, \nu_n$ be a multi-index and let

$$\Lambda^\nu = \Pi_j \Lambda_j^{\nu_j} \,.$$

If $\varphi = \varphi_0$, define $\|\varphi\|_\nu = |\varphi|$. Now suppose $\varphi = \varphi_n(k)$ where k has $3n$ components. Let τ be a vector with r components (each a real number) and with $\tau_1 = 0$ and let j be a function,

$$j : \{1, \ldots, n\} \to \{1, \ldots, r\} \,. \tag{8.6}$$

Define

$$\lambda + \tau = \lambda_1 + \tau_{j(1)}, \ldots, \lambda_n + \tau_{j(n)}$$

$$d\tau = d\tau_2 \ldots d\tau_r$$

$$\|\varphi\|_{j,\nu} = \sup_{\lambda,\Omega} \int |\Lambda^\nu F\, \varphi(\lambda+\tau, \Omega)|\, d\tau$$

$$\|\varphi\|_\nu = \sup_j \|\varphi\|_{j,\nu} \,. \tag{8.7}$$

Let $L^\wedge$ be the completion of $\mathscr{S}$ in the set of seminorms (8.7). Each φ in $L^\wedge$ is a rapidly decreasing continuous function. If σ is a real number and $\lambda + \sigma = \lambda_1 + \sigma, \ldots, \lambda_n + \sigma$ we define

$$\gamma\,\varphi = -iF^{-1} \int_{\sigma \leqq 0} F\,\varphi(\lambda+\sigma)\, d\sigma \,. \tag{8.8}$$

[We get this definition of $\lambda + \sigma$ if $j(i) \equiv 2 = r$ in (8.6).] The inverse Fourier transformation in (8.8) is taken in the sense of distributions. These definitions apply also with kernels v_{lm} replacing the function φ_n. Let

$$(k)_n = k_1, \ldots, k_n$$

denote a vector with $3n$ components and let

$$\langle \varphi_n, \psi_m \rangle_s = \int \overline{\varphi_n}((k')_{n-s}, (k)_s)\, \psi_m((k'')_{m-s}, (k)_s)\, dk \tag{8.9}$$

if $n, m \geqq s$ and let $\langle \varphi_n, \psi_m \rangle_s = 0$ otherwise.

Lemma 8.2. Let φ_n and ψ_m be in $L^\wedge$. Then

$$\|\langle \varphi_n, \psi_m \rangle_s\|_\nu \leqq K^s \|\varphi_n\|_\nu \, \|\psi_m\|_\nu \, . \tag{8.10}$$

If $s \geqq 1$ then

$$\|\langle \gamma\, \varphi_n, \psi_m \rangle_s\|_\nu \leqq K^s \|\varphi_n\|_\nu \, \|\psi_m\|_\nu \tag{8.11}$$

$$\|\langle \varphi_n, \gamma\, \psi_m \rangle_s\|_\nu \leqq K^s \|\varphi_n\|_\nu \, \|\psi_m\|_\nu \, . \tag{8.12}$$

Proof. We use the Fourier transform to express the integration in (8.9) as integrals with respect to λ and Ω. The operator Λ^ν does not effect the proof and so we only consider the case $\nu = 0$. Let a function j be given. If an integration occurs with respect to some variable τ_i which affects φ but not ψ, we have

$$\int |\varphi\, \psi| \, d\tau_i = |\psi| \int |\psi| \, d\tau_i \, .$$

If τ_i affects both variables we use the bound

$$\int |\varphi\, \psi| \, d\tau_i = (\sup_{\tau_i} |\varphi|) \int |\psi| \, d\tau_i \, . \tag{8.13}$$

We find

$$\begin{aligned} \|\langle \varphi, \psi \rangle_s\|_j = {} & (2\pi)^r \sup \int |\overline{F}\, \varphi(\lambda' + \tau, \lambda) F \psi(\lambda'' + \tau, \lambda)| \, d\tau \, d\lambda \, d\Omega \\ \leqq {} & (2\pi)^r (\textstyle\int d\Omega) \sup \int |\overline{F}\, \varphi(\lambda' + \tau, \lambda) F \psi(\lambda'' + \tau, \lambda)| \, d\tau \, d\lambda \, . \end{aligned}$$

