**Communications in Mathematical Analysis**
Volume 19, Number 2, pp. 62–67 (2016) www.math-res-pub.org/cma
ISSN 1938-9787

# A Note on Closedness of the Sum of Two Closed Subspaces in a Banach Space*

**Zhe-Ming Zheng**†
College of Mathematics and Information Science
Jiangxi Normal University
Nanchang, Jiangxi 330022, People's Republic of China

**Hui-Sheng Ding**‡
College of Mathematics and Information Science
Jiangxi Normal University
Nanchang, Jiangxi 330022, People's Republic of China

(Communicated by Simeon Reich)

**Abstract**

Let $X$ be a Banach space, and $M,N$ be two closed subspaces of $X$. We collect several necessary and sufficient conditions for the closedness of $M+N$ ($M+N$ is not necessarily direct sum), and show that a necessary condition in a classical textbook is also sufficient.

**AMS Subject Classification:** 46B25.

**Keywords**: Banach spaces; sum of two closed subspaces; closedness.

## 1 Introduction

Let $X$ be a Banach space, and $M,N$ be two closed subspaces of $X$. Then, $M+N$ is not necessarily closed in $X$ even if $X$ is a Hilbert space and $M\cap N=\{0\}$ (see, e.g., [6, p.145, Exercise 9]). So, to study when $M+N$ is closed in $X$ is always an interesting problem.

For the case of $M\cap N=\{0\}$, a necessary and sufficient condition for $M+N$ being closed in $X$ is given by Kober [3] as follows:

**Theorem 1.1.** *Let $X$ be a Banach space, $M,N$ be two closed subspaces of $X$ and $M\cap N=\{0\}$. Then $M+N$ is closed in $X$ if and only if there exists a constant $A>0$ such that for all $x\in M$ and $y\in N$ we have $\|x\|\le A\cdot\|x+y\|$.*

*The work was supported by NSFC (11461034), and the Program for Cultivating Young Scientist of Jiangxi Province (20133BCB23009), and the NSF of Jiangxi Province (20143ACB21001).

†E-mail address: zhengzhemingn212@163.com

‡E-mail address: dinghs@mail.ustc.edu.cn

It seems that there are seldom published results concerning necessary and sufficient conditions for $M+N$ being closed in $X$ in the case of $M+N$ being not necessarily direct sum. To the best of our knowledge, the first result of a necessary and sufficient condition for $M+N$ (not necessarily direct sum) being closed in $X$ is given by Luxemburg:

**Theorem 1.2.** [5, Theorem 2.5] *Let $X$ be a Banach space, and $M,N$ be two closed subspaces of $X$. Then $M+N$ is closed in $X$ if and only if $T: M\times N \to X; (m,n) \longmapsto m+n$ is an open mapping.*

Luxemburg [5] obtain the above theorem in a more general setting. Theorem 1.2 is only one of the interesting results concerning this topic given by Luxemburg. We refer the reader to [5] for more details.

In addition, for the case of $X$ being a Banach lattice or a Hilbert space, there has been of great interest for some researchers to study if the sum of two closed subspaces of $X$ is still closed. We refer the reader to [4, 5, 8, 9] and references therein for the case of $X$ being a Banach lattice or a Fréchet space and to [2, 7] and references therein for the case of $X$ being a Hilbert space.

This short note is also devoted to this problem for the case of $X$ being a general Banach space. As one will see, we give a Kober-like theorem for the case of $M+N$ being not necessarily direct sum, and show that a necessary condition in the classical textbook [6] is also sufficient (see Remark 2.4).

# 2 Main Results

**Lemma 2.1.** *Let $X$ be a Banach space, $M$ and $N$ be two vector spaces of $X$. Assume that $N$ is closed in $X$ and contained in $M$. The following assertions are equivalent:*

*(1) $M$ is closed in $X$,*

*(2) $M/N$ is closed in $X/N$.*

*Proof.* The implication of (1)⇒(2) follows from the well-known fact that $M/N$ is a Banach space (see, e.g., [6]). On the other hand, the implication of (2)⇒(1) follows from the inequality: $\|x\| \geq \|x+N\|$ for every $x \in X$. □

**Theorem 2.2.** *Let $X$ be a Banach space, and $M,N$ be two closed subspaces of $X$. Then the following assertions are equivalent:*

*(i) $M+N$ is closed in $X$;*

*(ii) $(M+N)/N$ is closed in $X/N$;*

*(iii) there exists a constant $K>0$ such that for every $x \in M+N$, there is a decomposition $x = m+n$ such that*

$$\|m\| \leq K \cdot \|x\|,$$

*where $m \in M$ and $n \in N$;*

*(iv) $T: M\times N \to M+N; (m,n) \longmapsto m+n$ is an open mapping.*

*Proof.* "(i) $\Longrightarrow$ (ii)". Obviously, it follows from Lemma 2.1.

