Communications in **M**athematical **A**nalysis
Volume 14, Number 1, pp. 85–103 (2013) www.math-res-pub.org/cma
ISSN 1938-9787

Algebraic and ergodicity properties of the Berezin transform

Namita Das[*]
P.G. Department of Mathematics
Utkal University
Vanivihar, Bhubaneswar, 751004,
Odisha, India

Rajendra Prasad Lal[†]
Department of Computer and Information Sciences
University of Hyderabad
Hyderabad, India-500046

(Communicated by Palle Jorgensen)

Abstract

In this paper we derive certain algebraic and ergodicity properties of the Berezin transform defined on $L^2(\mathbb{B}_N, d\eta')$ where $\mathbb{B}_N$ is the open unit ball in $\mathbb{C}^N, N \geq 1, N \in \mathbb{Z}$, $d\eta'(z) = K_{\mathbb{B}_N}(z,z)d\nu(z)$ is the Mobius invariant measure, $K_{\mathbb{B}_N}$ is the reproducing kernel of the Bergman space $L_a^2(\mathbb{B}_N, d\nu)$ and $d\nu$ is the Lebesgue measure on $\mathbb{C}^N$, normalized so that $\nu(\mathbb{B}_N) = 1$. We establish that the Berezin transform B is a contractive linear operator on each of the spaces $L^p(\mathbb{B}_N, d\eta'(z)), 1 \leq p \leq \infty$, $B^n \to 0$ in norm topology and B is similar to a part of the adjoint of the unilateral shift. As a consequence of these results we also derive certain algebraic and asymptotic properties of the integral operator defined on $L^2[0,1]$ associated with the Berezin transform.

AMS Subject Classification: 47B38, 31B05.

Keywords: Bergman space, Berezin transform, positive operators, contraction, Helgason-Fourier transform.

1 Introduction

Let $\mathbb{B}_N$ be the open unit ball of $\mathbb{C}^N, N \geq 1, N \in \mathbb{Z}$, with respect to the Euclidean metric. The letter ν denotes the Lebesgue measure on $\mathbb{C}^N$, normalized so that $\nu(\mathbb{B}_N) = 1$ and $L^p(\mathbb{B}_N, d\nu), 1 \leq p \leq \infty$ are the usual Lebesgue spaces and the integration is with respect to

[*]E-mail address: namitadas440@yahoo.co.in
[†]E-mail address: rlal77@gmail.com

the measure ν. When $N = 1$, $d\nu = dA$, the normalized area measure on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Consider the space $L^2(\mathbb{B}_N, d\nu)$ for an integer $N \geq 1$. Let $L^2_a(\mathbb{B}_N, d\nu)$ be the Bergman space of holomorphic functions in $L^2(\mathbb{B}_N, d\nu)$ and $K_{\mathbb{B}_N}$ be the reproducing kernel for $L^2_a(\mathbb{B}_N, d\nu)$. Notice that for $z, \lambda \in \mathbb{B}_N$,

$$K_{\mathbb{B}_N}(z,\lambda) = \frac{N!}{(1 - z\cdot\overline{\lambda})^{N+1}} \tag{1.1}$$

where $z\cdot\overline{\lambda} = z_1\overline{\lambda}_1 + \cdots + z_N\overline{\lambda}_N$. For details see [15]. Let $d\eta'(z) = K_{\mathbb{B}_N}(z,z)d\nu(z)$. The reproducing kernel $K_{\mathbb{B}_N}(z,w)$ of $L^2_a(\mathbb{B}_N, d\nu)$ is holomorphic in z and antiholomorphic in w and

$$\int_{\mathbb{B}_N} |K_{\mathbb{B}_N}(z,w)|^2 d\nu(w) = K_{\mathbb{B}_N}(z,z) > 0 \tag{1.2}$$

for all $z \in \mathbb{B}_N$. Thus we define for each $\lambda \in \mathbb{B}_N$, a unit vector k_λ in $L^2_a(\mathbb{B}_N)$ by

$$k_\lambda(z) = \frac{K_{\mathbb{B}_N}(z,\lambda)}{\sqrt{K_{\mathbb{B}_N}(\lambda,\lambda)}}. \tag{1.3}$$