We write

$$\tau = \varrho_1 + \varrho_2$$

where ϱ_1 is the part of τ which affects the variables of φ alone. Then

$$\begin{aligned} & \int |F \varphi(\lambda' + \tau, \lambda) F \psi(\lambda'' + \tau, \lambda)| \, d\tau \, d\lambda \\ & \quad = \int [\textstyle\int |F \varphi(\lambda' + \varrho_1 + \varrho_2, \lambda)| \, d\varrho_1] \, |F \psi(\lambda'' + \varrho_2, \lambda)| \, d\varrho_2 \, d\lambda \\ & \quad \leqq [\sup_{\lambda', \lambda} \textstyle\int |F \varphi(\lambda' + \varrho_1, \lambda) d\varrho_1] \int |F \psi(\lambda'' + \varrho_2, \lambda)| d\varrho_2 \, d\lambda \\ & \quad \leqq \|\varphi\|_{j_*} \|\psi\|_j \end{aligned}$$

for some new function j_*. Thus (8.10) holds with $K = 2\pi \int d\Omega_1 = 8\pi^2$.

We now suppose $s \geqq 1$ and let σ be a real number.

$$\begin{aligned} \|\langle \gamma\, \varphi, \psi \rangle_s\|_j & \leqq K^s \sup \int |F\, \gamma\, \varphi(\lambda' + \varrho_1, \lambda)| d\varrho_1 \|\psi\|_j \\ & \leqq K^s \|\psi\|_j \sup \int \int |F \varphi(\lambda' + \varrho_1 + \sigma, \lambda + \sigma)| \, d\sigma \, d\varrho_1 \, . \end{aligned}$$

We substitute $\varrho_1 - \sigma$ for ϱ_1, which eliminates σ from the variables affected by ϱ_1 and gives us

$$\|\langle \gamma\, \varphi, \psi \rangle_s\|_j = K^s \|\psi\|_j \|\varphi\|_{j_{**}}$$

for some new function j_{**}. This proves (8.11). The proof of (8.12) is similar but requires reversing the roles of φ and ψ in (8.13).

This lemma applies with φ and/or ψ replaced by kernels v_{ij} and/or v_{lm}. Thus the ν norm of any of the following products,

$$V_{ij}V_{lm},\ [\Gamma V_{ij}, V_{lm}],\ V_{lm}\varphi_n$$

or

$$\Gamma V_{lm}\varphi_n \qquad \text{(provided } m > 0),$$

is bounded in terms of a product

$$\|v_{ij}\|_\nu \|v_{lm}\|_\nu \quad \text{or} \quad \|v_{lm}\|_\nu \|\varphi_n\|_\nu .$$

The restriction $s \geqq 1$ in (8.11) and (8.12) interferes with the products

$$\Gamma V_{l0}\varphi_n,\ (\Gamma V_{ij})V_{lm},\ :(\Gamma V_{ij})V_{lm}: .$$

The next lemma yields some continuity for $\Gamma V_{l0}\varphi_n$ provided $v_{l0} \in \mathcal{S}$, since it shows that in this case $\gamma v_{l0} \in \mathcal{S} \subset L^\wedge$.

Lemma 8.3. Let v_{l0} and v_{0m} be in $L^\wedge$, $l \neq 0 \neq m$. Then

$$\gamma v_{l0}(k) = (\textstyle\sum \omega_j)^{-1} v_{l0}(k)$$
$$\gamma v_{0m}(k) = -(\Sigma\omega_j)^{-1} v_{0m}(k) .$$

Proof. Let φ be in $\mathcal{S}$. Then for $v = v_{l0}$,

$$\langle\varphi, \gamma v\rangle = \text{const.} \langle F\varphi, F\gamma v\rangle = -\text{const.}\, i \int\limits_{\sigma \leqq 0} \overline{F}\varphi(\lambda) Fv(\lambda + \sigma)\, d\lambda\, d\sigma,$$