"(ii) $\Longrightarrow$ (iii)". Define a mapping $\phi : (M+N)/N \to M/(M\cap N)$ by

$$\phi(x+N) = m + (M\cap N),$$

where $x = m+n \in M+N$, $m \in M$ and $n \in N$. It is easy to see that $\phi$ is well-defined. Moreover, $\phi$ is linear and bijective. Noting that

$$\|\phi(x+N)\| = \|m+(M\cap N)\| \geq \|m+N\| = \|x+N\|,$$

we conclude that $\phi^{-1}$ is a bounded linear operator from $M/(M\cap N)$ to $(M+N)/N$. Since $(M+N)/N$ and $M/(M\cap N)$ are both Banach spaces, it follows from the open mapping theorem that $\phi$ is also a bounded linear operator from $(M+N)/N$ to $M/(M\cap N)$. Taking $K = \|\phi\|+1$, the assertion (iii) follows. In fact, letting $x = m'+n' \in M+N$ and $x \neq 0$, where $m' \in M$ and $n' \in N$, we have

$$\|m' + (M\cap N)\| = \|\phi(x+N)\| \leq \|\phi\|\cdot\|x+N\| \leq \|\phi\|\cdot\|x\| < K\|x\|.$$

Then, there exists $y \in M\cap N$ such that

$$\|m'+y\| < K\|x\|.$$

Letting $m = m'+y$ and $n = n'-y$, we get $x = m+n$ and $\|m\| < K\|x\|$.

"(iii) $\Longrightarrow$ (iv)". It is easy to see that

$$kerT = \{(x,-x) : x \in M\cap N\}.$$

Let $\pi$ be the quotient map from $M\times N$ to $(M\times N)/kerT$, and $\widetilde{T} : (M\times N)/kerT \to M+N$ be defined as follows

$$\widetilde{T}[(m,n)+kerT] = m+n, \quad (m,n) \in M\times N.$$

Then $\widetilde{T}$ is linear and bijective. For every $(m,n) \in M\times N$, by (iii), there exist $m' \in M$ and $n' \in N$ such that $m+n = m'+n'$ and

$$\|m'\| \leq K\|m+n\|,$$

which yields that

$$\|m'\| + \|n'\| \leq (2K+1)\|m+n\|.$$

Then, we have

$$\|\widetilde{T}[(m,n)+kerT]\| = \|m+n\| \geq \frac{\|m'\|+\|n'\|}{2K+1} \geq \frac{1}{2K+1}\|(m,n)+kerT\|,$$

which means that $\widetilde{T}$ is an open mapping. Combing this with the fact that $\pi$ is open, we conclude that $T = \widetilde{T}\circ\pi$ is also open.

"(iv) $\Longrightarrow$ (i)". As noted in the Introduction, (i) is equivalent to (iv) has been shown by Luxemburg using a more general setting. Here, we give a different proof (maybe a more direct proof in the setting of Banach spaces).

Let $\pi$, $kerT$, $\widetilde{T}$ be as in the proof of "(iii) $\Longrightarrow$ (iv)". For every $(m,n) \in M \times N$ and $x \in M \cap N$, there holds

$$\|m+n\| \le \|m+x\| + \|n-x\| = \|(m+x, n-x)\| = \|(m,n)+(x,-x)\|,$$

which yields

$$\|\widetilde{T}[(m,n)+kerT]\| = \|m+n\| \le \inf_{x \in M \cap N} \|(m,n)+(x,-x)\| = \|(m,n)+kerT\|,$$

i.e., $\|\widetilde{T}\| \le 1$. On the other hand, since $\pi : M \times N \to (M \times N)/kerT$ is continuous and $T$ is an open mapping, for every open set $U \subset (M \times N)/kerT$,

$$\widetilde{T}(U) = T(\pi^{-1}(U))$$

is also an open set. Thus, $\widetilde{T}$ is an open mapping, which means that $\left(\widetilde{T}\right)^{-1}$ is continuous, and so bounded. Now, we conclude that as normed linear spaces, $M+N$ and $(M \times N)/kerT$ are topological isomorphic. Therefore, it follows that $(M \times N)/kerT$ is a Banach space that $M+N$ is also a Banach space. This completes the proof. □

*Remark* 2.3. Very recently, Blot and Cieutat [1] prove that (i) is equivalent to (iii) by a different and interesting proof (see [1, Theorem 3.1]). Also, by applying this result, they obtain a class of interesting and important results about sufficient conditions for the closeness of the sum of two closed subspaces of the Banach space of bounded functions.