The Bergman space $L^2_a(\mathbb{B}_N, d\nu)$ is a closed subspace [5], [23] of $L^2(\mathbb{B}_N, d\nu)$. Let P be the orthogonal projection of $L^2(\mathbb{B}_N, d\nu)$ onto $L^2_a(\mathbb{B}_N, d\nu)$. For $\phi \in L^\infty(\mathbb{B}_N)$, define the Toeplitz operator T_ϕ from $L^2_a(\mathbb{B}_N)$ into itself as $T_\phi f = P(\phi f)$. The operator T_ϕ is a bounded linear operator and $\|T_\phi\| \leq \|\phi\|_\infty$. Toeplitz operators can also be defined for unbounded symbols. Since the Bergman projection P can be extended to the space $L^1(\mathbb{B}_N, d\nu)$, we also have $T_\phi f = P(\phi f)$, $f \in H^\infty(\mathbb{B}_N)$, even for $\phi \in L^1(\mathbb{B}_N, d\nu)$. It is easy to see that $H^\infty(\mathbb{B}_N)$, the space of bounded analytic functions on $\mathbb{B}_N$ is dense in $L^2_a(\mathbb{B}_N)$. The Berezin transform plays an important role [22],[12] in the theory of Toeplitz and Hankel operators on the Bergman space.

The group of all one-to-one holomorphic maps of $\mathbb{B}_N$ onto $\mathbb{B}_N$ (the automorphisms of $\mathbb{B}_N$) will be denoted by $Aut(\mathbb{B}_N)$. It is generated by the unitary operators on $\mathbb{C}^N$ and the involutions ϕ_a of the form

$$\phi_a(z) = \frac{a - \mathcal{P}z - (1-|a|^2)^{\frac{1}{2}}Qz}{1 - \langle z, a\rangle} \tag{1.4}$$

where $a \in \mathbb{B}_N$, $\mathcal{P}$ is the orthogonal projection onto the space spanned by a, $Qz = z - \mathcal{P}z$,

$$\langle z, a\rangle = \sum_{i=1}^{n} z_i\overline{a_i}, \text{ and } |a|^2 = \langle a, a\rangle.$$

Let G_0 be the isotropy subgroup of $Aut(\mathbb{B}_N)$ at 0; i.e.

$$G_0 = \{\psi \in Aut(\mathbb{B}_N) : \psi(0) = 0\}.$$

It is well known [21] that G_0 is compact and that G_0 is a subgroup of the unitary group $\mathcal{U}_N$ of $\mathbb{C}^N$. Given $\psi \in Aut(\mathbb{B}_N)$, let $a = \psi^{-1}(0)$, then we have,

$$\psi \circ \phi_a(0) = \psi(a) = 0,$$

thus $\psi \circ \phi_a \in G_0$ and so there exists a unitary matrix U such that $\psi = U\phi_a$ where $U \in G_0$. It is also not difficult to verify that the identity

$$1 - |\phi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z, w\rangle|^2}$$

holds and that the (real) Jacobian of ϕ_z is

$$(J_{\mathbb{R}}\phi_z)(w) = \frac{(1-|z|^2)^{N+1}}{|1-\langle z, w\rangle|^{2N+2}}.$$

For details see [1] and [15].
The invariant Laplacian $\widetilde{\triangle}$ is defined [19] for $f \in C^2(\mathbb{B}_N)$ by

$$(\widetilde{\triangle} f)(z) = \triangle(f \circ \phi_z)(0),$$

where $\triangle$ is the ordinary Laplacian. It commutes with every $\psi \in Aut(\mathbb{B}_N)$:

