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Variational Inequality with Evolutional Curl Constraint in a Multi-Connected Domain

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Abstract

We consider a system of quasilinear parabolic type equations involving operator curl associated with the Maxwell equations in a multi-connected domain. The paper is a continuation of the author's previous paper. We deal with a variational inequality with curl constraint. It is an extension of the results of Miranda et al. for p-curl system.

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1 Introduction

In the previous paper Aramaki [4], we considered the Maxwell equations in an electromagnetic field. If we denote the electric and the magnetic fields by $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively, it is well known that $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the classical Maxwell equations

$$\begin{cases} \varepsilon \boldsymbol{E}_t + \sigma \boldsymbol{j} = \operatorname{curl} \boldsymbol{H}, \\ \mu \boldsymbol{H}_t + \operatorname{curl} \boldsymbol{E} = \boldsymbol{F}, \\ \varepsilon \operatorname{div} \boldsymbol{E} = q, \\ \operatorname{div} \boldsymbol{H} = 0 \end{cases} \tag{1.1}$$

in $Q_T := (0,T)\times\Omega$ where Ω is a bounded domain in $\mathbb{R}^3$, ε is the permittivity of the electric field, μ is the permeability of the magnetic field, σ is the electric conductivity of the material, $\boldsymbol{j}$ is the total current density and q is the density of electric charge. Since the displace

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current $\varepsilon \boldsymbol{E}_t$ is small in comparison of eddy currents, we neglect the term. We use the non-linear extension of Ohm's law $|\boldsymbol{j}|^{p-2}\boldsymbol{j} = \sigma \boldsymbol{E}$ where σ is the electric conductivity. Then $\boldsymbol{H}$ satisfies the following equation

$$\begin{cases} \mu \boldsymbol{H}_t + \operatorname{curl}[\frac{1}{\sigma}|\operatorname{curl}\boldsymbol{H}|^{p-2}\operatorname{curl}\boldsymbol{H}] = \boldsymbol{F}, \\ \operatorname{div}\boldsymbol{H} = 0 \text{ in } Q_T. \end{cases} \tag{1.2}$$

Natural tangent boundary conditions are

$$\boldsymbol{H}\cdot\boldsymbol{n} = 0,\ \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{G} \text{ on } \Sigma_T := \Gamma\times(0,T)$$

where $\Gamma = \partial\Omega$ is the boundary of Ω and $\boldsymbol{n}$ denotes the outer normal unit vector field to Γ. Here the second boundary condition corresponds to consider a superconductive wall. Putting $\nu = 1/\sigma$, we must consider the following problem.

$$\begin{cases} \mu \boldsymbol{H}_t + \operatorname{curl}[\nu|\operatorname{curl}\boldsymbol{H}|^{p-2}\operatorname{curl}\boldsymbol{H}] = \boldsymbol{F}, & \\ \operatorname{div}\boldsymbol{H} = 0 & \text{in } Q_T, \\ \boldsymbol{H}\cdot\boldsymbol{n} = 0,\ \nu|\operatorname{curl}\boldsymbol{H}|^{p-2}\operatorname{curl}\boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{G} & \text{on } \Sigma_T, \\ \boldsymbol{H}(0) = \boldsymbol{H}_0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

For such formulation, see Yin et al. [15], Yin [14], Miranda et al. [10] and Aramaki [3, 4].

We consider a constitutive law arising in type-II superconductors and we know it as an extension of the Bean critical-state model in Prigozhin [11]. In this case the current density cannot exceed the critical value $\Psi(x,t) > 0$ and we have

$$\boldsymbol{E} = \begin{cases} \nu|\operatorname{curl}\boldsymbol{H}|^{p-2}\operatorname{curl}\boldsymbol{H} & \text{if } |\operatorname{curl}\boldsymbol{H}| < \Psi(x,t), \\ (\nu\Psi^{p-2}+\lambda)\operatorname{curl}\boldsymbol{H} & \text{if } |\operatorname{curl}\boldsymbol{H}| = \Psi(x,t) \end{cases}$$

where $\nu = \nu(x) \geq 0$ and $\lambda = \lambda(x,t) \geq 0$ is an unkown Lagrange multiplier.

The authors of [10] considered the system of equations containing such problem. They replaced the term $\nu|\boldsymbol{u}|^{p-2}\boldsymbol{u}$ with $\boldsymbol{u} = \operatorname{curl}\boldsymbol{H}$ in (1.3) with a Carathéodry function $\boldsymbol{a}(x,t,\boldsymbol{u})$ which satisfies some structure conditions, and they showed the existence theorem. They also examine the variational inequality with evolutional curl constraint for the case where $\boldsymbol{a}(x,t,\boldsymbol{u}) = \boldsymbol{a}(x,\boldsymbol{u}) = \nu|\boldsymbol{u}|^{p-2}\boldsymbol{u}$. This associates with the solution to the Bean model. However, they assumed that Ω is a simply-connected domain Ω.

In this paper, we extend the results of the variatonal inequality to the case where Ω is multi-connected domain with tangential or normal boundary conditions. Furthermore, it will be seen that the results in [10] are extended. The authors of [10] treated only the case $\boldsymbol{a}(x,t,\boldsymbol{u}) = \nu(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$, but we shall extend the result to the more general form of $\boldsymbol{a}(x,t,\boldsymbol{u}) = \boldsymbol{a}(x,\boldsymbol{u})$.

The paper is organized as follows. In section 2, since we allow the domain Ω to be multi-connected and we need some Poincaré type inequalities, we must set the domain appropriately. Moreover, since we consider more general formulation $\boldsymbol{a}(x,t,\boldsymbol{u})$ than $\nu|\boldsymbol{u}|^{p-2}\boldsymbol{u}$, we must impose the structure conditions for $\boldsymbol{a}(x,t,\boldsymbol{u})$. Furthermore, we shall review the existence theorem and the regularity of solution which will be used in section 3. In section 3, we show that the variational inequality with evolutional curl constraint. We shall extend the results of [10] who considered only the case $\boldsymbol{a}(x,\boldsymbol{u}) = \nu|\boldsymbol{u}|^{p-2}\boldsymbol{u}$. Section 4 is devoted with the continuous dependence on the data. Finally, in section 5, we consider the limit problem as the time variable tending to infinity.

2 Preliminaries

In this section, we state some preliminaries and the existence and regularity theorems which were proved in the previous paper [4] (cf. also [10]) for the genelarized system containing the system (1.2) with some boundary and initial conditions.

Since we allow that Ω is multi-connected, we assume that Ω has the following conditions as in Amrouche and Seloula [1] (cf. also see Amrouche and Seloula [2], Dautray and Lions [6] and Girault and Raviart [7]). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{1,1}$ with the boundary $\Gamma = \partial\Omega$ and Ω is locally situated on one side of Γ.

(i) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(ii) There exist n connected open surfaces Σ_j, $(j = 1, \dots, n)$, called cuts, contained in Ω such that

  (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

  (b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \dots, n)$ and Σ_i is non-tangential to Γ.

  (c) $\overline{\Sigma_i} \cap \overline{\Sigma_j} = \emptyset (i \neq j)$.

  (d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{i=1}^n \Sigma_i)$ is simply connected and Lipschitz.

Put $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$. Let $\boldsymbol{a} : Q_T \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathéodry function with values in $\mathbb{R}^3$ satisfying the following structure conditions. There exist constants $a_*, a^* > 0$ and $1 < p < \infty$ such that for all $(x,t) \in Q_T$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$,

$$\boldsymbol{a}(x,t,\boldsymbol{u}) \cdot \boldsymbol{u} \geq a_* |\boldsymbol{u}|^p, \tag{2.1}$$

$$|\boldsymbol{a}(x,t,\boldsymbol{u})| \leq a^* |\boldsymbol{u}|^{p-1}, \tag{2.2}$$

$$(\boldsymbol{a}(x,t,\boldsymbol{u}) - \boldsymbol{a}(x,t,\boldsymbol{v})) \cdot (\boldsymbol{u} - \boldsymbol{v}) > 0 \text{ if } \boldsymbol{u} \neq \boldsymbol{v}, \tag{2.3}$$

or

$$(\boldsymbol{a}(x,t,\boldsymbol{u}) - \boldsymbol{a}(x,t,\boldsymbol{v})) \cdot (\boldsymbol{u} - \boldsymbol{v}) \geq \begin{cases} a_* |\boldsymbol{u} - \boldsymbol{v}|^p & \text{if } p \geq 2, \\ a_* (|\boldsymbol{u}| + |\boldsymbol{v}|)^{p-2} |\boldsymbol{u} - \boldsymbol{v}|^2 & \text{if } p < 2. \end{cases} \tag{2.3$'$}$$

Here for any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, we denote the Euclidean inner product by $\boldsymbol{u} \cdot \boldsymbol{v}$. Note that (2.3)$'$ implies (2.3).

Example 2.1. *If* $\boldsymbol{a}(x,t,\boldsymbol{u}) = \nu(x,t) |\boldsymbol{u}|^{p-2} \boldsymbol{u}$ *where* $\nu(x,t)$ *is a measurable function in* $\Omega \times (0,T)$ *satisfying* $0 < \nu_* \leq \nu(x,t) \leq \nu^* < \infty$, *then* $\boldsymbol{a}$ *satisfies* (2.1)-(2.3). *Furthermore* $\boldsymbol{a}$ *also satisfies* (2.3)$'$.

For the proof, see Aramaki [4].

We consider the following two systems.

$$\begin{cases} \partial_t \boldsymbol{h} + \operatorname{curl} [\boldsymbol{a}(x,t,\operatorname{curl} \boldsymbol{h})] = \boldsymbol{f}(x,t) & \text{in } Q_T, \\ \operatorname{div} \boldsymbol{h} = 0 & \text{in } Q_T, \\ \boldsymbol{n} \cdot \boldsymbol{h}(x,t) = 0 & \text{on } \Sigma_T, \\ \boldsymbol{a}(x,t,\operatorname{curl} \boldsymbol{h}) \times \boldsymbol{n} = \boldsymbol{g}(x,t) & \text{on } \Sigma_T, \\ \boldsymbol{h}(x,0) = \boldsymbol{h}_0(x) & \text{in } \Omega \end{cases} \tag{2.4$_N$}$$

where $\boldsymbol{n}$ is the unit outer normal vector field to Γ.

$$\begin{cases} \partial_t \boldsymbol{h} + \operatorname{curl}[\boldsymbol{a}(x,t,\operatorname{curl}\boldsymbol{h})] = \boldsymbol{f}(x,t) & \text{in } Q_T, \\ \operatorname{div}\boldsymbol{h} = 0 & \text{in } Q_T, \\ \boldsymbol{n}\times\boldsymbol{h}(x,t) = \boldsymbol{0} & \text{on } \Sigma_T, \\ \boldsymbol{a}(x,t,\operatorname{curl}\boldsymbol{h}) = \boldsymbol{g}(x,t) & \text{on } \Sigma_T, \\ \boldsymbol{h}(x,0) = \boldsymbol{h}_0(x) & \text{in } \Omega. \end{cases} \quad . \tag{2.4$_T$}$$

Throughout this paper, if E is a function space, we denote the vector space E^3 by $\boldsymbol{E}$. Define the following closed subspaces of $\boldsymbol{W}^{1,p}(\Omega)$ where $W^{1,p}(\Omega)$ is the standard Sobolev space.

$$\mathbb{W}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0, i = 1,\dots,n\},$$

and

$$\mathbb{W}^p_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}\times\boldsymbol{n} = 0 \text{ on } \Gamma, \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_j} = 0, j = 1,\dots,m\}.$$

Then we have

Proposition 2.2. *Let $1 < p < \infty$. Then $\mathbb{W}^p_N(\Omega)$ and $\mathbb{W}^p_T(\Omega)$ are reflexive, separable Banach spaces, and the semi-norm $\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is the norm in $\mathbb{W}^p_N(\Omega)$ and $\mathbb{W}^p_T(\Omega)$, and it is equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.*

For the proof, see Brezis [5] and [1, 2].

From now, we write $\mathbb{W}^p_*(\Omega)$ for $\mathbb{W}^p_N(\Omega)$ or $\mathbb{W}^p_T(\Omega)$ and $\|\boldsymbol{v}\|_{\mathbb{W}^p_*(\Omega)} = \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$.

It is well known that the following Sobolev type inequalities and the trace theorem hold.

Proposition 2.3. *There exist positive constants C_q and C_r such that for all $\boldsymbol{v} \in \mathbb{W}^p_*(\Omega)$,*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \le C_q \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \text{ with } \begin{cases} q \le 3p/(3-p) & \text{if } 1 < p < 3, \\ \text{any } q < \infty & \text{if } p = 3, \\ q = \infty & \text{if } p > 3, \end{cases}$$

and

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Gamma)} \le C_r \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \text{ with } \begin{cases} r \le 2p/(3-p) & \text{if } 1 < p < 3, \\ \text{any } r < \infty & \text{if } p = 3, \\ r = \infty & \text{if } p > 3, \end{cases}$$

In particular, if $p \ge 6/5$, then the embedding $\mathbb{W}^p_*(\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is continuous.