since the integrand is in L_1. Thus

$$\begin{aligned}
\langle\varphi, \gamma v\rangle &= -\lim_{N\to\infty} \int_{-N}^{0} \langle\varphi, i e^{-i(\Sigma\omega)\sigma} v\rangle\, d\sigma \\
&= \lim_{N\to\infty} \int_{-N}^{0} (d/d\sigma) \langle\varphi, (\textstyle\sum\omega)^{-1} e^{-i(\Sigma\omega)\sigma} v\rangle\, d\sigma \\
&= \langle\varphi, (\textstyle\sum\omega)^{-1} v\rangle - \lim_{N\to\infty} \int \overline{\varphi} (\textstyle\sum\omega)^{-1} e^{i(\Sigma\omega)N} v\, dk \\
&= \langle\varphi, (\textstyle\sum\omega)^{-1} v\rangle - \lim_{N\to\infty} \int \overline{\varphi} (\textstyle\sum\omega)^{-1} e^{i(\Sigma\omega)N} \times \\
&\qquad \times v\, \Pi_j (k_j\omega_j)^{1/2} d\Omega\, d\omega_1 \ldots d\omega_{l-1} d(\textstyle\sum \cdot \omega) \\
&= \langle\varphi, (\textstyle\sum\omega)^{-1} v\rangle
\end{aligned}$$

by the Riemann Lebesgue lemma.

We set $\mathfrak{D}_0 = \mathfrak{D}_0(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{D}(\mathcal{S})$. Let $\mathfrak{D}(L^\wedge)$ be defined as the space of sequences

$$\Phi = \{\varphi_0, \varphi_1, \ldots\}$$

for which φ_j is a symmetric function of the variables $k_1, \ldots, k_j$ and $\varphi_j \in L^\wedge$. Let $\mathfrak{D}_0(L^\wedge)$ be the subspace consisting of those Φ for which $\varphi_j = 0$ for all large j. We set $\mathfrak{u}_1 = \mathfrak{u}_1(\mathcal{S})$, etc. and we define $\mathfrak{u}_1(L^\wedge)$, etc. in a similar fashion.

If $V \in \mathfrak{u}(L^\wedge)$ and if $V = \sum_{(l,m) \neq 0,0} V_{lm}$, we define

$$\Gamma V = \sum \Gamma V_{lm}$$

where ΓV_{lm} is defined as an operator with kernel γv_{lm}. We define the seminorms

$$\|\Phi\|_{\nu,n} = \sup_{j \leqq n} \|\varphi_j\|_\nu,\ \Phi \in \mathfrak{H}(L^\wedge) \tag{8.14}$$

$$\|V\|_{\nu,n} = \sup_{l \leqq n} \|v_{lm}\|_\nu,\ V \in \mathfrak{u}_{1,r}(L^\wedge)\ . \tag{8.15}$$

These seminorms define a topology in $\mathfrak{H}(L^\wedge)$, $\mathfrak{u}_{1,r}(L^\wedge)$ and

$$\mathfrak{w}_{1,r}(L^\wedge) = \{\lambda H_0\} \oplus \mathfrak{u}_{1,r}(L^\wedge)\ .$$

We give

$$\mathfrak{w}_1(L^\wedge) = \cup_r\, \mathfrak{w}_{1,r}(L^\wedge)$$

the inductive limit topology.

Proposition 8.4. Let $V_1 \in \mathfrak{u}_{1,-r}(L^\wedge)$ and $V_2 \in \mathfrak{u}_{1,-s}(L^\wedge)$. Suppose $(V_1)_{00} = 0$. Then

$$\|V_1 \Phi\|_{\nu,n} \leqq \text{const.}\ \|V_1\|_{\nu,n} \|\Phi\|_{\nu,n+r}$$

$$\|V_1 V_2\|_{\nu,n} \leqq \text{const.}\ \|V_1\|_{\nu,n} \|V_2\|_{\nu,n+r},\ r \geqq 0$$

$$\|[\Gamma V_1, V_2]\|_{\nu,n} \leqq \text{const.}\ \|V_1\|_{\nu,n} \|V_2\|_{\nu,n+r},\ r \geqq 0\ .$$

If $(V_1)_{l0} = 0$ then

$$\|\Gamma V_1 \Phi\|_{\nu,n} \leqq \text{const.}\ \|V_1\|_{\nu,n} \|\Phi\|_{\nu,n+r}\ .$$

The constant depends only on n, r and s.

Proof. This follows from Lemma 8.2 together with the observation that the left hand sides of these inequalities depend only on certain low order terms of V_1, V_2 and Φ. (Cf. Lemma 3.3.)