$$(\widetilde{\triangle} f) \circ \psi = \widetilde{\triangle}(f \circ \psi).$$

The $\mathcal{M}$-harmonic functions in $\mathbb{B}_N$ are those for which $\widetilde{\triangle} f = 0$. We recall that "$\mathcal{M}$-harmonic" is the same as "harmonic" when $N = 1$, but not when $N > 1$. For more details see [1],[3] and [2]. If $\widetilde{\triangle} f = 0$ then the mean value of f on spheres of radius $r < 1$ is $f(0)$. If f is also in $L^1(\mathbb{B}_N)$ it follows that

$$\int_{\mathbb{B}_N} (f \circ \psi) d\nu = f(\psi(0)) \tag{1.5}$$

for every $\psi \in Aut(\mathbb{B}_N)$. It happens as $\widetilde{\triangle} f = 0$ implies $\widetilde{\triangle}(f \circ \psi) = 0$ for all $\psi \in Aut(\mathbb{B}_N)$. The property described in equation (1.5) is called the invariant mean value property. It is invariant in the sense that $f \circ \psi$ has it for every $\psi \in Aut(\mathbb{B}_N)$ whenever f has it.
Let $\Gamma(s)$ stand for the usual Gamma function, which is an analytic function of s in the whole complex plane except for simple poles at the points $\{0, -1, -2, \cdots\}$. In fact

$$\Gamma(z) = \frac{e^{-\beta z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

where β is the Euler's constant; its approximate value is 0.57722.

If $f \in L^1(\mathbb{B}_N, d\nu)$, the Berezin transform of f is defined by

$$(Bf)(w) = \int_{\mathbb{B}_N} f(z)|k_w(z)|^2 d\nu(z) \tag{1.6}$$

where $k_w(z)$ is the normalized reproducing kernel at $w \in \mathbb{B}_N$. Notice that $k_w \in L^\infty(\mathbb{B}_N)$ for all $w \in \mathbb{B}_N$, so the definition makes sense and $(Bf)(w) = \langle T_f k_w, k_w\rangle$ for $f \in L^1(\mathbb{B}_N, d\nu)$. Let $\tilde{f}(w) = (Bf)(w)$. The function $\tilde{f}$ is called the Berezin symbol of the Toeplitz operator T_f and Bf is called the Berezin transform of f. If f is a bounded $\mathcal{M}$-harmonic function then since $\langle T_f k_w, k_w\rangle = \tilde{f}(w) = (Bf)(w) = f(w)$, hence the Berezin symbol of T_f is the function f itself. Ahern, Flores and Rudin [1] proved that if $Bf = f, f \in L^1(\mathbb{B}_N, d\nu)$, then f is $\mathcal{M}$-harmonic if $N \leq 11$, but not if $N \geq 12$. In what follows, we present some basic properties of the operator

B. It is known [1] that if f is radial, $f \in L^1(\mathbb{B}_N, d\nu)$ and $f(z) = g(|z|^2)$ for all $z \in \mathbb{B}_N$ then $Bf = f$ if and only if $Tg = g$ where T is the integral operator given by

$$(Tg)(x) = (1-x)^{N+1} \int_0^1 \frac{N+tx}{(1-tx)^{N+2}} g(t) t^{N-1} dt. \tag{1.7}$$

Now

$$(Bf)(z) = \int_{\mathbb{B}_N} \frac{(1-|z|^2)^{N+1}}{|1-\langle z,w\rangle|^{2(N+1)}} f(w) d\nu(w). \tag{1.8}$$

Thus we obtain, if f is radial and $f(z) = g(|z|^2)$ then

$$\int_{\mathbb{B}_N} \frac{(1-|z|^2)^{N+1}}{|1-\langle z,w\rangle|^{2(N+1)}} f(w) d\nu(w) = f$$

if and only if

$$g(x) = (1-x)^{N+1} \int_0^1 \frac{N+tx}{(1-tx)^{N+2}} g(t) t^{N-1} dt = Tg(x). \tag{1.9}$$

In this paper, we derive certain algebraic and ergodicity properties of the Berezin transform. The layout of this paper is as follows.