Throughout this paper, we assume that $p \ge 6/5$ for the brevity.

Define $\mathbb{L}^2_*(\Omega) =$ the closure of $\mathbb{W}^p_*(\Omega)$ in $\boldsymbol{L}^2(\Omega)$. Then we have

$$\mathbb{W}^p_*(\Omega) \subset \mathbb{L}^2_*(\Omega) \subset \mathbb{W}^p_*(\Omega)'$$

where $\mathbb{W}^p_*(\Omega)'$ is the dual space of $\mathbb{W}^p_*(\Omega)$. Here we note $\mathbb{W}^p_*(\Omega)$ is dense in $\mathbb{L}^2_*(\Omega)$ and the inclusion maps are continuous.

For a.e. $t \in (0,T)$, define an operator $A(t) : \mathbb{W}^p_*(\Omega) \to \mathbb{W}^p_*(\Omega)'$ by

$$\langle A(t)\boldsymbol{h}, \boldsymbol{\phi}\rangle = \int_\Omega \boldsymbol{a}(x,t,\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{\phi}\, dx \text{ for all } \boldsymbol{h},\boldsymbol{\phi} \in \mathbb{W}^p_*(\Omega).$$

Then we see that $A(t)$ is a bounded operator, and

$$\|A(t)\boldsymbol{h}\|_{\mathbb{W}^p_*(\Omega)'} \leq a^* \|\mathrm{curl}\,\boldsymbol{h}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

In fact, by the structure condition (2.2) and the Hölder inequality, we have

$$\begin{aligned} |\langle A(t)\boldsymbol{h},\boldsymbol{\phi}\rangle| &\leq \int_\Omega a^* |\mathrm{curl}\,\boldsymbol{h}|^{p-1} |\mathrm{curl}\,\boldsymbol{\phi}|\,dx \\ &\leq a^* \left(\int_\Omega |\mathrm{curl}\,\boldsymbol{h}|^p\,dx\right)^{1/p'} \left(\int_\Omega |\mathrm{curl}\,\boldsymbol{\phi}|^p\,dx\right)^{1/p} \\ &= a^* \|\mathrm{curl}\,\boldsymbol{h}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \|\mathrm{curl}\,\boldsymbol{\phi}\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Here and hereafter, for any $1 \leq p \leq \infty$, we denote the conjugate index by p', that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Since the function $\boldsymbol{a}(x,t,\boldsymbol{u})$ is the Carathéodry function, we easily see that $A(t)$ is hemicontinuous, i.e., for any $\boldsymbol{h},\boldsymbol{\phi},\boldsymbol{\psi} \in \mathbb{W}^p_*(\Omega), \lambda \in \mathbb{R}$, $\langle A(t)(\boldsymbol{h}+\lambda\boldsymbol{\phi}),\boldsymbol{\psi}\rangle$ is continuous with respect to λ. Moreover, we can see that $A(t)$ is coercive, i.e., for $\boldsymbol{v} \in \mathbb{W}^p_*(\Omega)$, $\langle A(t)\boldsymbol{v},\boldsymbol{v}\rangle \geq \alpha\|\boldsymbol{v}\|^p_{\mathbb{W}^p_*(\Omega)}$ for some $\alpha > 0$. In fact, from structure condition (2.1), we have

$$\langle A(t)\boldsymbol{v},\boldsymbol{v}\rangle = \int_\Omega \boldsymbol{a}(x,t,\mathrm{curl}\,\boldsymbol{v}) \cdot \mathrm{curl}\,\boldsymbol{v}\,dx \geq a_* \int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p\,dx = a_*\|\boldsymbol{v}\|^p_{\mathbb{W}^p_*(\Omega)}.$$

Let $\boldsymbol{f}(t) \in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega)), \boldsymbol{g}(t) \in L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))$ where p',q' and r' are conjugate index of p,q and r, respectively, and q and r are as in Proposition 2.3. For a.e., $t \in (0,T)$, define $\mathbb{L}_*(t) \in \mathbb{W}^p_*(\Omega)'$ where $* = N$ or $* = T$ by

$$\langle \mathbb{L}_*(t),\boldsymbol{\phi}\rangle = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{\phi}\,dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot\boldsymbol{\phi}\,dS$$

for $\boldsymbol{\phi} \in \mathbb{W}^p_*(\Omega)$. Here we denote

$$\boldsymbol{g}^*(t) = \begin{cases} \boldsymbol{g}(t) & \text{if } * = N, \\ \boldsymbol{n}\times\boldsymbol{g}(t) & \text{if } * = T \end{cases} \tag{2.5}$$

Well-definedness follows from Proposition 2.2 and following inequalities.

$$\int_\Omega |\boldsymbol{f}(t)\cdot\boldsymbol{\phi}|\,dx \leq \|\boldsymbol{f}(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{\phi}\|_{\boldsymbol{L}^q(\Omega)} \leq C\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{\phi}\|_{\mathbb{W}^p_*(\Omega)}$$

and

$$\int_\Gamma |\boldsymbol{g}^*(t)\cdot\boldsymbol{\phi}|\,dx \leq \|\boldsymbol{g}(t)\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\boldsymbol{\phi}\|_{\boldsymbol{L}^r(\Gamma)} \leq C\|\boldsymbol{g}(t)\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\boldsymbol{\phi}\|_{\mathbb{W}^p_*(\Omega)}.$$

Therefore we can apply the result of Zheng [16, Theorem 3.2.1] (see also Lions [9]) as follows. For any $\boldsymbol{h}_0 \in \mathbb{L}^2_*(\Omega)$, $\mathbb{L}_*(t) \in Ł^{p'}(0,T;\mathbb{W}^p_*(\Omega)')$, the system

$$\begin{cases} \partial_t\boldsymbol{h} + A(t)\boldsymbol{h} = \mathbb{L}_*(t) & \text{in } Q_T, \\ \boldsymbol{h}(0) = \boldsymbol{h}_0 & \text{in } \Omega \end{cases} \tag{2.6}$$

has a unique solution $\boldsymbol{h}\in C([0,T];\mathbb{L}^2_*(\Omega))\cap L^p(0,T;\mathbb{W}^p_*(\Omega))$, and $\partial_t\boldsymbol{h}=\boldsymbol{h}_t\in L^{p'}(0,T;\mathbb{W}^p_*(\Omega)')$ in the sense of $L^{p'}(0,T;\mathbb{W}^p_*(\Omega)')$. That is to say, for all $\boldsymbol{\phi}\in L^p(0,T;\mathbb{W}^p_*(\Omega))$,

$$\int_0^T\int_\Omega \boldsymbol{h}_t\cdot\boldsymbol{\phi}\,dxdt+\int_0^T\int_\Omega \boldsymbol{a}(x,t,\operatorname{curl}\boldsymbol{h}(t))\cdot\operatorname{curl}\boldsymbol{\phi}(t)\,dxdt$$
$$=\int_0^T\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{\phi}(t)\,dxdt+\int_0^T\int_\Gamma \boldsymbol{g}^*(t)\cdot\boldsymbol{\phi}(t)\,dS\,dt \quad (2.7)$$

where dS denotes the surface area of Γ and we interpret $\boldsymbol{h}_t\cdot\boldsymbol{\phi}$ as the duality of $\mathbb{W}^p_*(\Omega)'$ and $\mathbb{W}^p_*(\Omega)$. If we choose any $\boldsymbol{\phi}\in\mathbb{W}^p_*(\Omega)$ and $\eta\in C_0^\infty(0,T)$ and take $\boldsymbol{\phi}(t)=\eta(t)\boldsymbol{\phi}$ as a test function in (2.7), we can see that (2.7) means that for a.e. $t\in(0,T)$,

$$\begin{cases}\int_\Omega \boldsymbol{h}_t\cdot\boldsymbol{\phi}\,dx+\int_\Omega \boldsymbol{a}(x,t,\operatorname{curl}\boldsymbol{h}(t))\cdot\operatorname{curl}\boldsymbol{\phi}\,dx\\ \quad=\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{\phi}\,dx+\int_\Gamma \boldsymbol{g}^*(t)\cdot\boldsymbol{\phi}\,dS \ \text{ for all } \boldsymbol{\phi}\in\mathbb{W}^p_*(\Omega),\\ \boldsymbol{h}(0)=\boldsymbol{h}_0.\end{cases} \quad (2.8)_*$$

The first term of the left hand side in the first equation of $(2.8)_*$ is satisfied in the duality sense. This is the weak formulation of $(2.4)_N$ or $(2.4)_T$.

We have the following existence theorem of unique solution of $(2.4)_*$ and an estimate (cf. [10] or [4]).

Theorem 2.4. *Assume that* (2.1)-(2.3) *hold and let* $\boldsymbol{f}\in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))$, $\boldsymbol{g}\in L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))$, $\boldsymbol{h}_0\in\mathbb{L}^2_*(\Omega)$. *Then the problem* $(2.4)_*$ *has a unique solution*

$$\boldsymbol{h}\in L^p(0,T;\mathbb{W}^p_*(\Omega))\cap C([0,T];\mathbb{L}^2_*(\Omega)),\ \partial_t\boldsymbol{h}\in L^{p'}(0,T;\mathbb{W}^p_*(\Omega)')$$

in the sense of $(2.8)_*$. *Moreover there exists a constant* $C>0$ *independent of* T *such that*

$$\|\boldsymbol{h}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\operatorname{curl}\boldsymbol{h}\|^p_{\boldsymbol{L}^p(Q_T)}\le C(\|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}+\|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}+\|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)}).$$

When $\boldsymbol{a}(x,t,\boldsymbol{u})$ is independent of t, we can get more regularity. In order to do so, we assume that $\boldsymbol{a}(x,t,\boldsymbol{u})=\boldsymbol{a}(x,\boldsymbol{u})$ is a Carathéodry function on $\Omega\times\mathbb{R}^3$ satisfying (2.1)-(2.3). Moreover we assume that there exists a scalar function $b(x,\boldsymbol{u})$ which is measurable in x and C^1 class in $\boldsymbol{u}$ such that

$$\nabla_{\boldsymbol{u}}b(x,\boldsymbol{u})=\boldsymbol{a}(x,\boldsymbol{u}). \quad (2.9)$$

We may assume that $b(x,\boldsymbol{0})=0$. By a simple calculations, we have

$$\frac{1}{p}a_*|\boldsymbol{u}|^p\le b(x,\boldsymbol{u})\le\frac{1}{p}a^*|\boldsymbol{u}|^p.$$

Example 2.5. *If* $\boldsymbol{a}(x,\boldsymbol{u})=\nu(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$ *satisfies that* $0<a_*\le\nu(x)\le a^*<\infty$, *then* $b(x,\boldsymbol{u})=\frac{1}{p}\nu(x)|\boldsymbol{u}|^p$ *satisfies* (2.9).

Theorem 2.6. *Assume that* $\boldsymbol{a}(x,\boldsymbol{u})$ *satisfies* (2.1)-(2.3) *and* (2.9). *Let* $\boldsymbol{f}\in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))\cap\boldsymbol{L}^2(Q_T)$, $\boldsymbol{g}\in L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma))\cap W^{1,p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))$, $\boldsymbol{h}_0\in\mathbb{W}^p_*(\Omega)$. *Then the solution in Theorem* 2.4 *satisfies that*

$$\partial_t\boldsymbol{h}\in\boldsymbol{L}^2(Q_T),\ \operatorname{curl}\boldsymbol{h}\in L^\infty(0,T;\boldsymbol{L}^p(\Omega)).$$

For the proof, see [4] (cf. see also [10]).

3 Variational inequality with evolutional curl constraint.

In this section, we consider a variational inequality with evolutional curl constraint. In order to do so, we assume that $\boldsymbol{a}(x,t,\boldsymbol{u}) = \boldsymbol{a}(x,\boldsymbol{u})$ satisfies (2.1), (2.2), (2.3)′ and (2.9). The authors in [10] only considered the case where $\boldsymbol{a}(x,\boldsymbol{u}) = \nu(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$, however we shall extend their results to the more general function $\boldsymbol{a}(x,\boldsymbol{u})$.

Let $\Psi(x,t) \in W^{1,\infty}(0,T;\boldsymbol{L}^\infty(\Omega))$ with $\Psi(x,t) \geq \alpha > 0$ for some constant $\alpha > 0$ and define for a.e. $t \in (0,T)$,

$$\mathbb{K}(t) = \{\boldsymbol{v} \in \mathbb{W}_*^p(\Omega); b(x,\operatorname{curl}\boldsymbol{v}) \leq \Psi(x,t)^p \text{ a.e. in } \Omega\}.$$

Our problem is as follows: for given $\boldsymbol{f} \in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega)) \cap \boldsymbol{L}^2(Q_T)$, $\boldsymbol{g} \in L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma)) \cap W^{1,p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))$ and $\boldsymbol{h}_0 \in \mathbb{K}(0)$, find a function $\boldsymbol{h}$ in a suitable class such that $\boldsymbol{h}(t) \in \mathbb{K}(t)$ a.e. $t \in (0,T)$, $\boldsymbol{h}(0) = \boldsymbol{h}_0$ and

$$\begin{aligned}\int_\Omega \partial_t\boldsymbol{h}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}(t))\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}(t))dx \\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Gamma \boldsymbol{g}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dS \qquad (3.1)\end{aligned}$$

for any $\boldsymbol{\phi} \in \mathbb{K}(t)$ a.e. $t \in (0,T)$.