We consider the elements of $\mathfrak{w}_1(L^\wedge)$ as operators on $\mathfrak{H}(L^\wedge)$. $\mathfrak{w}_1(L^\wedge)$ is a Lie algebra, $\mathfrak{u}_1(L^\wedge)$ is an algebra and

$$[\Gamma V_1, V_2] \in \mathfrak{u}_1(L^\wedge)$$

if $V_i \in \mathfrak{u}_1(L^\wedge)$ and $(V_1)_{l0} = 0$. (If $(V_1)_{00} = 0$ but $(V_1)_{l0} \neq 0$ for some l then the commutator is formally an element of $\mathfrak{u}_1(L^\wedge)$ but might not be everywhere defined.)

Theorem 8.5. Let

$$P = Q_1 + \Gamma Q_2$$

be given with $Q_i \in \mathfrak{u}_{1,0}(L^\wedge)$ and $(Q_2)_{l0} = 0$. Then the power series for $\exp P$ and $\exp(\operatorname{ad} P)$ converge uniformly on bounded sets in $\mathfrak{H}(L^\wedge)$ and in $\mathfrak{w}_1(L^\wedge)$ respectively. If $H \in \mathfrak{w}_1(L^\wedge)$ then

$$\exp(P) H \exp(-P) = \exp(\operatorname{ad} P) H\ .$$

Proof. The inequalities in Proposition 8.4 show that there is uniform convergence with respect to each seminorm.

Theorem 8.6. Let $V \in \mathfrak{u}_1(L^\wedge)$ and let $v_{l0} \in \mathscr{S}$ and $V_{00} = 0$. Then

$$[H_0, \Gamma V] = V\,. \tag{8.16}$$

Proof. We have

$$F H_0 \varphi_n = F(\textstyle\sum_{j=1}^{n} \omega_j)\varphi_n = \sum \Lambda_j F \varphi_n$$

and for some constant $\alpha = \alpha(l, m, n)$ we have

$$F V_{lm}\varphi_n = \alpha \operatorname{Sym} \textstyle\int F v_{lm}(\lambda', \lambda'') F \varphi_n(\lambda'', (\lambda)_{n-m}) d\lambda'' d\Omega''\,.$$

After integrating by parts and cancelling terms we find

$$\begin{aligned} &F[H_0, \Gamma V_{lm}]\varphi_n \\ &\quad = -i\alpha \operatorname{Sym} \int \int_{\sigma \leqq 0} [(\textstyle\sum_{j=1}^{l} \Lambda_j' + \sum_{j=1}^{m} \Lambda_j'') F v_{lm}(\lambda' + \sigma, \lambda'' + \sigma)] \\ &\qquad F\varphi_n(\lambda'', (\lambda)_{n-m}) d\sigma\, d\lambda''\, d\Omega'' \\ &\quad = \alpha \operatorname{Sym} \int \int_{\sigma \leqq 0} (d/d\sigma) F v_{lm}(\lambda' + \sigma, \lambda'' + \sigma) F \varphi_n(\lambda'', (\lambda)_{n-m}) d\sigma\, d\lambda'' d\Omega'' \\ &\quad = F V_{lm} \varphi_n\,. \end{aligned}$$

§ 9. Solutions of the Schrödinger equation which have a partial perturbation expansion

We now consider a Hamiltonian of the form

$$H = H_0 + \varepsilon V \in \mathfrak{w}_1(\mathscr{S}) \tag{9.1}$$

$$V = \textstyle\sum_{l-m > -r} V_{lm} + V_{0r} \tag{9.2}$$

where ε is a parameter, $r > 0$ and $V_{0r} \neq 0$. Let

$$H_{\mathrm{ren}} = H - E \tag{9.3}$$

denote the renormalized Hamiltonian. E is a (finite) multiple of the identity operator. In this section we find a T which intertwines H_{ren} and H_0,

$$H_{\mathrm{ren}} T = T H_0\,, \tag{9.4}$$

and the "low order" parts of T depend analytically on ε.

Theorem 9.1. Let V, ε and a positive integer n be given. There is a renormalization constant E and a continuous transformation T from $D_0(L^\wedge)$ into $D(L^\wedge)$ which solve (9.4). If

$$\Psi = T\Phi = T\{\varphi_0, \ldots, \varphi_j, 0, \ldots\}$$

is in the range of T then the terms

$$\psi_0, \ldots, \psi_n \tag{9.5}$$

depend analytically on ε.