In section 2 we establish certain algebraic properties of the Berezin transform. We present an alternative formula for Bf. Given $a \in \mathbb{B}_N$ and f any measurable function on $\mathbb{B}_N$, we define $C_a f = f(\phi_a(z))$. We prove that the Berezin transform B commutes with all the composition operators C_a, $a \in \mathbb{B}_N$ and extending this result we also show that $C_\psi B = BC_\psi$ where C_ψ is the composition operator defined on $L^1(\mathbb{B}_N, d\nu)$ defined by $C_\psi f = f \circ \psi, \psi \in Aut(\mathbb{B}_N)$. We further show that the Berezin transform B is a contractive linear operator on each of the spaces $L^p(\mathbb{B}_N, d\eta'(z)), 1 \le p \le \infty$. In this section we also show that if $f \in L^1(\mathbb{D}, dA)$ is radial then Bf is radial and if $f \in L^1(\mathbb{D}, dA)$ then $\widetilde{f}$ is real analytic on $\mathbb{D}$. As a consequence of these results we also derive certain algebraic properties of the integral operator T defined on $L^1[0,1]$ associated with the Berezin transform. In section 3 we show that the Berezin transform B defined on $L^2(\mathbb{B}_N, d\eta')$ into itself is a positive operator and has spectral radius less than 1. We also show that $\|B\| = \Phi_N(\frac{N}{2}) < 1$ where $\Phi_N(\gamma) = \frac{\Gamma(\gamma+1)\Gamma(N+1-\gamma)}{\Gamma(N+1)}, \gamma \in \mathbb{N}$. Further we establish that B is similar to a part of the adjoint of the unilateral shift and $B^n \to 0$ in norm topology. From these results we derive many ergodicity properties of the Berezin transform and the corresponding integral operator T defined on $L^1[0,1]$. Applications of these results are also discussed.

2 Algebraic properties of the Berezin transform

In this section we establish certain algebraic properties of the Berezin transform. We present an alternative formula for Bf. Given $a \in \mathbb{B}_N$ and f any measurable function on $\mathbb{B}_N$, we define $C_a f = f(\phi_a(z))$. We prove that the Berezin transform B commutes with all the composition operators C_a, $a \in \mathbb{B}_N$ and extending this result we also show that $C_\psi B = BC_\psi$ where C_ψ is the composition operator defined on $L^1(\mathbb{B}_N, d\nu)$ defined by $C_\psi f = f \circ \psi, \psi \in Aut(\mathbb{B}_N)$. We further show that the Berezin transform B is a contractive linear operator on each of the spaces $L^p(\mathbb{B}_N, d\eta'(z)), 1 \le p \le \infty$. We also derive certain algebraic properties of the integral operator T defined on $L^1[0,1]$ associated with the Berezin transform. Let $\mathcal{L}(H)$ denote the set of all bounded linear operators from the Hilbert space H into itself.

Lemma 2.1. *The operator B satisfies the following algebraic properties:*

(i) *The operator B is a contraction in $L^\infty(\mathbb{B}_N)$.*

(ii) *If $f \geq 0$, then $Bf \geq 0$; if $f \geq g$, then $Bf \geq Bg$.*

(iii) *Constants are fixed points of B on $L^1(\mathbb{B}_N, d\nu)$.*

(iv) *If $f \in L^1(\mathbb{B}_N, d\nu)$, then*

$$(Bf)(z) = \int_{\mathbb{B}_N} f(\phi_z(w))d\nu(w).$$

(v) *For every $f \in L^2(\mathbb{B}_N, d\nu), a \in \mathbb{B}_N$, $BC_a f = C_a Bf$. That is, B commutes with all the composition operators $C_a, a \in \mathbb{B}_N$.*

(vi) *If $\Psi \in Aut(\mathbb{B}_N)$, $f \in L^1(\mathbb{B}_N, d\nu)$ then $(Bf) \circ \Psi = B(f \circ \Psi)$.*