We will consider an approximation of the solution $\boldsymbol{h}$. In order to do so, for any $0 < \varepsilon < 1$, choose a function $k_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ which is continuous and increasing such that

$$k_\varepsilon(s) = \begin{cases} 1 & s \leq 0, \\ e^{s/\varepsilon} & \varepsilon \leq s < 1/\varepsilon - \varepsilon, \\ e^{1/\varepsilon^2} & s \geq 1/\varepsilon, \end{cases}$$

and define

$$\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u}) = k_\varepsilon((b(x,\boldsymbol{u}) - \Psi(x,t)^p)\boldsymbol{a}(x,\boldsymbol{u})$$

where $b(x,\boldsymbol{u})$ is a function as in (2.9). We note that $\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})$ satisfies the structure condition (2.1), (2.2) and (2.3)′ with a^* replaced with $e^{1/\varepsilon^2}a^*$. In fact, since $1 \leq k_\varepsilon \leq e^{1/\varepsilon^2}$,

$$\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})\cdot\boldsymbol{u} \geq \boldsymbol{a}(x,\boldsymbol{u})\cdot\boldsymbol{u} \geq a_*|\boldsymbol{u}|^p,$$

and

$$|\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})| \leq e^{1/\varepsilon^2}|\boldsymbol{a}(x,\boldsymbol{u})| \leq e^{1/\varepsilon^2}a^*.$$

Thus it is clear that (2.1) and (2.2) hold. For (2.3)′, we have

$$\begin{aligned}&(\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u}) - \boldsymbol{a}_\varepsilon(x,t,\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v}) \\ &\quad = \big(k_\varepsilon(b(x,\boldsymbol{u}) - \Psi(x,t)^p)\boldsymbol{a}(x,\boldsymbol{u}) - k_\varepsilon(b(x,\boldsymbol{v}) - \Psi(x,t)^p)\boldsymbol{a}(x,\boldsymbol{v})\big)\cdot(\boldsymbol{u}-\boldsymbol{v}). \qquad (3.2)\end{aligned}$$

We claim that if $b(x,\boldsymbol{u}) \geq b(x,\boldsymbol{v})$, then $\boldsymbol{a}(x,\boldsymbol{u})\cdot(\boldsymbol{u}-\boldsymbol{v}) \geq 0$. Because, if $\boldsymbol{a}(x,\boldsymbol{u})\cdot(\boldsymbol{u}-\boldsymbol{v}) < 0$, then we have

$$\begin{aligned}b(x,\boldsymbol{u}) - b(x,\boldsymbol{v}) &= (\boldsymbol{u}-\boldsymbol{v})\cdot\int_0^1 \nabla_{\boldsymbol{u}} b(x,\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))d\theta \\ &= (\boldsymbol{u}-\boldsymbol{v})\cdot\int_0^1 \boldsymbol{a}(x,\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))d\theta. \qquad (3.3)\end{aligned}$$

From the monotonicity of $\boldsymbol{a}$,

$$(\boldsymbol{a}(x,\boldsymbol{u}) - \boldsymbol{a}(x,\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))\cdot(\boldsymbol{u}-(\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))\geq 0,$$

so

$$\boldsymbol{a}(x,\boldsymbol{v}+\theta(\boldsymbol{u}-\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v})\leq \boldsymbol{a}(x,\boldsymbol{u})\cdot(\boldsymbol{u}-\boldsymbol{v})<0$$

for $\theta<1$. Thus from (3.3), we have $b(x,\boldsymbol{u})-b(x,\boldsymbol{v})<0$. This is a contradiction. Therefore it follows from (3.2) and the increasingness of k_ε that

$$\begin{aligned}(\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})-\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v}) &\geq (k_\varepsilon(b(x,\boldsymbol{v})-\Psi(x,t)^p)(\boldsymbol{a}(x,\boldsymbol{u})-\boldsymbol{a}(x,\boldsymbol{v}))\cdot(\boldsymbol{u}-\boldsymbol{v})\\ &\geq (\boldsymbol{a}(x,\boldsymbol{u})-\boldsymbol{a}(x,\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}).\end{aligned}$$

Thus if $\boldsymbol{a}(x,\boldsymbol{u})$ satisfies (2.3)$'$, then $\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})$ also satisfies (2.3)$'$. The case where $b(x,\boldsymbol{u})\leq b(x,\boldsymbol{v})$ is similar.

Thus from Theorem 2.4 with $\boldsymbol{a}(x,t,\boldsymbol{u})=\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})$, there exists a unique solution $\boldsymbol{h}_\varepsilon\in L^p(0,T;\mathbb{W}^p_*(\Omega))\cap C([0,T];\mathbb{L}_*(\Omega))$, $\partial_t\boldsymbol{h}_\varepsilon\in L^{p'}(0,T;\mathbb{W}^p_*(\Omega)')$ such that for a.e. $t\in(0,T)$,

$$\int_\Omega \partial_t\boldsymbol{h}_\varepsilon(t)\cdot\boldsymbol{\phi}\,dx+\int_\Omega \boldsymbol{a}_\varepsilon(x,t,\operatorname{curl}\boldsymbol{h}_\varepsilon(t))\cdot\operatorname{curl}\boldsymbol{\phi}\,dx=\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{\phi}\,dx+\int_\Gamma \boldsymbol{g}^*(t)\cdot\boldsymbol{\phi}\,dS \quad (3.4)$$

for any $\boldsymbol{\phi}\in\mathbb{W}^p_*(\Omega)$.

Lemma 3.1. *There exists a constant $C_1>0$ independent of $0<\varepsilon<1$ and $T>0$ such that*

$$\begin{aligned}&\|k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t))-\Psi(x,t)^p)\|_{L^1(Q_T)}\\ &\qquad\leq C_1\Big(\frac{1}{\alpha^p}\|\Psi\|^p_{L^p(Q_T)}+\|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}+\|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}+\|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)}\Big).\end{aligned}$$

Proof. If we take $\boldsymbol{\phi}=\boldsymbol{h}_\varepsilon(t)$ as a test function of (3.4), then using Hölder inequality we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\boldsymbol{h}_\varepsilon(t)|^2dx+\int_\Omega\boldsymbol{a}_\varepsilon(x,t,\operatorname{curl}\boldsymbol{h}_\varepsilon(t))\cdot\operatorname{curl}\boldsymbol{h}_\varepsilon(t)dx\\ &\quad\leq\int_\Omega|\boldsymbol{f}(t)\cdot\boldsymbol{h}_\varepsilon(t)|dx+\int_\Gamma|\boldsymbol{g}(t)\cdot\boldsymbol{h}_\varepsilon(t)|dS\\ &\quad\leq\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{h}_\varepsilon(t)\|_{\boldsymbol{L}^q(\Omega)}+\|\boldsymbol{g}(t)\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\boldsymbol{h}_\varepsilon(t)\|_{\boldsymbol{L}^r(\Gamma)}.\end{aligned}$$

From the monotonicity of $\boldsymbol{a}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\boldsymbol{h}_\varepsilon(t)|^2dx+a_*\int_\Omega k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t))-\Psi(x,t)^p)|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^pdx\\ &\qquad\leq C_q\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)\|_{L^p(\Omega)}+C_r\|\boldsymbol{g}(t)\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)\|_{L^p(\Omega)}.\quad(3.5)\end{aligned}$$

Integrating over $(0,T)$ and using Hölder inequality, for any $\delta > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{h}_\varepsilon(T)|^2 dx + a_* \int_0^T \int_\Omega k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^p dxdt \\
&\le C_q\left(\int_0^T \|\boldsymbol{f}(t)\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} dt\right)^{1/p'} \left(\int_0^T \|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)\|^p_{\boldsymbol{L}^p(\Omega)} dt\right)^{1/p} \\
&\quad + C_r\left(\int_0^T \|\boldsymbol{g}(t)\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} dt\right)^{1/p'} \left(\int_0^T \|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)\|^p_{\boldsymbol{L}^p(\Omega)} dt\right)^{1/p} + \frac{1}{2}\int_\Omega |\boldsymbol{h}_0|^2 dx \\
&\le C_{q,\delta}\int_0^T \|\boldsymbol{f}(t)\|^{p'}_{L^{q'}(\Omega)} dt + C_{r,\delta}\int_0^T \|\boldsymbol{g}(t)\|^{p'}_{L^{r'}(\Gamma)} dt + \delta\int_0^T\int_\Omega |\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^p dxdt \\
&\quad + \frac{1}{2}\int_\Omega |\boldsymbol{h}_0|^2 dx \\
&\le C_{q,\delta}\int_0^T \|\boldsymbol{f}(t)\|^{p'}_{L^{q'}(\Omega)} dt + C_{r,\delta}\int_0^T \|\boldsymbol{g}(t)\|^{p'}_{L^{r'}(\Gamma)} dt \\
&\quad + \delta\int_0^T\int_\Omega k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^p dxdt + \frac{1}{2}\int_\Omega |\boldsymbol{h}_0|^2 dx.
\end{aligned}$$

Choosing $\delta = a_*/2$ and using $b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) \le \frac{1}{p}a^*|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^p$, we have

$$\begin{aligned}
&\iint_{Q_T} k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p) b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) dxdt \\
&\qquad\qquad \le C(\|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))} + \|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)}
\end{aligned} \tag{3.6}$$

where C is a constant independent of $0 < \varepsilon < 1$ and T. On the other hand, if we put

$$\begin{aligned}
D_\varepsilon &= \{(x,t) \in Q_T; b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) \le \Psi(x,t)^p\}, \\
E_\varepsilon &= \{(x,t) \in Q_T; b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) \ge \Psi(x,t)^p\},
\end{aligned}$$

and we note that $k_\varepsilon(s) = 1$ for $s \le 0$ and $k_\varepsilon(s)s \ge 0$, we have

$$\begin{aligned}
&\iint_{Q_T} k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p) dxdt \\
&\quad \ge \iint_{D_\varepsilon} b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) dxdt - \iint_{D_\varepsilon} \Psi(x,t)^p dxdt \ge -\iint_{Q_T} \Psi(x,t)^p dxdt.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\iint_{Q_T} k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)(\Psi(x,t)^p - b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(t))) dxdt \\
&\qquad\qquad \le \iint_{Q_T} \Psi(x,t)^p dxdt.
\end{aligned} \tag{3.7}$$

Since $\Psi(x,t) \geq \alpha > 0$ and using (3.6) and (3.7),

$$\begin{aligned}
&\iint_{Q_T} k_\varepsilon(b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) - \Psi^p) dxdt \\
&\leq \iint_{Q_T} k_\varepsilon(b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) - \Psi^p) \frac{\Psi^p}{\alpha^p} dxdt \\
&= \frac{1}{\alpha^p} \iint_{Q_T} k_\varepsilon(b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) - \Psi^p)(\Psi^p - b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) dxdt \\
&\quad + \frac{1}{\alpha^p} \iint_{Q_T} k_\varepsilon(b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p) b(x, \operatorname{curl} \boldsymbol{h}_\varepsilon(t)) dxdt \\
&\leq \frac{1}{\alpha^p} \|\Psi\|^p_{L^p(Q_T)} + C(\|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))} + \|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)}).
\end{aligned}$$

□

Lemma 3.2. *There exists a constant $C > 0$ independent of $0 < \varepsilon < 1$ and T such that*

$$\begin{aligned}
&\|\boldsymbol{h}_\varepsilon\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\operatorname{curl} \boldsymbol{h}_\varepsilon\|^p_{\boldsymbol{L}^p(Q_T)} \\
&\qquad \leq C_1 \Big(\frac{1}{\alpha^p} \|\Psi\|^p_{L^p(Q_T)} + \|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))} + \|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)} \Big).
\end{aligned}$$

Proof. We use (3.5). Integrating (3.5) over $(0,t)$ and using $k_\varepsilon \geq 1$ and Proposition 2.3, we have