As a consequence of this theorem,

$$\Psi(t, \varepsilon) = T(\varepsilon) \exp(-itH_0)\Phi(0)$$

is a solution of the renormalized Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi = H_{\mathrm{ren}} \Psi$$

in the space $\mathfrak{H}(L^\wedge)$, and the low order terms (9.5) of Ψ depend analytically on ε. We conjecture that these terms agree with standard perturbation theory up to any desired finite order (for example, $n-1$) if T is suitably chosen.

Lemma 9.2. There are polynomials

$$Q(\varepsilon) = \varepsilon Q^{(1)} + \cdots + \varepsilon^{2n} Q^{(2n)}$$

and

$$R(\varepsilon) = \varepsilon R^{(1)} + \cdots + \varepsilon^{2n} R^{(2n)}$$

with coefficients

$$Q^{(j)} = \sum_{l>0} Q_{l0}^{(j)} \in \mathfrak{u}_1(\mathscr{S})$$

and

$$R^{(j)} = \sum_{l \geqq m > 0} R_{lm}^{(j)} \in \mathfrak{u}_1(L^\wedge)$$

and there is a constant E and there is an H'' in $\mathfrak{w}_1(L^\wedge)$; for these operators the relation

$$\exp(\Gamma R)\exp(\Gamma Q) H_{\text{ren}} \exp(-\Gamma Q)\exp(-\Gamma R) = H'' \tag{9.6}$$

is valid. Furthermore they can be chosen so that

$$(H'' - H_0)_{lm} = 0\,, \qquad l - m < -r \text{ or } l = m = 0 \tag{9.7}$$

$$(H'' - H_0)_{lm} = \varepsilon W_{lm}^{(1)} + \cdots, \qquad -r \leqq l - m < 0 \tag{9.8}$$

$$(H'' - H_0)_{lm} = \varepsilon^{2n+1} W_{lm}^{(2n)}) + \cdots, \quad 0 \leqq l - m,\, l \neq 0\,. \tag{9.9}$$

The $+\cdots$ signify higher order terms in a convergent series.

Proof. Let $Q^{(0)} = 0$ and $Q_j = \sum_{0 \leqq i \leqq j} \varepsilon^i Q^{(i)}$. We proceed by induction and so we may suppose Q_j defined for some $j \geqq 0$. Let

$$\exp(\Gamma Q_j) H \exp(-\Gamma Q_j) = H'(j)\,. \tag{9.10}$$

We suppose

$$(H'(j) - H_0)_{l0} = \varepsilon^{j+1} Y_{l0}^{(j+1)} + \cdots, \qquad l > 0 \tag{9.11}$$

and we suppose $H'(j) \in \mathfrak{w}_1(\mathscr{S})$. Let

$$Q^{(j+1)} = \sum_{0<l} Y_{l0}^{(j+1)}\,. \tag{9.12}$$

Then

$$\begin{aligned} H'(j+1) &= \exp(\Gamma Q_{j+1}) H \exp(-\Gamma Q_{j+1}) \\ &\equiv \exp(\Gamma Q_j) H \exp(-\Gamma Q_j) + \textstyle\sum_{0<l} \varepsilon^{j+1} [\Gamma Y^{(j+1)}, H_0] \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\operatorname{mod} \varepsilon^{j+2}) \\ &= H'(j) - \varepsilon^{j+1} Q^{(j+1)}\,. \end{aligned}$$

This proves (9.11) for $j+1$. By the definition (9.12), $Q^{(j+1)} \in \mathfrak{u}_1(\mathscr{S})$. By Lemma 8.3, $\Gamma Q^{(j+1)} \in \mathfrak{u}_1(\mathscr{S})$ and by Theorem 5.5, $H'(j+1) \in \mathfrak{w}_1(\mathscr{S})$. This completes the inductive definition of Q. Let $H' = H(2n)$. Then

$$\exp(\Gamma Q) H \exp(-\Gamma Q) = H'\,. \tag{9.13}$$

Let $R^{(0)} = 0$ and let $R_1 = \sum_{0 \leqq i \leqq j} \varepsilon^i R^{(i)}$. We use a similar induction; let