*Remark* 2.4. In the classical textbook [6] (see p.137, Theorem 5.20), it has been shown that (iii) is a necessary condition for (i) by using the open mapping theorem. Here, we show that (iii) is also a sufficient condition for (i). In fact, the fact that (i) is equivalent to (iii) is a Kober-like result for the case of $M+N$ being not necessarily direct sum. Moreover, by using the idea in the proof of [10, Theorem 2.3], we will give a direct proof of "(iii) $\Longrightarrow$ (i)" in the following. We think that it may be of interest for some readers. Here is our proof:

Let $\{x_j\}_{j=1}^{\infty} \subset M+N$ and $x_j \to x$ in $X$ as $j \to \infty$. Then, we can choose a subsequence $\{x_k\}$ of $\{x_j\}$ such that

$$\|x_{k+1} - x_k\| \le \frac{1}{2^k \cdot K}, \quad k = 1,2\ldots.$$

By taking $x = x_2 - x_1$ in the assertion (iii), there exist $m_1 \in M$ and $n_1 \in N$ such that $x_2 - x_1 = m_1 + n_1$ and

$$\|m_1\| \le K \cdot \|x_2 - x_1\| \le \frac{1}{2}.$$

Similarly, by taking $x = x_3 - x_2$ in the assertion (iii), there exist $m_2 \in M$ and $n_2 \in N$ such that $x_3 - x_2 = m_2 + n_2$ and

$$\|m_2\| \le K \cdot \|x_3 - x_2\| \le \frac{1}{2^2}.$$

Continuing by this way, we get two sequences $\{m_k\} \subset M$ and $\{n_k\} \subset N$ such that

$$x_{k+1} - x_k = m_k + n_k, \quad k = 1,2\ldots,$$

and

$$\|m_k\| \le \frac{1}{2^k}, \quad k = 1,2\ldots.$$

Then, we have $\sum_{k=1}^{\infty} \|m_k\| < \infty$. Also, we can get $\sum_{k=1}^{\infty} \|n_k\| < \infty$. Since $M$ and $N$ are both Banach spaces, there exist $m \in M$ and $n \in N$ such that

$$m = \sum_{k=1}^{\infty} m_k, \quad n = \sum_{k=1}^{\infty} n_k.$$

Recalling that $x_k \to x$, we get

$$x - x_1 = \sum_{k=1}^{\infty} (x_{k+1} - x_k) = m + n,$$

which yields that $x = x_1 + m + n \in M + N$.

**Corollary 2.5.** *Let $X$ be a Banach space, and $M, N$ be two closed subspaces of $X$. Then the following assertions are equivalent:*

*(a) $M + N$ is closed in $X$;*

*(b) $(M + N)/(M \cap N)$ is closed in $X/(M \cap N)$.*

*Proof.* One can show this corollary by directly using Lemma 2.1. Here, we give another proof by using Theorem 2.2.

Noting that $(M + N)/(M \cap N) = M/(M \cap N) + N/(M \cap N)$, it follows from Theorem 2.2 that the closeness of $(M + N)/(M \cap N)$ is equivalent to the closedness of

$$[(M + N)/(M \cap N)]/[M/(M \cap N)].$$

On the other hand, it is not difficult to show that $(M + N)/M$ is isometric to $[(M + N)/(M \cap N)]/[M/(M \cap N)]$, and so their closedness are equivalent. Thus, the closedness of $(M + N)/(M \cap N)$ is equivalent to the closedness of $(M + N)/M$. Again by Theorem 2.2, we complete the proof. □

*Remark* 2.6. By Corollary 2.5, whenever we find an example of non-direct sum $M + N$, which is not closed, we can get an example of direct sum $M/(M \cap N) + N/(M \cap N) = (M + N)/(M \cap N)$, which is still not closed.

### Acknowledgments

The authors are grateful to the anonymous reviewer for his/her professional comments and suggestions. Especially, Lemma 2.1 is put forward by the anonymous reviewer.

## References

[1] J. Blot and P. Cieutat, Completeness of sums of subspace of bounded functions and applications, arXiv:1510.01160 [math.FA].

[2] I. S. Feshchenko, On closeness of the sum of $n$ subspaces of a Hilbert space. *Ukr. Math. J.* **63** (2012), pp 1566-1622.

[3] H. Kober, A theorem on Banach spaces. *Compos. Math.* **7** (1940), pp 135-140.

[4] H. P. Lotz, A note on the sum of two closed lattice ideals. *Proc. Am. Math. Soc.* **44** (1974), pp 389-390.

[5] W. A. J. Luxemburg, A note on the sum of two closed linear subspaces. *Indag. Math. (Proceedings)* **88** (1985), pp 235-242.

[6] W. Rudin, *Functional Analysis*, McGraw-Hill, second edition, 1991.

[7] I. E. Schochetman, R. L. Smith, and S-K. Tsui, On the closure of the sum of closed subspaces. *Int. J. Math. Math. Sci.* **26** (2001), pp 257-267.

[8] J. Voigt, On the sum of two closed subspaces. *Indag. Math.* **25** (2014), pp 575-578.

[9] W. Wnuk, A note on the sum of closed ideals and Riesz subspaces. *Positivity* **19** (2015), pp 137-147.

[10] Z.-M. Zheng and H.-S. Ding, On completeness of the space of weighted pseudo almost automorphic functions. *J. Funct. Anal.* **268** (2015), pp 3211-3218.