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |\boldsymbol{h}_\varepsilon(t)|^2 dx + a_* \int_0^t \int_\Omega |\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)|^p dx d\tau \\
&\leq C_q \int_0^t \|\boldsymbol{f}(\tau)\|_{\boldsymbol{L}^{q'}(\Omega)} \|\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)\|_{\boldsymbol{L}^p(\Omega)} d\tau \\
&\quad + C_r \int_0^t \|\boldsymbol{g}(\tau)\|_{\boldsymbol{L}^{r'}(\Gamma)} \|\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)\|_{\boldsymbol{L}^p(\Omega)} d\tau + \frac{1}{2} \|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C_q \Big(\int_0^t \|\boldsymbol{f}(\tau)\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} d\tau \Big)^{1/p'} \Big(\int_0^t \|\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau \Big)^{1/p} \\
&\quad + C_r \Big(\int_0^t \|\boldsymbol{g}(\tau)\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} d\tau \Big)^{1/p'} \Big(\int_0^t \|\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau \Big)^{1/p} + \frac{1}{2} \|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C_{q,\delta} \int_0^t \|\boldsymbol{g}(\tau)\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} d\tau + C_{r,\delta} \int_0^t \|\boldsymbol{g}(\tau)\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} d\tau \\
&\quad + \delta \int_0^t \|\operatorname{curl} \boldsymbol{h}_\varepsilon(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau + \frac{1}{2} \|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned}$$

for any $\delta > 0$. Choosing $\delta = a_*/2$, and then taking supremum of both hand side, we see that there exists a constant $C > 0$ independent of $0 < \varepsilon < 1$ and T such that the conclusion holds. □

Lemma 3.3. *There exists a constant $C > 0$ depending only on $\|\Psi\|_{L^p(Q_T)}$, $\|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}$, $\|\boldsymbol{g}\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}$ and $\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}$ but independent of $0 < \varepsilon < 1$ such that*

$$\|\partial_t \boldsymbol{h}_\varepsilon\|_{\boldsymbol{L}^2(Q_T)} \leq C, \quad \operatorname*{ess\,sup}_{0 \leq t \leq T} \|\operatorname{curl} \boldsymbol{h}_\varepsilon(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C.$$

Proof. By tha Galerkin approximation, we may take $\partial_t \boldsymbol{h}_\varepsilon$ as a test function of (3.4). Integrating (3.4) with $\boldsymbol{\phi} = \boldsymbol{h}_\varepsilon$ over $(0,t)$,

$$\iint_{Q_t} |\partial_\tau \boldsymbol{h}_\varepsilon(\tau)|^2 dxd\tau + \iint_{Q_t} \boldsymbol{a}_\varepsilon(x,t,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) \cdot \partial_\tau \mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau) dxd\tau \\ = \iint_{Q_t} \boldsymbol{f}(\tau)\cdot \partial_\tau \boldsymbol{h}_\varepsilon(\tau) dxd\tau + \iint_{\Sigma_t} \boldsymbol{g}^*(\tau)\cdot\partial_\tau \boldsymbol{h}_\varepsilon(\tau) dS\, d\tau. \quad (3.8)$$

If we define

$$\phi_\varepsilon(s) = \int_0^s k_\varepsilon(s')ds',$$

then we have

$$\begin{aligned}
&\partial_\tau \phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p) \\
&\quad = k_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p)\partial_\tau(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p) \\
&\quad = k_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p)(\boldsymbol{a}(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau))\cdot\partial_\tau \mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau) - p\Psi(x,\tau)^{p-1}\partial_\tau\Psi(x,\tau)).
\end{aligned}$$

Thus from (3.8), we see that

$$\iint_{Q_t} |\partial_\tau \boldsymbol{h}_\varepsilon(\tau)|^2 dxd\tau + \iint_{Q_t} \partial_\tau \phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau) - \Psi(x,t)^p) dxd\tau \\ + \iint_{Q_t} pk_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p)\Psi(x,\tau)^{p-1}\partial_\tau\Psi(x,\tau)dxd\tau \\ = \iint_{Q_t} \boldsymbol{f}(\tau)\cdot\partial_\tau \boldsymbol{h}_\varepsilon(\tau)dxd\tau + \iint_{\Sigma_t} \boldsymbol{g}^*(\tau)\cdot\partial_\tau\boldsymbol{h}_\varepsilon(\tau)dS\,d\tau.$$

Here we have

$$\iint_{Q_t} \partial_\tau\phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau) - \Psi(x,t)^p)dxd\tau \\ = \int_\Omega \phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)dx - \int_\Omega \phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_0) - \Psi(x,0)^p)dx.$$

Since

$$\phi_\varepsilon(s)\begin{cases} = s & \text{if } s \le 0, \\ \ge s & \text{if } s \ge 0, \end{cases}$$

and $\boldsymbol{h}_0 \in \mathbb{K}(0)$, i.e., $b(x,\mathrm{curl}\,\boldsymbol{h}_0) - \Psi(x,0) \le 0$, we have

$$\iint_{Q_t} \partial_\tau\phi_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,t)^p)dxd\tau \\ \ge \int_\Omega (b(x,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t)) - \Psi(x,t)^p)dx \ge \frac{1}{p}a_* \int_\Omega |\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t)|^p dx - \|\Psi(t)\|^p_{L^p(\Omega)}.$$

On the other hand, it follows from Lemma 3.1 that

$$\begin{aligned}
&\left|\iint_{Q_t}(k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(\tau))-\Psi(x,\tau)^p)\Psi(x,\tau)^{p-1}\partial_\tau\Psi(x,\tau)dxd\tau\right|\\
&\le \|k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon)-\Psi(x,t)^p)\|_{L^1(Q_T)}\|\Psi^{p-1}\|_{L^\infty(Q_T)}\|\partial_t\Psi\|_{L^\infty(Q_T)}\\
&\le C\Big(\frac{1}{\alpha^p}\|\Psi\|^p_{L^p(Q_T)}+\|\boldsymbol{f}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}+\|\boldsymbol{g}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}\\
&\quad+\|\boldsymbol{h}_0\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\|\Psi\|^{p-1}_{L^\infty(Q_T)}\|\partial_t\Psi\|_{L^\infty(Q_T)}.
\end{aligned}$$

Moreover, we can see that for any $\delta>0$,

$$\left|\iint_{Q_t}\boldsymbol{f}(\tau)\cdot\partial_\tau\boldsymbol{h}_\varepsilon(\tau)dxd\tau\right|\le C_\delta\iint_{Q_t}|\boldsymbol{f}(\tau)|^2dxd\tau+\delta\iint_{Q_t}|\partial_\tau\boldsymbol{h}_\varepsilon(\tau)|^2dxd\tau.$$

By the integration by parts,

$$\begin{aligned}
\iint_{\Sigma_t}\boldsymbol{g}^*(\tau)\cdot\partial_\tau\boldsymbol{h}_\varepsilon(\tau)dxd\tau=\int_\Gamma\boldsymbol{g}^*(t)\cdot\boldsymbol{h}_\varepsilon(t)dS-\int_\Gamma\boldsymbol{g}^*(0)\cdot\boldsymbol{h}_0dS&\\
-\iint_{\Sigma_t}\partial_\tau\boldsymbol{g}^*(\tau)\cdot\boldsymbol{h}_\varepsilon(\tau)dS\,d\tau.&
\end{aligned}$$

Here from Proposition 2.3, the Hölder and Young inequalities,

$$\left|\int_\Gamma\boldsymbol{g}^*(t)\cdot\boldsymbol{h}_\varepsilon(t)dS\right|\le C(\delta)\|\boldsymbol{g}\|^{p'}_{L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma))}+\delta\|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)\|^p_{\boldsymbol{L}^p(\Omega)},$$

$$\left|\int_\Gamma\boldsymbol{g}^*(0)\cdot\boldsymbol{h}_0dS\right|\le\|\boldsymbol{g}\|_{L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma))}\|\operatorname{curl}\boldsymbol{h}_0\|_{\boldsymbol{L}^p(\Omega)},$$

and

$$\left|\iint_{\Sigma_t}\partial_\tau\boldsymbol{g}^*(\tau)\cdot\boldsymbol{h}_\varepsilon(\tau)dS\,d\tau\right|\le\|\partial_t\boldsymbol{g}\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}\|\operatorname{curl}\boldsymbol{h}_\varepsilon\|_{\boldsymbol{L}^p(Q_T)}.$$

Here it follows from Lemma 3.2 that $\|\operatorname{curl}\boldsymbol{h}_\varepsilon\|_{\boldsymbol{L}^p(\Omega)}\le C$ where C is a constant depending only on $\|\Psi\|_{L^p(Q_T)}$, $\|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}$, $\|\boldsymbol{g}\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}$ and $\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}$. If we choose $\delta>0$ small enough, we can see that

$$\iint_{Q_t}|\partial_\tau\boldsymbol{h}_\varepsilon(\tau)|^2dxd\tau+\int_\Omega|\operatorname{curl}\boldsymbol{h}_\varepsilon(t)|^pdx\le C$$

where C depends only on $\|\Psi\|_{L^p(Q_T)}$, $\|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}$, $\|\boldsymbol{g}\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}$ and $\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}$. □

Theorem 3.4. *Assume that*

$$\boldsymbol{f}\in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))\cap\boldsymbol{L}^2(Q_T),\boldsymbol{g}\in L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma))\cap W^{1,p'}(0,T;\boldsymbol{L}^{r'}(\Gamma)),\boldsymbol{h}_0\in\mathbb{K}(0).$$

Then the variational inequality (3.1) *has a unique solution*

$$\boldsymbol{h}\in L^p(0,T;\mathbb{W}^\infty_*(\Omega))\cap H^1(0,T;\boldsymbol{L}^2(\Omega)).$$

Proof. From Lemma 3.2 and 3.3,

$$\|\boldsymbol{h}_\varepsilon\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathrm{curl}\,\boldsymbol{h}_\varepsilon\|_{L^\infty(0,T;\boldsymbol{L}^p(\Omega))} \leq C.$$

Thus $\{\boldsymbol{h}_\varepsilon\}$ is bounded in $L^\infty(0,T;W^{1,1}(\Omega))$. By Sobolev embedding theorem, the injection $\boldsymbol{W}^{1,1}(\Omega) \hookrightarrow \boldsymbol{L}^1(\Omega)$ is compact. It follows from Lemma 3.3 that $\{\boldsymbol{h}_\varepsilon\}$ is bounded in $H^1(0,T;\boldsymbol{L}^2(\Omega))$. We apply Simon [12, Corollary 5] with $X = \boldsymbol{W}^{1,1}(\Omega), B = Y = \boldsymbol{L}^1(\Omega)$, $p = \infty, s = 1, r = 2$. Since $\{\boldsymbol{h}_\varepsilon\}$ is bounded in $L^\infty(0,T;\boldsymbol{W}^{1,1}(\Omega)) \cap W^{1,2}(0,T;\boldsymbol{L}^1(\Omega))$, we can see that $\{\boldsymbol{h}_\varepsilon\}$ is relatively compact in the space $C([0,T];\boldsymbol{L}^1(\Omega))$. Passing to a subsequence, we may assume that $\boldsymbol{h}_\varepsilon \to \boldsymbol{h}$ weak star in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, strongly in $C([0,T];\boldsymbol{L}^1(\Omega))$, $\mathrm{curl}\,\boldsymbol{h}_\varepsilon \to \mathrm{curl}\,\boldsymbol{h}$ weakly in $\boldsymbol{L}^p(Q_T)$, $\partial_t\boldsymbol{h}_\varepsilon \to \partial_t\boldsymbol{h}$ weakly in $\boldsymbol{L}^2(Q_T)$. Let $\boldsymbol{\phi} \in \mathbb{K}(t)$. Then $b(x,\mathrm{curl}\,\boldsymbol{\phi}) \leq \Psi(x,t)^p$. By the monotonicity of $\boldsymbol{a}_\varepsilon(x,t,\boldsymbol{u})$, we have

$$(\boldsymbol{a}_\varepsilon(x,t,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t)) - \boldsymbol{a}_\varepsilon(x,t,\mathrm{curl}\,\boldsymbol{\phi}))\cdot(\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t) - \mathrm{curl}\,\boldsymbol{\phi}) \geq 0.$$

Since $k_\varepsilon(b(x,\mathrm{curl}\,\boldsymbol{\phi}) - \Psi(x,t)^p) = 1$, we can see that

$$\boldsymbol{a}_\varepsilon(x,t,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t))\cdot\mathrm{curl}\,(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t)) \leq \boldsymbol{a}(x,\mathrm{curl}\,\boldsymbol{\phi})\cdot(\mathrm{curl}\,\boldsymbol{\phi} - \mathrm{curl}\,\boldsymbol{h}_\varepsilon(t)). \tag{3.9}$$

Taking $\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t)$ as a test function in (3.4),

$$\begin{aligned}\int_\Omega \partial_t\boldsymbol{h}_\varepsilon(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))\,dx + \int_\Omega \boldsymbol{a}_\varepsilon(x,t,\mathrm{curl}\,\boldsymbol{h}_\varepsilon(t))\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))\,dx \\ = \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))\,dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))\,dS.\end{aligned}$$

Using (3.9), we have

$$\begin{aligned}\iint_{Q_T} \partial_t\boldsymbol{h}_\varepsilon(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))dxdt + \iint_{Q_T} \boldsymbol{a}(x,\mathrm{curl}\,\boldsymbol{\phi})\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))dxdt \\ \geq \int_{Q_T} \boldsymbol{f}(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))dxdt + \int_{\Sigma_T} \boldsymbol{g}^*(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon(t))dS\,dt.\end{aligned} \tag{3.10}$$