$$\exp(\Gamma R_j) H' \exp(-\Gamma R_j) = H''(j) .$$

Then

$$(H''(j) - H_0)_{lm} = \varepsilon^{j+1} Z_{lm}^{(j+1)} + \cdots , \qquad l \geqq m, l \neq 0 \quad (9.14)$$

and $(H''(j) - H_0)_{l0} = \varepsilon^{2n+1} Z_{l0}^{(2n+1)} + \cdots$ for $l > 0$. Let

$$R^{(j+1)} = \sum_{l \geqq m > 0} Z_{lm}^{(j+1)}$$
$$E = (H''(2n))_{00}$$
$$H'' = H''(2n) - E .$$

Then

$$\exp(\Gamma R)\,(H' - E) \exp(-\Gamma R) = H''$$

and this combined with (9.13) yields (9.6) and completes the proof.

Let W'' be the sum of the terms occuring in (9.9) and let

$$\begin{aligned} H_2 &= H'' - W'' , \\ H_1 &= \exp(-\Gamma R) \exp(-\Gamma Q) H_2 \exp(\Gamma Q) \exp(\Gamma R) \\ &= H_{\text{ren}} - \exp(-\Gamma R) \exp(-\Gamma Q)\, W'' \exp(\Gamma Q) \exp(\Gamma R) \\ &= H_{\text{ren}} - \exp(-\Gamma R) \exp(-\Gamma Q)\, W'' \exp(\Gamma Q) \exp(\Gamma R) \end{aligned}$$

and let

$$W_1 = -\exp(-\Gamma R) \exp(-\Gamma Q)\, W'' \exp(\Gamma Q) \exp(\Gamma R) .$$

Then $W_1 \in \mathfrak{u}_{1,0}(L^\wedge)$ and

$$W_1 = \varepsilon^{2n+1} W_1^{(2n+1)} + \cdots$$

Lemma 9.3. There is a $P = P(\varepsilon)$ in $\mathfrak{u}_{1,1}(L^\wedge)$ such that

$$\exp(P) H_{\text{ren}} \exp(-P) = H_1 . \qquad (9.15)$$

If $l - m = \mu$ then P_{lm} has an expansion

$$P_{lm} = \varepsilon^{2n-\mu} P_{lm}^{(2n-\mu)} + \cdots \qquad (9.16)$$

which converges for $\varepsilon \neq 0$ (and for $\varepsilon = 0$ if $\mu \leqq 2n$).

Proof. First we discuss the equation

$$([P_{l+r,m}, \varepsilon V_{0r}])_{lm} = F_{lm} \qquad (9.17)$$

where F_{lm} is given and $P_{l+r,m}$ is the unknown. The proof of Theorem 4.1 gives us

Lemma 9.4. Let $F_{lm} \in \mathfrak{u}_{1,0}(L^\wedge)$ and let $\varepsilon \neq 0$. A solution $P_{l+r,m}$ in $\mathfrak{u}_{1,1}(L^\wedge)$ to (9.17) can be found which is a continuous linear function of F_{lm}.

It follows that if

$$F_{lm} = \varepsilon^j F_{lm}^{(j)} + \cdots \qquad (9.18)$$

then

$$P_{l+r,m} = \varepsilon^{j-1} P_{l+r,m}^{(j-1)} + \cdots \qquad (9.19)$$

and (9.19) converges if (9.18) does. We construct P by induction on μ. If $l - m < r$ set $P_{lm} = 0$. Suppose

$$P_\mu = \sum_{l-m \leqq \mu} P_{lm} \tag{9.20}$$

has been defined so that (9.16) holds and

$$(\exp(P_\mu) H_{\mathrm{ren}} \exp(-P_\mu))_{lm} = (H_1)_{lm} \tag{9.21}$$

if $l - m \leqq \mu - r$. (This is true if $\mu = r - 1$.) In order to achieve (9.21) for $\mu + 1$, we only need

$$\begin{aligned}[P_{m+\mu+1,m}, \varepsilon V_{0r}]_{m+\mu+1-r,m} &= \{-\exp(P_\mu) H_{\mathrm{ren}} \exp(-P_\mu) + H_{\mathrm{ren}} + W_1 - \\ &\quad - \textstyle\sum_{j=1}^{r-1} [P_{m+\mu+1-j,m-j}, V_{0r}]\}_{m+\mu+1-r,m}.\end{aligned} \tag{9.22}$$

This is solved by induction on m, using Lemma 9.4.