Proof. The proof of (i),(ii) and (iii) is a straightforward generalization of the unit disk case given in [9]. We shall now establish (iv). For any $\Psi \in \text{Aut}(\mathbb{B}_N)$, we denote by $J_\Psi(z)$ the complex Jacobian determinant of the mapping $\Psi : \mathbb{B}_N \to \mathbb{B}_N$. If $a \in \mathbb{B}_N$, then by a result of [15], [21] there exists a unimodular constant $\theta(a)$ such that

$$J_{\phi_a}(z) = \theta(a)k_a(z)$$

for all $z \in \mathbb{B}_N$. In fact if $a \in \mathbb{B}_N$ then $\theta(a) = (-1)^N$. Thus $|J_{\phi_a}(z)|^2 = |k_a(z)|^2$. Hence $(Bf)(z) = \int_{\mathbb{B}_N} f(w)|k_z(w)|^2 d\nu(w) = \int_{\mathbb{B}_N} (f \circ \phi_z)(w)d\nu(w)$. Now we shall prove (v). By a change of variable,

$$\begin{aligned} Bf(\phi_a(z)) &= \int_{\mathbb{B}_N} f(w)|k_{\phi_a(z)}(w)|^2 d\nu(w) \\ &= \int_{\mathbb{B}_N} f(\phi_a(w))|k_{\phi_a(z)} \circ \phi_a(w)|^2 |k_a(w)|^2 d\nu(w). \end{aligned}$$

Let $U = \phi_{\phi_a(z)} \circ \phi_a \circ \phi_z$. Then $U \in \text{Aut}(\mathbb{B}_N), U(0) = 0$ and U is unitary. Further,

$$\phi_{\phi_a(z)} \circ \phi_a = U\phi_{\phi_a \circ \phi_a(z)} = U\phi_z.$$

Taking the real Jacobian determinant of the above equation, we get

$$|k_{\phi_a(z)} \circ \phi_a(w)|^2 |k_a(w)|^2 = |k_z(w)|^2$$

for all a, z, and w in $\mathbb{B}_N$. Therefore,

$$\begin{aligned} (Bf)(\phi_a(z)) &= \int_{\mathbb{B}_N} f(\phi_a(w))|k_z(w)|^2 d\nu(w) \\ &= B(f \circ \phi_a)(z). \end{aligned}$$

Thus $BC_a f = C_a Bf$ for $f \in L^2(\mathbb{B}_N, d\nu)$. We shall now establish (vi). For every $z \in \mathbb{B}_N$, the automorphism $\phi_{\Psi(z)} \circ \Psi \circ \phi_z$ takes 0 to 0, hence is some unitary U. Thus

$$\begin{aligned} B(f\circ\Psi)(z) &= \int_{\mathbb{B}_N} f(\Psi(\phi_z(w)))d\nu(w) \\ &= \int_{\mathbb{B}_N} f(\phi_{\Psi(z)}Uw)d\nu(w) \\ &= (Bf)(\Psi(z)) \end{aligned}$$

since ν is rotation invariant. □

It follows from Lemma 2.1 that if $g_1, g_2 \in L^1[0,1], g_1 \geq 0, g_1 \geq g_2$ then $Tg_1 \geq 0$ and $Tg_1 \geq Tg_2$. We shall show below that the Berezin transform is a contractive linear operator on $L^p(\mathbb{B}_N, d\eta'(z))$ where $d\eta'(z) = K_{\mathbb{B}_N}(z,z)d\nu(z)$, and $1 \leq p \leq \infty$.

Lemma 2.2. *The Berezin transform B is a contractive linear operator on each of the spaces $L^p(\mathbb{B}_N, d\eta'(z)), 1 \leq p \leq \infty$.*

Proof. Notice that $L^1(\mathbb{B}_N, d\eta') \subset L^1(\mathbb{B}_N, d\nu)$. Since the Berezin transform is defined on the space $L^1(\mathbb{B}_N, d\nu)$ hence B is defined on $L^1(\mathbb{B}_N, d\eta')$. Further

$$|(Bf)(w)| = \left|\int_{\mathbb{B}_N} f(z)|k_w(z)|^2 d\nu(z)\right| \leq B(|f|)(w).$$