Since $\boldsymbol{h}_\varepsilon \to \boldsymbol{h}$ weak star in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, $\boldsymbol{h}_\varepsilon(T) \to \boldsymbol{h}(T)$ weakly in $\boldsymbol{L}^2(\Omega)$, we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \iint_{Q_T} \partial_t\boldsymbol{h}_\varepsilon\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon)dxdt \\ &= \limsup_{\varepsilon\to 0} \iint_{Q_T} \partial_t\boldsymbol{h}_\varepsilon\cdot\boldsymbol{\phi}dxdt - \frac{1}{2}\liminf_{\varepsilon\to 0}\int_\Omega |\boldsymbol{h}_\varepsilon(T)|^2dx + \frac{1}{2}\int_\Omega |\boldsymbol{h}_0|^2dx.\end{aligned}$$

Since $\boldsymbol{\phi} \in \mathbb{K}(t)$ and $p \geq 6/5$, $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$. Moreover, since $\partial_t\boldsymbol{h}_\varepsilon \to \partial_t\boldsymbol{h}$ weakly in $\boldsymbol{L}^2(Q_T)$,

$$\iint_{Q_T} \partial_t\boldsymbol{h}_\varepsilon\cdot\boldsymbol{\phi}dxdt \to \iint_{Q_T} \partial_t\boldsymbol{h}\cdot\boldsymbol{\phi}dxdt \text{ as } \varepsilon \to 0.$$

Thus we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \iint_{Q_T} \partial_t\boldsymbol{h}_\varepsilon\cdot(\boldsymbol{\phi} - \boldsymbol{h}_\varepsilon)dxdt \\ &\leq \iint_{Q_T} \partial_t\boldsymbol{h}\cdot\boldsymbol{\phi}dxdt - \frac{1}{2}\int_\Omega |\boldsymbol{h}(T)|^2dx + \frac{1}{2}\int_\Omega |\boldsymbol{h}_0|^2dx = \iint_{Q_T} \partial_t\boldsymbol{h}\cdot(\boldsymbol{\phi} - \boldsymbol{h})dxdt.\end{aligned} \tag{3.11}$$

Since $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\Omega)) \subset L^1(0,T;\boldsymbol{L}^2(\Omega))$ and $\boldsymbol{h}_\varepsilon \to \boldsymbol{h}$ weak star in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, we have

$$\iint_{Q_T} \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}_\varepsilon)dxdt \to \iint_{Q_T} \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h})dxdt.$$

Moreover, since $\operatorname{curl}\boldsymbol{h}_\varepsilon \to \operatorname{curl}\boldsymbol{h}$ weakly in $\boldsymbol{L}^p(Q_T)$ and $|\boldsymbol{a}(x,\operatorname{curl}\boldsymbol{\phi})| \le a^*|\operatorname{curl}\boldsymbol{\phi}|^{p-1} \in \boldsymbol{L}^{p'}(Q_T)$, we have

$$\iint_{Q_T} \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{\phi})\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}_\varepsilon)dxdt \to \iint_{Q_T} \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{\phi})\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h})dxdt.$$

Since $\operatorname{curl}\boldsymbol{h}_\varepsilon \to \operatorname{curl}\boldsymbol{h}$ weakly in $\boldsymbol{L}^p(Q_T)$, $\boldsymbol{h}_\varepsilon \to \boldsymbol{h}$ weakly in $L^p(0,T;\boldsymbol{L}^r(\Gamma))$. Since $\boldsymbol{g} \in L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))$, we have

$$\iint_{\Sigma_T} \boldsymbol{g}^*\cdot(\boldsymbol{\phi}-\boldsymbol{h}_\varepsilon)dS\,dt \to \iint_{\Sigma_T} \boldsymbol{g}^*\cdot(\boldsymbol{\phi}-\boldsymbol{h})dS\,dt.$$

Therefore, from (3.10) and (3.11), we get

$$\iint_{Q_T} \partial_t\boldsymbol{h}\cdot(\boldsymbol{\phi}-\boldsymbol{h})dxdt + \iint_{Q_T} \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{\phi})\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h})dxdt$$
$$\ge \iint_{Q_T} \boldsymbol{f}\cdot(\boldsymbol{\phi}-\boldsymbol{h})dxdt + \iint_{\Sigma_T} \boldsymbol{g}^*\cdot(\boldsymbol{\phi}-\boldsymbol{h})dS\,dt. \quad (3.12)$$

If we assume that $\boldsymbol{h}(t) \in \mathbb{K}(t)$ a.e. $t \in (0,T)$ which will be shown in the next lemma, we have a.e. $t \in (0,T)$,

$$\int_\Omega \partial_t\boldsymbol{h}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{\phi}(t))\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}(t))dx$$
$$\ge \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dS.$$

If we replace $\boldsymbol{\phi}$ with $\boldsymbol{h}+\lambda(\boldsymbol{\phi}-\boldsymbol{h})$ $(0<\lambda<1)$, we have

$$\int_\Omega \partial_t\boldsymbol{h}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}(t)+\lambda(\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}(t))\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}(t))dx$$
$$\ge \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dS.$$

Since $\boldsymbol{a}(x,\boldsymbol{u})$ is a Carathéodory function, letting $\lambda \to 0$, we finally get

$$\int_\Omega \partial_t\boldsymbol{h}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}(t))\cdot\operatorname{curl}(\boldsymbol{\phi}-\boldsymbol{h}(t))dx$$
$$\ge \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{\phi}-\boldsymbol{h}(t))dS$$

for any $\boldsymbol{\phi} \in \mathbb{K}(t)$ a.e. $t \in (0,T)$.

Finally we show the uniqueness of solution. Let $\boldsymbol{h}^1, \boldsymbol{h}^2$ be two solutions of (3.1). Then by (3.1) with $\boldsymbol{h} = \boldsymbol{h}^1, \boldsymbol{\phi} = \boldsymbol{h}^2$ or $\boldsymbol{h} = \boldsymbol{h}^2, \boldsymbol{\phi} = \boldsymbol{h}^1$, we have

$$\begin{aligned}\int_\Omega \partial_t \boldsymbol{h}^1(t)\cdot(\boldsymbol{h}^2(t)-\boldsymbol{h}^1(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}^1(t))\cdot\operatorname{curl}(\boldsymbol{h}^2(t)-\boldsymbol{h}^1(t))dx\\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{h}^2(t)-\boldsymbol{h}^1(t))dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{h}^2(t)-\boldsymbol{h}^1(t))dS\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \partial_t \boldsymbol{h}^2(t)\cdot(\boldsymbol{h}^1(t)-\boldsymbol{h}^2(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}^2(t))\cdot\operatorname{curl}(\boldsymbol{h}^1(t)-\boldsymbol{h}^2(t))dx\\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{h}^1(t)-\boldsymbol{h}^2(t))dx + \int_\Gamma \boldsymbol{g}^*(t)\cdot(\boldsymbol{h}^1(t)-\boldsymbol{h}^2(t))dS.\end{aligned}$$

If we put $\boldsymbol{w}(t) = \boldsymbol{h}^1(t) - \boldsymbol{h}^2(t)$, $\boldsymbol{w}$ satisfies

$$\int_\Omega \partial_t \boldsymbol{w}(t)\cdot\boldsymbol{w}(t)dx + \int_\Omega (\boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}^1(t)) - \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}^2(t))\cdot\operatorname{curl}(\boldsymbol{h}^1(t)-\boldsymbol{h}^2(t))dx \leq 0.$$

a.e. $t \in (0,T)$. Since $\boldsymbol{a}$ is monotone, we have

$$\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx \leq 0.$$

Thus

$$\int_\Omega |\boldsymbol{w}(t)|^2 dx \leq \int_\Omega |\boldsymbol{w}(0)|^2 dx = 0$$

so we have $\boldsymbol{w}(t) \equiv 0$. □

Lemma 3.5. *Let $\boldsymbol{h}_\varepsilon$ be the solution of* (3.4) *and $\boldsymbol{h}_\varepsilon \to \boldsymbol{h}$ weakly in $L^p(0,T;\mathbb{W}^p(\Omega)) \cap H^1(0,T;\boldsymbol{L}^2(\Omega))$. Then $\boldsymbol{h}(t) \in \mathbb{K}(t)$ a.e. $t \in (0,T)$.*

Proof. Define

$$\begin{aligned}A_\varepsilon &= \{(x,t)\in Q_T; b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)) - \Psi(x,t)^p < \sqrt{\varepsilon}\},\\ B_\varepsilon &= \{(x,t)\in Q_T; \sqrt{\varepsilon} \leq b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)) - \Psi(x,t)^p \leq 1/\varepsilon\},\\ C_\varepsilon &= \{(x,t)\in Q_T; b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)) - \Psi(x,t)^p > 1/\varepsilon\}.\end{aligned}$$

Then

$$\iint_{A_\varepsilon} \sqrt{\varepsilon}dxdt \leq \sqrt{\varepsilon}|Q_T| \to 0 \text{ as } \varepsilon \to 0,$$

and from Lemma 3.1,

$$\iint_{C_\varepsilon} \frac{1}{\varepsilon}dxdt \leq \frac{1}{\varepsilon}\iint_{C_\varepsilon} \frac{k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)-\Psi(x,t)^p)}{e^{1/\varepsilon^2}}dxdt \leq C\frac{1}{\varepsilon}e^{-1/\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0.$$

For small $\varepsilon > 0$, on B_ε

$$k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)) - \Psi(x,t)^p) \geq k_\varepsilon(\sqrt{\varepsilon}) = e^{1/\sqrt{\varepsilon}}.$$

Thus using again Lemma 3.1,

$$|B_\varepsilon| = \iint_{B_\varepsilon} 1dxdt \le \iint_{B_\varepsilon} \frac{k_\varepsilon(b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon(x,t)) - \Psi(x,t)^p)}{e^{1/\sqrt{\varepsilon}}}dxdt \le Ce^{-1/\sqrt{\varepsilon}} \to 0 \text{ as } \varepsilon \to 0.$$

Moreover, $\frac{1}{\varepsilon}|B_\varepsilon| \le C\frac{1}{\varepsilon}e^{-1/\sqrt{\varepsilon}} \to 0$ as $\varepsilon \to 0$. Since $\operatorname{curl}\boldsymbol{h}_\varepsilon \to \operatorname{curl}\boldsymbol{h}$ weakly in $L^p(0,T;\boldsymbol{L}^p(\Omega))$, we have

$$\iint_{Q_T} b(x,\operatorname{curl}\boldsymbol{h})dxdt \le \liminf_{\varepsilon\to 0} \iint_{Q_T} b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon)dxdt.$$

Therefore,

$$\begin{aligned}
&\iint_{Q_T} (b(x,\operatorname{curl}\boldsymbol{h}) - \Psi^p)^+ dxdt \\
&\le \liminf_{\varepsilon\to 0} \iint_{Q_T} (b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon) - \Psi^p) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon}dxdt \\
&= \liminf_{\varepsilon\to 0}\Big(\iint_{A_\varepsilon} \sqrt{\varepsilon}dxdt + \iint_{B_\varepsilon} (b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon) - \Psi^p)dxdt + \iint_{C_\varepsilon} \frac{1}{\varepsilon}dxdt\Big) \\
&= \liminf_{\varepsilon\to 0} \iint_{B_\varepsilon} (b(x,\operatorname{curl}\boldsymbol{h}_\varepsilon) - \Psi^p)\chi_{B_\varepsilon}dxdt \\
&\le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}|B_\varepsilon| = 0.
\end{aligned}$$

Hence we have $b(x,\operatorname{curl}\boldsymbol{h}(x,t)) \le \Psi(x,t)^p$ a.e. in Q_T. That is, $\boldsymbol{h}(t) \in \mathbb{K}(t)$ a.e. $t \in (0,T)$. □

4 Continuous dependence on the data

In this section, we examine the continuous dependence on the data of the solution of (3.1). Let $\Psi_i \in W^{1,\infty}(0,T;\boldsymbol{L}^\infty(\Omega))$ satisfy $\Psi_i(x,t) \ge \alpha > 0$ for $i = 1,2$, and let

$$\boldsymbol{f}_i \in L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega)) \cap \boldsymbol{L}^2(Q_T), \boldsymbol{g}_i \in L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma)) \cap W^{1,p'}(0,T;\boldsymbol{L}^{r'}(\Gamma)).$$

Define

$$\mathbb{K}_i(t) = \{\boldsymbol{v} \in \mathbb{W}^p_*(\Omega); b(x,\operatorname{curl}\boldsymbol{v}) \le \Psi_i(x,t)^p \text{ a.e. in } \Omega\},$$

and let $\boldsymbol{h}_{i0} \in \mathbb{K}_i(0)$.