To complete the induction on μ we must verify (9.16) for $\mu + 1$. The first two terms in the right member of (9.22) contribute a sum of terms of the form

$$\operatorname{ad} P_{m_1+k_1,m_1} \ldots \operatorname{ad} P_{m_j+k_j,m_j} (H_{\mathrm{ren}})_{m_0+\varrho,m_0}, \tag{9.23}$$

where $j \geqq 1$, $0 \leqq k_i \leqq \mu$ and

$$\sum_{i=1}^{j} k_i + \varrho = \mu + 1 - r\,.$$

However, $-\varrho \leqq r$ and so

$$\sum_{i=1}^{j} k_i = \mu + 1 - (\varrho + r) \leqq \mu + 1\,. \tag{9.24}$$

If $\varrho = -r$ then $(H_{\mathrm{ren}})_{m_0+\varrho,m_0} = \varepsilon V_{0r}$ if $m_0 = r$ and is zero otherwise. Thus in this case H_{ren} contributes a power of ε to (9.23). If $\varrho > -r$ then the inequality prevails in (9.24). Combining these two cases and using (9.16) we see that the minimum exponent of ε occuring in (9.23) is at least $2jn - \mu \leqq 2n - \mu$. It follows that (9.16) is true for $P_{\mu+1,0}$, and by an induction on m, it is true for $P_{m+\mu+1,m}$. This completes the induction on μ. (9.15) follows from (9.21) and the proof is complete.

Let

$$T_1 = \exp(-P)\exp(-\Gamma Q)\exp(-\Gamma R)\,.$$

We now have

$$H_{\mathrm{ren}} T_1 = T_1 H_2\,,$$

or in other words, H_{ren} has been put in "triangular form".

Lemma 9.5. There is an operator

$$U = \sum_{l<m} U_{lm} \tag{9.25}$$

such that

$$\exp(\Gamma U) H_2 \exp(-\Gamma U) = H_0\,. \tag{9.26}$$

The terms

$$U_{lm} = \varepsilon U_{lm}^{(1)} + \cdots$$

are analytic functions of ε with coefficients $U_{lm}^{(j)}$ in $\mathfrak{u}_1(L^\wedge)$.

We give

$$\mathfrak{D}_0(L^\wedge) = \cup_k \mathfrak{D}^k(L^\wedge)$$

the inductive limit topology. Each subspace $\mathfrak{D}^k(L^\wedge)$ of states with at most k particles gets its topology from the seminorms (8.14), $\nu = 1, 2, \ldots$, $n = k$. We realize expressions of the form (9.25) as operators on $\mathfrak{D}_0(L^\wedge)$. We assert that the power series $\exp(\Gamma U)$ converges and that ΓU and $\exp \Gamma U$ are continuous operators. It is sufficient to prove this on each subspace $\mathfrak{D}^k(L^\wedge)$. However on such a subspace at most $k+1$ terms of the power series are nonzero. Thus the convergence is trivial and continuity follows from Lemma 8.2 (or Proposition 8.4). Let

$$T = T_1 \exp(-\Gamma U)\,.$$

Our lemmas show that (9.4) holds, and so Theorem 9.1 follows from Lemma 9.5.

Proof of Lemma 9.5. Let

$$U_j = \sum_{l-m=-j} U_{lm}\,.$$

Let $U_j = 0$ for $j \leqq 0$ and by induction on j choose U_j so that

$$\exp(\Gamma \textstyle\sum_{i=1}^{j} U_i) H_2 \exp(-\Gamma \sum_{i=1}^{j} U_i) - H_0 \qquad (9.27)$$

only contains nonzero terms $W_{lm}(j)$ for which $l + j + 1 \leqq m$. We set

$$U_{j+1} = \sum_{l-m=-(j+1)} W_{lm}(j)$$

and check that (9.27) has the correct form. Thus U is defined and (9.26) holds. The analyticity of U follows from that of H_2 together with the fact that each term U_{lm} of U depends (in a continuous manner) on only a finite number of the terms of $H_2 - H_0$.

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