Thus

$$\begin{aligned} \int_{\mathbb{B}_N} |(Bf)(w)| K_{\mathbb{B}_N}(w,w)d\nu(w) &\leq \int_{\mathbb{B}_N}\left(\int_{\mathbb{B}_N} |f(z)||k_w(z)|^2 d\nu(z)\right)K_{\mathbb{B}_N}(w,w)d\nu(w) \\ &= \int_{\mathbb{B}_N} |f(z)|\left(\int_{\mathbb{B}_N} |K_{\mathbb{B}_N}(z,w)|^2 d\nu(w)\right)d\nu(z) \\ &= \int_{\mathbb{B}_N} |f(z)| K_{\mathbb{B}_N}(z,z)d\nu(z). \end{aligned}$$

The change of the order of integration being justified by the positivity of the integrand. Hence it follows that B is a contraction on $L^1(\mathbb{B}_N, d\eta')$. The same is true for $L^\infty(\mathbb{B}_N)$ by Lemma 2.1 and so the result follows from the Marcinkiewicz interpolation theorem. □

Thus by Lemma 2.2, the integral operator T is a contractive linear operator on each of the spaces $L^p([0,1], \frac{t^{N-1}dt}{(1-t)^{N+1}}), 1 \leq p \leq \infty, N \geq 1$.

Notice that the Berezin transform B does not carry $L^1(\mathbb{B}_N, d\nu)$ into $L^1(\mathbb{B}_N, d\nu)$, because

$$\int_{\mathbb{B}_N} \frac{(1-|z|^2)^{N+1}}{|1-\langle z,w\rangle|^{2N+2}} d\nu(z)$$

tends to ∞ when $|w| \to 1$. It is not difficult to verify [1], [4] that B is bounded as an operator from $L^1(\mathbb{B}_N, d\nu)$ to $L^1(\mathbb{B}_N, (1-|z|)d\nu)$. Again we know in $\mathbb{D}$, the only measure left invariant by all Mobius transformations is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}$. Therefore, the only harmonic function in $L^p(\mathbb{D}, d\eta)$ is constant zero. Thus even though one can show that every space $L^p((0,1), \frac{dt}{(1-t)^2}), 1 \le p \le \infty$ is an invariant subspace [1], [8] of the operator T (when $N = 1$) but these spaces are no good in this context. This is because (except for L^∞) the corresponding spaces $L^p(\mathbb{D}, d\eta)$ do not contain nonzero harmonic functions, even no nonzero constants. Similar is the case for $\mathbb{B}_N$.

Lemma 2.3. **(i)** *If a function $f \in L^1(\mathbb{B}_N, d\nu)$ is $\mathcal{M}$-harmonic then $Bf = f$.*

(ii) *Suppose $N \in \mathbb{Z}_+$ and $N \le 11$. If $f \in L^1(\mathbb{B}_N, d\nu)$ and $Bf = f$ then f is $\mathcal{M}$-harmonic.*

(iii) *If $f \in L^1(\mathbb{B}_N, d\eta'), N \in \mathbb{Z}_+, N \le 11$ then $Bf = f$ if and only if f is $\mathcal{M}$-harmonic.*

(iv) *If $f \in L^2(\mathbb{B}_N, d\eta')$ is $\mathcal{M}$-harmonic then $f = 0$.*

Proof. (i) If $f \in L^1(\mathbb{B}_N, d\nu)$ is $\mathcal{M}$-harmonic, then so is $f \circ \phi_a$ for any $a \in \mathbb{B}_N$; by the mean value property,

$$(Bf)(z) = \int_{\mathbb{B}_N} f(\phi_z(w))d\nu(w) = (f \circ \phi_z)(0) = f(z).$$

(ii)The result follows from [1]. (iii) Since $L^1(\mathbb{B}_N, d\eta') \subset L^1(\mathbb{B}_N, d\nu)$, the result follows. (iv)Denote the unit sphere, the boundary of the open unit ball $\mathbb{B}_N$ in $\mathbb{C}^N$ by S_N. Let $d\sigma$ be the normalized surface-area measure (Hausdorff measure) of S_N such that $\sigma(S_N) = 1$. Let $M(r) = \int_{\partial\mathbb{B}_N} |f(r\xi)|^2 d\sigma(\xi)$. Then