Lemma 4.1. *If $\boldsymbol{h}_1 \in L^p(0,T;\mathbb{W}^p_*(\Omega))$ satisfies $\boldsymbol{h}_1(t) \in \mathbb{K}_1(t)$ a.e. $t \in (0,T)$, then there exists $\widehat{\boldsymbol{h}}_2 \in L^p(0,T;\mathbb{W}^p_*(\Omega))$ such that $\widehat{\boldsymbol{h}}_2(t) \in \mathbb{K}_2(t)$ a.e. $t \in (0,T)$, and there exists a constant $C > 0$ such that*

$$\|\operatorname{curl}(\boldsymbol{h}_1 - \widehat{\boldsymbol{h}}_2)\|_{L^p(Q_T)} \le C\|\Psi_1 - \Psi_2\|_{L^\infty(Q_T)}.$$

Proof. Let $\beta(t) = \|\Psi_1 - \Psi_2\|_{L^\infty(\Omega)}$ and $\eta(t) = \alpha/(\alpha+\beta(t))$ and define $\widehat{\boldsymbol{h}}_2(t) = \eta(t)^p\boldsymbol{h}_1(t)$. Since $b(x,\boldsymbol{u})$ is convex in $\boldsymbol{u}$, $b(x,\boldsymbol{0}) = 0$ and $0 \le \eta(t) \le 1$, we have

$$b(x,\operatorname{curl}\widehat{\boldsymbol{h}}_2(t)) = b(x,\eta(t)^p\operatorname{curl}\boldsymbol{h}_1(t)) \le \eta(t)^p b(x,\operatorname{curl}\boldsymbol{h}_1(t)) \le (\eta(t)\Psi_1(x,t))^p.$$

Now we have

$$\frac{\Psi_1(t)}{\Psi_2(t)} = \frac{\Psi_1(t) - \Psi_2(t) + \Psi_2(t)}{\Psi_2(t)} \le \frac{\beta(t)}{\alpha} + 1 = \frac{\alpha+\beta(t)}{\alpha} = \frac{1}{\eta(t)}.$$

Thus $\eta(t)\Psi_1(t) \le \Psi_2(t)$. Therefore we have $b(x, \mathrm{curl}\,\widehat{\boldsymbol{h}}_2(t)) \le \Psi_2(t)^p$, that is, $\widehat{\boldsymbol{h}}_2(t) \in \mathbb{K}_2(t)$ a.e. $t \in (0,T)$. Moreover, we have

$$\begin{aligned}
\|\mathrm{curl}\,(\boldsymbol{h}_1(t) - \widehat{\boldsymbol{h}}_2(t)\|^p_{\boldsymbol{L}^p(\Omega)} &= \int_\Omega |\mathrm{curl}\,\boldsymbol{h}_1(t) - \mathrm{curl}\,\widehat{\boldsymbol{h}}_2(t)|^p dx \\
&= \int_\Omega |\mathrm{curl}\,\boldsymbol{h}_1(t) - \eta(t)^p \mathrm{curl}\,\boldsymbol{h}_1(t)|^p dx \\
&= \int_\Omega ((1-\eta(t)^p)^p |\mathrm{curl}\,\boldsymbol{h}_1(t)|^p dx.
\end{aligned}$$

Here

$$1-\eta(t)^p = 1 - \frac{\alpha^p}{(\alpha+\beta(t))^p} = \frac{(\alpha+\beta(t))^p - \alpha^p}{(\alpha+\beta(t))^p} \le \frac{1}{\alpha^p}\beta(t)\int_0^1 p(\alpha+\theta\beta(t))^{p-1}d\theta.$$

If we put $C_1 = \|\Psi_1\|_{L^\infty(Q_T)} + \|\Psi_2\|_{L^\infty(Q_T)}$, taking $\beta(t) \le C_1$ into consideration, we have

$$1-\eta(t)^p \le \frac{p(\alpha+C_1)^{p-1}}{\alpha^p}\beta(t).$$

Thus if we put $C = \frac{p(\alpha+C_1)^{p-1}}{\alpha^p}$, we get

$$\|\mathrm{curl}\,(\boldsymbol{h}_1(t) - \widehat{\boldsymbol{h}}_2(t)\|_{\boldsymbol{L}^p(\Omega)} \le C\beta(t)\|\mathrm{curl}\,\boldsymbol{h}_1(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

□

Remark 4.2. *If we replace $\boldsymbol{h}_1$ by $\boldsymbol{h}_2$, we can construct the corresponding function denoted by $\widehat{\boldsymbol{h}}_1$.*

Our variational inequality problem is as follows: for $i = 1,2$, find $\boldsymbol{h}_i$ satisfying $\boldsymbol{h}_i(t) \in \mathbb{K}_i(t)$ a.e. $t \in (0,T)$ and $\boldsymbol{h}_i(0) = \boldsymbol{h}_{i0}$ such that

$$\begin{aligned}
\int_\Omega \partial_t \boldsymbol{h}_i(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_i(t))dx + \int_\Omega \boldsymbol{a}(x, \mathrm{curl}\,\boldsymbol{h}_i(t))\cdot \mathrm{curl}\,(\boldsymbol{\phi} - \boldsymbol{h}_i(t))dx& \\
\ge \int_\Omega \boldsymbol{f}_i(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_i(t))dx + \int_\Gamma \boldsymbol{g}_i^*(t)\cdot(\boldsymbol{\phi} - \boldsymbol{h}_i(t)dS& \quad (4.1)_i
\end{aligned}$$

for any $\boldsymbol{\phi} \in \mathbb{K}_i(t)$ a.e. $t \in (0,T)$ where $\boldsymbol{g}_i^*(t) = \boldsymbol{g}_i(t)$ if $* = N$ and $\boldsymbol{g}_i^*(t) = \boldsymbol{n}\times\boldsymbol{g}_i(t)$ if $* = T$.

Then we have the following theorem.

Theorem 4.3. *Let $\boldsymbol{h}_i$ $(i = 1,2)$ be solutions of* (4.1)*. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|\boldsymbol{h}_1 - \boldsymbol{h}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathrm{curl}\,(\boldsymbol{h}_1 - \boldsymbol{h}_2)\|^{p\vee 2}_{\boldsymbol{L}^p(Q_T)} \\
&\qquad \le C(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|^{p'\wedge 2}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|^{p'\wedge 2}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))} \\
&\qquad\qquad + \|\boldsymbol{h}_{10} - \boldsymbol{h}_{20}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\Psi_1 - \Psi_2\|_{L^\infty(Q_T)}).
\end{aligned}$$

Here and hearafter we denote $a\vee b = \max\{a,b\}$ and $a\wedge b = \min\{a,b\}$ for any real numbers a and b.

Proof. If we take $\boldsymbol{\phi} = \widehat{\boldsymbol{h}}_1(t)$ as a test function in $(4.1)_1$, we have

$$\int_\Omega \partial_t \boldsymbol{h}_1(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_1(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_1(t))\cdot\operatorname{curl}(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_1(t))dx \\ \leq \int_\Omega \boldsymbol{f}_1(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_1(t))dx + \int_\Gamma \boldsymbol{g}_1^*(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_1(t))dS.$$

Thus we have

$$\begin{aligned}
&\int_\Omega \partial_t \boldsymbol{h}_1(t)\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_1(t))\cdot\operatorname{curl}(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx \\
&\leq \int_\Omega \boldsymbol{f}_1(t)\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx + \int_\Gamma \boldsymbol{g}_1^*(t)\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dS \\
&\quad + \int_\Omega \partial_t \boldsymbol{h}_1(t)\cdot(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_1(t))\cdot\operatorname{curl}(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))dx \\
&\quad + \int_\Omega \boldsymbol{f}_1(t)\cdot(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))dx + \int_\Gamma \boldsymbol{g}_1^*(t)\cdot(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))dS.
\end{aligned}$$

Similarly, if we take $\boldsymbol{\phi} = \widehat{\boldsymbol{h}}_2(t)$ as a test function in $(4.1)_2$, we have

$$\begin{aligned}
&\int_\Omega \partial_t \boldsymbol{h}_2(t)\cdot(\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_2(t))\cdot\operatorname{curl}(\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t))dx \\
&\leq \int_\Omega \boldsymbol{f}_2(t)\cdot(\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t))dx + \int_\Gamma \boldsymbol{g}_2^*(t)\cdot(\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t))dS \\
&\quad + \int_\Omega \partial_t \boldsymbol{h}_2(t)\cdot(\widehat{\boldsymbol{h}}_2(t)-\boldsymbol{h}_1(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_2(t))\cdot\operatorname{curl}(\widehat{\boldsymbol{h}}_2(t)-\boldsymbol{h}_1(t))dx \\
&\quad + \int_\Omega \boldsymbol{f}_2(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_2(t))dx + \int_\Gamma \boldsymbol{g}_2^*(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_2(t))dS.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\int_\Omega \partial_t(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx \\
&\qquad + \int_\Omega (\boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_1(t))-\boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_2(t))\cdot\operatorname{curl}(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx \\
&\leq \int_\Omega (\boldsymbol{f}_1(t)-\boldsymbol{f}_2(t))\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dx + \int_\Gamma (\boldsymbol{g}_1^*(t)-\boldsymbol{g}_2^*(t))\cdot(\boldsymbol{h}_1(t)-\boldsymbol{h}_2(t))dS + \Theta(t)
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
\Theta(t) \;=\; &\int_\Omega \partial_t \boldsymbol{h}_1(t)\cdot(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_1(t))\cdot\operatorname{curl}(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))dx \\
&+ \int_\Omega \boldsymbol{f}_1(t)\cdot(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))dx + \int_\Gamma \boldsymbol{g}_1^*(t)\cdot(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))dS \\
&+ \int_\Omega \partial_t \boldsymbol{h}_2(t)\cdot(\widehat{\boldsymbol{h}}_2(t)-\boldsymbol{h}_1(t))dx + \int_\Omega \boldsymbol{a}(x,\operatorname{curl}\boldsymbol{h}_2(t))\cdot\operatorname{curl}(\widehat{\boldsymbol{h}}_2(t)-\boldsymbol{h}_1(t))dx \\
&+ \int_\Omega \boldsymbol{f}_2(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_2(t))dx + \int_\Gamma \boldsymbol{g}_2^*(t)\cdot(\boldsymbol{h}_1(t)-\widehat{\boldsymbol{h}}_2(t))dS.
\end{aligned}$$

The case $p \geq 2$. Integrating (4.2) over $(0,t)$, taking essential supremum and using the Young inequality, for any $\delta > 0$, we have

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{h}_1-\boldsymbol{h}_2\|^2_{L^\infty((0,T);\boldsymbol{L}^2(\Omega))}+a_*\|\mathrm{curl}\,(\boldsymbol{h}_1-\boldsymbol{h}_2)\|^p_{\boldsymbol{L}^p(Q_T)}\\
&\leq C(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}+\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))})\\
&\quad\times\|\mathrm{curl}\,(\boldsymbol{h}_1-\boldsymbol{h}_2)\|_{\boldsymbol{L}^p(Q_T)}+\frac{1}{2}\|\boldsymbol{h}_{10}-\boldsymbol{h}_{20}\|^2_{\boldsymbol{L}^2(\Omega)}+\int_0^T|\Theta(t)|dt\\
&\leq C(\delta)(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}+\|\boldsymbol{g}_1-\boldsymbol{g}_2\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))})\\
&\quad+\delta\|\mathrm{curl}\,(\boldsymbol{h}_1-\boldsymbol{h}_2)\|^p_{\boldsymbol{L}^p(Q_T)}+\frac{1}{2}\|\boldsymbol{h}_{10}-\boldsymbol{h}_{20}\|^2_{\boldsymbol{L}^2(\Omega)}+\int_0^T|\Theta(t)|dt.
\end{aligned}$$

We estimate the term $\int_0^T|\Theta(t)dt$. First we note that since $p \geq 2$, it follows from Proposition 2.3 that

$$\|\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t)\|_{\boldsymbol{L}^2(\Omega)}\leq C\|\mathrm{curl}\,\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

Now we have

$$\begin{aligned}
&\int_0^T\int_\Omega|\partial_t\boldsymbol{h}_1(t)\cdot(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))|dxdt\\
&\quad\leq\int_0^T\|\partial_t\boldsymbol{h}_1(t)\|_{\boldsymbol{L}^2(\Omega)}\|\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t)\|_{\boldsymbol{L}^2(\Omega)}dt\\
&\quad\leq\|\partial_t\boldsymbol{h}_1\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\left(\int_0^T\|\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t)\|^2_{\boldsymbol{L}^2(\Omega)}dt\right)^{1/2}\\
&\quad\leq C\|\partial_t\boldsymbol{h}_1\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\left(\int_0^T\|\mathrm{curl}\,(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))\|^2_{\boldsymbol{L}^p(\Omega)}dt\right)^{1/2}\\
&\quad\leq C_T\|\partial_t\boldsymbol{h}_1\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\mathrm{curl}\,(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))\|_{\boldsymbol{L}^p(Q_T)}\\
&\quad\leq C'_T\|\partial_t\boldsymbol{h}_1\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\Psi_1-\Psi_2\|_{L^\infty(Q_T)}.
\end{aligned}$$

By the Hölder inequality and Lemma 4.1, we have

$$\begin{aligned}
&\int_\Omega|\boldsymbol{a}(x,\mathrm{curl}\,\boldsymbol{h}_1(t))\cdot\mathrm{curl}\,(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))|dx\\
&\quad\leq\int_0^T\int_\Omega a^*|\mathrm{curl}\,\boldsymbol{h}_1(t)|^{p-1}|\mathrm{curl}\,(\widehat{\boldsymbol{h}}_1(t)-\boldsymbol{h}_2(t))|dxdt\\
&\quad\leq a^*\|\mathrm{curl}\,\boldsymbol{h}_1\|^{p/p'}_{\boldsymbol{L}^p(Q_T)}\|\mathrm{curl}\,(\widehat{\boldsymbol{h}}_1-\boldsymbol{h}_2)\|_{\boldsymbol{L}^p(Q_T)}\\
&\quad\leq C\|\mathrm{curl}\,\boldsymbol{h}_1\|^{p/p'}_{\boldsymbol{L}^p(Q_T)}\|\Psi_1-\Psi_2\|_{L^\infty(Q_T)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\int_0^T\int_\Omega|\boldsymbol{f}_1(t)\cdot(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))|dxdt &\leq \int_0^T\|\boldsymbol{f}_1(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t)\|_{\boldsymbol{L}^q(\Omega)}dt\\
&\leq C\int_0^T\|\boldsymbol{f}_1(t)\|_{\boldsymbol{L}^{q'}(\Omega)}\|\mathrm{curl}\,(\boldsymbol{h}_2(t)-\widehat{\boldsymbol{h}}_1(t))\|_{\boldsymbol{L}^p(\Omega)}dt\\
&\leq C\|\boldsymbol{f}_1\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}\|\mathrm{curl}\,(\boldsymbol{h}_2-\widehat{\boldsymbol{h}}_1)\|_{\boldsymbol{L}^p(Q_T)}\\
&\leq C\|\boldsymbol{f}_1\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))}\|\Psi_1-\Psi_2\|_{L^\infty(Q_T)}.
\end{aligned}$$

Similarly

$$\int_0^T \int_\Gamma |\boldsymbol{g}_1^*(t) \cdot (\boldsymbol{h}_2(t) - \widehat{\boldsymbol{h}}_1(t))| dxdt \le C\|\boldsymbol{g}_1\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))}\|\Psi_1 - \Psi_2\|_{L^\infty(Q_T)}.$$

The other terms are also estimated by $\|\Psi_1 - \Psi_2\|_{L^\infty(Q_T)}$.

The case $6/5 \le p < 2$. It follows from (4.3) and (2.3)$'$ that

$$\begin{aligned}
&\frac{1}{2} \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)|^2 dx \\
&\qquad + a_* \iint_{Q_T} (|\mathrm{curl}\, \boldsymbol{h}_1(t)| + |\mathrm{curl}\, \boldsymbol{h}_2(t)|)^{p-2} |\mathrm{curl}\,(\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)|^2 dxdt \\
&\le C(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))})\|\mathrm{curl}\,(\boldsymbol{h}_1 - \boldsymbol{h}_2)\|_{\boldsymbol{L}^p(Q_T)} \\
&\qquad + \frac{1}{2}\|\boldsymbol{h}_{10} - \boldsymbol{h}_{20}\|_{\boldsymbol{L}^2(\Omega)} + \int_0^T |\Theta(t)| dt.
\end{aligned}$$

We put

$$\widehat{Q}_T = \{(x,t) \in Q_T; \mathrm{curl}\, \boldsymbol{h}_1(x,t) \ne \boldsymbol{0}, \mathrm{curl}\, \boldsymbol{h}_2(x,t) \ne \boldsymbol{0}\},$$

and we use the inverse Hölder inequality (cf. Sobolev [13, p. 8]):

$$\begin{aligned}
&\iint_{\widehat{Q}_T} |\mathrm{curl}\, \boldsymbol{h}_1 - \mathrm{curl}\, \boldsymbol{h}_2|^2 (|\mathrm{curl}\, \boldsymbol{h}_1| + |\mathrm{curl}\, \boldsymbol{h}_2|)^{p-2} dxdt \\
&\ge \left(\iint_{\widehat{Q}_T} |\mathrm{curl}\, \boldsymbol{h}_1 - \mathrm{curl}\, \boldsymbol{h}_2|^p dxdt \right)^{2/p} \left(\iint_{\widehat{Q}_T} (|\mathrm{curl}\, \boldsymbol{h}_1| + |\mathrm{curl}\, \boldsymbol{h}_2|)^p dxdt \right)^{(p-2)/p} \\
&= c \left(\iint_{\widehat{Q}_T} |\mathrm{curl}\, \boldsymbol{h}_1 - \mathrm{curl}\, \boldsymbol{h}_2|^p dxdt \right)^{2/p} \\
&= c\|\mathrm{curl}\, \boldsymbol{h}_1 - \mathrm{curl}\, \boldsymbol{h}_2\|^2_{\boldsymbol{L}^p(\widehat{Q}_T)}.
\end{aligned}$$

By similar arguments as the case $p \ge 2$, we have

$$\begin{aligned}
&\|\boldsymbol{h}_1 - \boldsymbol{h}_2\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathrm{curl}\,(\boldsymbol{h}_1 - \boldsymbol{h}_2)\|^2_{\boldsymbol{L}^p(Q_T)} \\
&\quad \le C(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|^2_{L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|^2_{L^{p'}(0,T;\boldsymbol{L}^{r'}(\Gamma))} + \|\boldsymbol{h}_{10} - \boldsymbol{h}_{20}\|^2_{\boldsymbol{L}^2(\Omega)}) + \int_0^T |\Theta(t)| dt.
\end{aligned}$$

The rest is similar as the case $p \ge 2$. □

5 The asymptotic behavior as $t \to \infty$ of the solution of the variable inequality

In this section, we shall show that the solution of the variationary inequality (3.1) converges to a solution of the stable variable inequality.

Let $\Psi_\infty \in L^\infty(\Omega)$ satisfy $\Psi_\infty(x) \ge \alpha > 0$ and define

$$\mathbb{K}_\infty = \{\boldsymbol{v} \in \mathbb{W}^p_*(\Omega); b(x, \mathrm{curl}\, \boldsymbol{v}) \le \Psi_\infty(x)^p \text{ a.e. in } \Omega\}.$$

Let $\boldsymbol{f}_\infty \in \boldsymbol{L}^{q'}(\Omega)$ and $\boldsymbol{g}_\infty \in \boldsymbol{L}^{r'}(\Gamma)$. We consider the following problem: find $\boldsymbol{h}_\infty \in \mathbb{K}_\infty$ such that

$$\int_\Omega \boldsymbol{a}(x, \operatorname{curl} \boldsymbol{h}_\infty) \cdot \operatorname{curl}(\boldsymbol{\phi} - \boldsymbol{h}_\infty) dx \geq \int_\Omega \boldsymbol{f}_\infty \cdot (\boldsymbol{\phi} - \boldsymbol{h}_\infty) dx + \int_\Gamma \boldsymbol{g}^*_\infty \cdot (\boldsymbol{\phi} - \boldsymbol{h}_\infty) dS \tag{5.1}$$

for any $\boldsymbol{\phi} \in \mathbb{K}_\infty$ where $\boldsymbol{g}^*_\infty = \boldsymbol{g}_\infty$ if $* = N$ and $\boldsymbol{g}^*_\infty = \boldsymbol{n} \times \boldsymbol{g}_\infty$ if $* = T$.

Then we have the following theorem.

Theorem 5.1. *Let $p \geq 6/5$ and $\boldsymbol{h}_\infty$ be a solution of (5.1). Moreover, assume that $\boldsymbol{f} \in L^\infty(0, \infty; \boldsymbol{L}^{q' \vee 2}(\Omega))$, $\boldsymbol{g} \in W^{1,\infty}(0, \infty; \boldsymbol{L}^{r'}(\Gamma))$, $\Psi \in W^{1,\infty}(0, \infty; L^\infty(\Omega))$ and $\boldsymbol{h}(t)$ be a solution of (3.1). Define*

$$\xi(t) = \|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\|^{p' \vee 2}_{L^s(\Omega)} + \|\boldsymbol{g}(t) - \boldsymbol{g}_\infty\|^{p' \vee 2}_{\boldsymbol{L}^{r'}(\Gamma)}$$

where $s = 2$ if $6/5 \leq p < 2$ and $s = q'$ if $p \geq 2$. Assume that

$$\begin{aligned} \int_{t/2}^{t} \xi(\tau) d\tau &\quad\rightarrow\quad 0 \text{ as } t \to \infty \text{ if } p > 2, \\ \int_{t}^{t+1} \xi(\tau) d\tau &\quad\rightarrow\quad 0 \text{ as } t \to \infty \text{ if } 6/5 \leq p \leq 2. \end{aligned}$$

Furthermore, we assume that there exists $D > 0$ such that

$$\|\Psi(t) - \Psi_\infty\|_{L^\infty(\Omega)} \leq D t^{-\gamma}$$

where

$$\gamma > \begin{cases} 3/2 & \text{if } p > 2, \\ 1/2 & \text{if } 6/5 \leq p \leq 2. \end{cases}$$

Then we have $\|\boldsymbol{h}(t) - \boldsymbol{h}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0$ as $t \to \infty$.

Proof. Put $\beta(t) = \|\Psi(t) - \Psi_\infty\|_{L^\infty(\Omega)}$, $\eta(t) = \alpha/(\alpha + \beta(t))$ and define $\overline{\boldsymbol{h}}(t) = \eta(t)^p \boldsymbol{h}_\infty$, $\overline{\boldsymbol{h}}_\infty(t) = \eta(t)^p \boldsymbol{h}(t)$. Then from Lemma 4.1, we can see that $\overline{\boldsymbol{h}}(t) \in \mathbb{K}(t), \overline{\boldsymbol{h}}_\infty(t) \in \mathbb{K}_\infty$ a.e. $t \in (0, T)$. From (4.2) with $\boldsymbol{h}_1 = \boldsymbol{h}$, $\boldsymbol{h}_2 = \boldsymbol{h}_\infty$,

$$\begin{aligned} &\int_\Omega \partial_t(\boldsymbol{h}(t) - \boldsymbol{h}_\infty) \cdot (\boldsymbol{h}(t) - \boldsymbol{h}_\infty) dx + \int_\Omega (\boldsymbol{a}(x, \operatorname{curl} \boldsymbol{h}(t)) - \boldsymbol{a}(x, \operatorname{curl} \boldsymbol{h}_\infty) \cdot \operatorname{curl}(\boldsymbol{h}(t) - \boldsymbol{h}_\infty) dx \\ &\qquad \leq \int_\Omega (\boldsymbol{f}(t) - \boldsymbol{f}_\infty) \cdot (\boldsymbol{h}(t) - \boldsymbol{h}_\infty) dx + \int_\Gamma (\boldsymbol{g}^*(t) - \boldsymbol{g}^*_\infty) \cdot (\boldsymbol{h}(t) - \boldsymbol{h}_\infty) dS + \Theta(t) \end{aligned}$$

where

$$\begin{aligned} \Theta(t) \;=\; &\int_\Omega \partial_t \boldsymbol{h}(t) \cdot (\overline{\boldsymbol{h}}(t) - \boldsymbol{h}_\infty) dx + \int_\Omega \boldsymbol{a}(x, \operatorname{curl} \boldsymbol{h}(t)) \cdot \operatorname{curl}(\overline{\boldsymbol{h}}(t) - \boldsymbol{h}_\infty) dx \\ &+ \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{h}_\infty - \overline{\boldsymbol{h}}(t)) dx + \int_\Gamma \boldsymbol{g}^*(t) \cdot (\boldsymbol{h}_\infty - \overline{\boldsymbol{h}}(t)) dS \\ &+ \int_\Omega \boldsymbol{a}(x, \operatorname{curl} \boldsymbol{h}_\infty) \cdot \operatorname{curl}(\overline{\boldsymbol{h}}(t) - \boldsymbol{h}(t)) dx \\ &+ \int_\Omega \boldsymbol{f}_\infty \cdot (\boldsymbol{h}(t) - \overline{\boldsymbol{h}}(t)) dx + \int_\Gamma \boldsymbol{g}^*_\infty \cdot (\boldsymbol{h}(t) - \overline{\boldsymbol{h}}(t)) dS. \end{aligned}$$

Lemma 5.2. *There exist constants $C_1, C_2 > 0$ independent of t such that*

$$\|\partial_\tau \boldsymbol{h}\|_{\boldsymbol{L}^2(Q_t)} \le C_1 t^{1/2} + C_2.$$