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{B}_N, d\eta')} &= \int_{\mathbb{B}_N} |f(z)|^2 d\eta'(z) \\
&= \int_0^1 M(r) K_{\mathbb{B}_N}(z,z) 2N r^{2N-1} dr \\
&= 2N \int_0^1 M(r) N! \frac{r^{2N-1}}{(1-r^2)^{N+1}} dr \\
&= NN! \int_0^1 M(r) \frac{r^{2N-2}}{(1-r^2)^{N+1}} 2r dr \\
&= NN! \int_0^1 M(\sqrt{t}) \frac{t^{N-1}}{(1-t)^{N+1}} dt
\end{aligned}$$

where $t = r^2$. So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$. Hence since f is $\mathcal{M}$-harmonic, by maximum principle $f = 0$. □

Corollary 2.4. *If $f \in L^1([0,1], \frac{t^{N-1}dt}{(1-t)^{N+1}}), N \in \mathbb{Z}_+, N \le 11$ then $Tf = f$ if and only if f is a constant.*

Proof. It is not difficult to verify that if f is a constant then $Tf = f$. Now suppose $Tf = f$. Let $g(z) = f(|z|^2)$. Then g is radial and $Bg = g$. By Lemma 2.3, g is $\mathcal{M}$-harmonic. Since a radial $\mathcal{M}$-harmonic function on $\mathbb{B}_N$ is a constant, hence g and therefore, f is a constant. □

Lemma 2.5. **(i)** *If $f \in L^1(\mathbb{D}, dA)$, then $\widetilde{f}$ is real analytic on $\mathbb{D}$.*

(ii) *If $f \in L^1(\mathbb{D}, dA)$ is radial then Bf is radial.*

Proof. (i)Define a complex valued function F on $\mathbb{D}\times\mathbb{D}$ by $F(w,z)=\langle T_f K_{\overline{w}}, K_z\rangle$ for $w,z\in\mathbb{D}$. Here we are using the unnormalized reproducing kernels $K_z(w)=\overline{K(z,w)}=\frac{1}{(1-\overline{z}w)^2}$. Because F is analytic in each variable separately, we conclude that F is holomorphic on $\mathbb{D}\times\mathbb{D}$ and since $\widetilde{f}(z)=\langle T_f k_z,k_z\rangle=(1-|z|^2)^2F(\overline{z},z)$, the function $\widetilde{f}$ is real analytic on $\mathbb{D}$.

(ii) For $f\in L^1(\mathbb{D},dA)$, the Berezin transform Bf is defined as follows :

$$(Bf)(z)=\widetilde{f}(z)=\int_{\mathbb{D}} f(w)|k_z(w)|^2dA(w).$$

We need to show if $f\in L^1(\mathbb{D},dA)$ then $B(rad\ f)=rad\ (Bf)$. Because if f is radial then $rad\ f=f$. In that case $Bf=B(rad\ f)=rad\ (Bf)$. Therefore, this will imply Bf is radial.

$$\begin{aligned}
B(rad f)(z) &= \int_{\mathbb{D}} rad\ (f)(w)|k_z(w)|^2dA(w)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{D}} f(we^{it})|k_z(w)|^2dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{D}} f(we^{it})|k_{e^{it}z}(e^{it}w)|^2dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{D}} f(u)|k_{e^{it}z}(u)|^2dA(u)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\widetilde{f}(e^{it}z)dt\\
&= rad(\widetilde{f})(z)=rad\ (Bf)(z).
\end{aligned}$$