Proof. From the proof of Lemma 3.3, we have

$$\begin{aligned}&\iint_{Q_t} |\partial_\tau \boldsymbol{h}_\varepsilon(\tau)|^2 dx + \frac{1}{p} a_* \int_\Omega |\mathrm{curl}\, \boldsymbol{h}_\varepsilon(t)|^p dx \\ &\le \iint_{Q_t} pk_\varepsilon(b(x, \mathrm{curl}\, \boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p)\Psi(x,\tau)^{p-1}|\partial_\tau \Psi(x,\tau)| dxdt \\ &\quad + \iint_{Q_t} |\boldsymbol{f}(\tau)\cdot \partial_\tau \boldsymbol{h}_\varepsilon(\tau)| dxd\tau + \iint_{\Sigma_t} |\boldsymbol{g}^*(\tau)\cdot \partial_\tau \boldsymbol{h}_\varepsilon(\tau)| dS\, d\tau.\end{aligned}$$

From Lemma 3.1,

$$\begin{aligned}&\iint_{Q_t} pk_\varepsilon(b(x, \mathrm{curl}\, \boldsymbol{h}_\varepsilon(\tau)) - \Psi(x,\tau)^p)\Psi(x,\tau)^{p-1}|\partial_\tau \Psi(x,\tau)| dxdt \\ &\qquad \le \|k_\varepsilon(b(x, \mathrm{curl}\, \boldsymbol{h}_\varepsilon(\tau) - \Psi(x,\tau)^p\|_{L^1(Q_\infty)} \|\Psi\|^{p-1}_{L^\infty(Q_\infty)} \|\partial \Psi\|_{L^\infty(Q_\infty)} \le C_2.\end{aligned}$$

For any $\delta > 0$,

$$\begin{aligned}\iint_{Q_t} |\boldsymbol{f}(\tau)\cdot \partial_\tau \boldsymbol{h}_\varepsilon(\tau)| dxd\tau &\le C_\delta \iint_{Q_t} |\boldsymbol{f}(\tau)|^2 dxd\tau + \delta \iint_{Q_t} |\partial_\tau \boldsymbol{h}_\varepsilon(\tau)|^2 dxd\tau \\ &\le C_\delta t \|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))} + \delta \iint_{Q_t} |\partial_\tau \boldsymbol{h}_\varepsilon(\tau)|^2 dxd\tau.\end{aligned}$$

Using the integration by parts,

$$\iint_{Q_t} \boldsymbol{g}^*(t)\cdot \partial_\tau \boldsymbol{h}_\varepsilon(\tau) dS\, d\tau = \int_\Gamma \boldsymbol{g}^*(t)\cdot \boldsymbol{h}_\varepsilon(t) dS - \int_\Gamma \boldsymbol{g}^*(0)\cdot \boldsymbol{h}_0 dS - \iint_{\Sigma_t} \partial_\tau \boldsymbol{g}^*(\tau)\cdot \boldsymbol{h}_\varepsilon(\tau) dS\, d\tau.$$

We estimate each term of the right hand side. For any $\delta > 0$,

$$\int_\Gamma |\boldsymbol{g}^*(t)\cdot \boldsymbol{h}_\varepsilon(t)| dS \le C(\delta)\|\boldsymbol{g}\|^{p'}_{L^\infty(0,T;\boldsymbol{L}^{r'}(\Gamma))} + \delta \|\mathrm{curl}\, \boldsymbol{h}_\varepsilon(t)\|^p_{\boldsymbol{L}^p(\Omega)}.$$

Moreover we have

$$\int_\Gamma |\boldsymbol{g}^*(0)\cdot \boldsymbol{h}_0| dS \le \|\boldsymbol{g}\|_{L^\infty(0,T;\boldsymbol{L}^{r'}(\Omega))} \|\mathrm{curl}\, \boldsymbol{h}_0\|_{\boldsymbol{L}^p(\Omega)} \le C.$$

Since it follows from Lemma 3.2 that

$$\|\mathrm{curl}\, \boldsymbol{h}_\varepsilon\|^p_{\boldsymbol{L}^p(Q_T)} \le C_1 t + C_2$$

where C_1 and C_2 are constants independent of t, we have

$$\begin{aligned}\iint_{\Sigma_t} |\partial_\tau \boldsymbol{g}^*(\tau)\cdot \boldsymbol{h}_\varepsilon(\tau)| dS\, d\tau &\le C \int_0^t \|\partial_\tau \boldsymbol{g}(\tau)\|_{\boldsymbol{L}^{r'}(\Gamma)} \|\mathrm{curl}\, \boldsymbol{h}_\varepsilon\|_{\boldsymbol{L}^p(\Omega)} d\tau \\ &\le \left(\int_0^t \|\partial_\tau \boldsymbol{g}(\tau)\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} d\tau\right)^{1/p'} \left(\int_0^t \|\mathrm{curl}\, \boldsymbol{h}_\varepsilon(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau\right)^{1/p} \\ &\le C\|\partial_\tau \boldsymbol{g}\|^{p'}_{L^{p'}(0,t;\boldsymbol{L}^{r'}(\Gamma))} + C\|\mathrm{curl}\, \boldsymbol{h}_\varepsilon\|^p_{\boldsymbol{L}^p(Q_t)} \\ &\le \le C_1 t + C_2.\end{aligned}$$

Thus taking $\delta > 0$ small enough, we can see that

$$\|\partial_\tau \boldsymbol{h}\|_{\boldsymbol{L}^2(Q_t)} \le \liminf_{\varepsilon\to 0} \|\partial_\tau \boldsymbol{h}_\varepsilon\|_{\boldsymbol{L}^2(Q_T)} \le C_1 t^{1/2} + C_2.$$

This completes the proof. □

Define

$$\Phi(t) = \int_\Omega |\boldsymbol{h}(t) - \boldsymbol{h}_\infty|^2 dx.$$

When $p > 2$, it follow from (5.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{h}(t)-\boldsymbol{h}_\infty|^2 dx + a_* \int_\Omega |\mathrm{curl}\,(\boldsymbol{h}(t)-\boldsymbol{h}_\infty)|^p dx \\
&\quad \le \|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|_{\boldsymbol{L}^{q'}(\Omega)}\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|_{\boldsymbol{L}^q(\Omega)} + \|\boldsymbol{g}(t)-\boldsymbol{g}_\infty\|_{\boldsymbol{L}^{r'}(\Gamma)}\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|_{\boldsymbol{L}^r(\Gamma)} + \Theta(t) \\
&\quad \le C(\delta)(\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}(t)-\boldsymbol{g}_\infty\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)}) + \delta\|\mathrm{curl}\,(\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^p_{\boldsymbol{L}^p(\Omega)} + \Theta(t).
\end{aligned}$$

Here we note that the each term of $\Theta(t)$ is estimated by $(C_1 t^{1/2} + C_2)\|\Psi(t)-\Psi_\infty\|_{L^\infty(\Omega)}$. By Proposition 2.3, we can see that

$$\|\boldsymbol{h}_\varepsilon(t)-\boldsymbol{h}_\infty\|_{\boldsymbol{L}^2(\Omega)} \le C\|\mathrm{curl}\,(\boldsymbol{h}(t)-\boldsymbol{h}_\infty)\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus we have

$$\Phi'(t) + c\Phi(t)^{p/2} \le l(t)$$

where $c > 0$ is a constant and

$$l(t) = C_3(\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}(t)-\boldsymbol{g}_\infty\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)} + \|\Psi(t)-\Psi_\infty\|_{L^\infty(\Omega)}) + C_4 t^{1/2}\|\Psi(t)-\Psi_\infty\|_{L^\infty(\Omega)}$$

Here we have

$$\int_{t/2}^t l(\tau)d\tau \le C_3\int_{t/2}^t \xi(\tau)d\tau + C_3\int_{t/2}^t D\tau^{-\gamma}d\tau + C_4\int_{t/2}^t D\tau^{-\gamma+1/2}d\tau.$$

When $p > 2$, since $p' < 2$, $s = q'$ and $\gamma > 3/2$, we have

$$\int_{t/2}^t l(\tau)d\tau \to 0 \text{ as } t \to \infty.$$

If we apply Simon [12, Lemma 1, p. 591], we have

$$\int_\Omega |\boldsymbol{h}(t)-\boldsymbol{h}_\infty|^2 dx \le \left(\frac{c(p-2)}{4}t\right)^{-2/(p-2)} + \int_{t/2}^t l(\tau)d\tau \to 0 \text{ as } t \to \infty.$$

When $6/5 \le p \le 2$, since $p' \ge 2$, $s = 2$, we have

$$\Phi'(t) + c\Phi(t) \le l(t)$$

where

$$l(t) = C_3(\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}(t)-\boldsymbol{g}_\infty\|^2_{\boldsymbol{L}^{r'}(\Gamma)} + \|\Psi(t)-\Psi_\infty\|_{L^\infty(\Omega)}) + C_4 t^{1/2}\|\Psi(t)-\Psi_\infty\|_{L^\infty(\Omega)}$$

Here we note that since $p \ge 6/5$, we can take $q' = 2$. Since $\gamma > 1/2$, we have

$$\begin{aligned}\int_t^{t+1} \tau^{1/2}\|\Psi(\tau)-\Psi_\infty\|_{L^\infty(\Omega)}d\tau &\le D\int_t^{t+1}\tau^{-\gamma+1/2}d\tau \\ &= \frac{D}{-\gamma+3/2}((t+1)^{-\gamma+3/2}-t^{-\gamma+3/2}) \\ &= \frac{D}{-\gamma+3/2}\int_0^1 (t+\theta)^{-\gamma+1/2}d\theta \\ &\le Dt^{-\gamma+1/2} \to 0 \text{ as } t\to\infty.\end{aligned}$$

We apply the following variation of Lemma 4 in Haraux [8, p. 286] (cf. [10, Lemma 2.4]).

Lemma 5.3. *Let $\phi(t) \ge 0$ be absolutely continuous in any compact interval of $\mathbb{R}^+$, $0 \le l(t) \in L^1_{\rm loc}(\mathbb{R}^+)$ and $c > 0$. If the following inequality holds:*

$$\phi'(t) + c\phi(t) \le l(t) \ a.e.\ t \ge 0,$$

then for any t_0, t with $t_0 \le t$,

$$\phi(t) \le e^{c(t_0-t)}\phi(t_0) + \frac{1}{1-e^{-c}}\sup_{\tau\ge t_0}\int_\tau^{\tau+1} l(\sigma)d\sigma.$$

The proof is elementary and given in the Appendix.

From this lemma, for fixed $t_0 > 0$ and for all $t > t_0$, we have

$$\int_\Omega |\boldsymbol{h}(t)-\boldsymbol{h}_\infty|^2dx \le e^{c(t_0-t)}l_1 + \frac{1}{1-e^{-c}}\sup_{\tau\ge t_0}\int_\tau^{\tau+1} l(\sigma)d\sigma.$$

for some constant $l_1 > 0$. If we take $t_0 > 0$ large enough and let $t \to 0$, then we can see that

$$\int_\Omega |\boldsymbol{h}(t)-\boldsymbol{h}_\infty|^2dx \to 0 \text{ as } t\to\infty.$$

□

A Proof of Lemma 5.3

In this appendix, we shall prove Lemma 5.3.

We put

$$C = \sup_{\tau\ge t_0}\int_\tau^{\tau+1} l(\sigma)d\sigma$$

and $\psi(t) = e^{-c(t_0-t)}\phi(t)$. Then $\phi(t)$ satisfies

$$\psi'(t) \le e^{-c(t_0-t)}l(t). \tag{A.1}$$

Integrating (A.1) over $[\tau, \tau+1]$, we have

$$\psi(\tau+1)-\psi(\tau) \le e^{c(t_0-(\tau+1))}\int_\tau^{\tau+1} l(t)dt.$$

From this, it follows that

$$\phi(\tau+1) \le e^{-c}\phi(\tau) + C \text{ for any } \tau \ge 0. \qquad (A.2)$$

Let $t_0 \le t$ and write $t - t_0 = k + \delta$ where $k \ge 0$ is an integer and $\delta \in [0,1)$. Then from (A.2),

$$\begin{aligned} \phi(t) &\le e^{-c}\phi(t-1) + C \\ &\le e^{-c}(e^{-c}\phi(t-2) + C) + C \\ &= e^{-2c}\phi(t-2) + C(1+e^{-c}) \\ &\cdots \\ &\le e^{-kc}\phi(t_0+\delta) + C(1+e^{-c}+\cdots+e^{-(k-1)c}). \end{aligned}$$

Moreover, integrating (A.1) over $[t_0, t_0+\delta]$, we have

$$\phi(t_0+\delta) \le e^{-c\delta}\phi(t_0) + C.$$

Thus we have

$$\begin{aligned} \phi(t) &\le e^{-kc}\phi(t_0+\delta) + C(1+e^{-c}+\cdots+e^{-(k-1)c}) \\ &\le e^{-kc}(e^{-c\delta}\phi(t_0) + C) + C(1+e^{-c}+\cdots+e^{-(k-1)c}) \\ &\le e^{c(t-t_0)}\phi(t_0) + \frac{1}{1-e^{-c}}C. \end{aligned}$$

This completes the proof.

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