Thus $B(rad\ f)=rad\ (Bf)$. The theorem is proved. □

Recall that the invariant Laplacian $\widetilde{\triangle}$ is defined [19] for $f\in C^2(\mathbb{B}_N)$ by

$$(\widetilde{\triangle}f)(z)=\triangle(f\circ\phi_z)(0),$$

where $\triangle$ is the ordinary Laplacian. Let $M=\{f\in L^1(\mathbb{B}_N,d\nu):Bf=f\}$. If $f\in M$ then f is real analytic as f lies in the range of B. Thus $\widetilde{\triangle}f$ exists for all $f\in M$.
For $f\in L^1(\mathbb{B}_N,d\nu), z\in\mathbb{B}_N$ define

$$\begin{aligned}
(Af)(z) &= (N+1)\int_{\mathbb{B}_N}(1-|w|^2)f(\phi_z(w))d\nu(w)\\
&= (N+1)\int_{\mathbb{B}_N}\frac{(1-|z|^2)^{N+2}(1-|w|^2)f(w)}{|1-\langle z,w\rangle|^{2(N+2)}}d\nu(w).
\end{aligned}$$

It is shown in [1], [4] that $\|A\|\le N+2$ and $Af=\left(1-\frac{\widetilde{\triangle}}{4(N+1)}\right)Bf$. Further for $f\in L^1(\mathbb{B}_N,d\nu), BAf=ABf$. When $N=1$, let $A=A_1$. Then

$$\begin{aligned}
(A_1f)(z) &= 2\int_{\mathbb{D}}(1-|w|^2)f(\phi_z(w))dA(w)\\
&= 2\int_{\mathbb{D}}(1-|\phi_z(w)|^2)f(w)|k_z(w)|^2dA(w).
\end{aligned}$$

We show below that if f is radial on $\mathbb{D}$ then A_1f is radial.

Theorem 2.6. *If $f \in L^1(\mathbb{D}, dA)$ is radial, then $A_1 f$ is radial.*

Proof. It is sufficient to show that for $f \in L^1(\mathbb{D}, dA), A_1(rad\ f) = rad\ (A_1 f)$. For $z \in \mathbb{D}$,

$$\begin{aligned}
A_1(rad\ f)(z) &= 2\int_{\mathbb{D}}(1-|\phi_z(w)|^2)rad\ (f)(w)|k_z(w)|^2 dA(w)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(2\int_{\mathbb{D}} f(we^{it})|k_z(w)|^2(1-|\phi_z(w)|^2)dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(2\int_{\mathbb{D}} f(we^{it})|k_{e^{it}z}(e^{it}w)|^2(1-|\phi_{e^{it}z}(e^{it}w)|^2)dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(2\int_{\mathbb{D}} f(u)|k_{e^{it}z}(u)|^2(1-|\phi_{e^{it}z}(u)|^2)dA(u)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}(A_1 f)(e^{it}z)dt\\
&= rad\ (A_1 f)(z).
\end{aligned}$$

Thus if f is radial, we have $rad\ f = f$. Hence $A_1 f = A_1(rad\ f) = rad\ (A_1 f)$. Therefore $A_1 f$ is radial. □

Theorem 2.7. *If f is radial, $f \in L^1(\mathbb{B}_N, d\nu)$ and $f(z) = g(|z|^2)$ then $Af = f$ if and only if*

$$g(x) = N(1-x)^{N+2}\int_0^1 \frac{N+1+tx}{(1-tx)^{N+3}} g(t)(1-t)t^{N-1}dt.$$

Proof. We have seen that

$$(Af)(z) = (N+1)\int_{\mathbb{B}_N} \frac{(1-|z|^2)^{N+2}(1-|w|^2)}{|1-\langle z,w\rangle|^{2(N+2)}} f(w)d\nu(w).$$

If $f(w) = g(|w|^2) = g(r^2)$ then from [19] it follows that

$$(Af)(z) = (1-|z|^2)^{N+2}2(N+1)N\int_0^1 I_{N+3}(rz)(1-r^2)r^{2N-1}g(r^2)dr$$

and

$$I_{N+3}(rz) = \frac{\Gamma(N+1)}{\Gamma^2(N+2)}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+N+2)}{\Gamma(k+1)\Gamma(k+N+1)}|rz|^{2k}$$

where we use polar coordinates $w = r\rho, \rho \in S_N$(the sphere that bounds $\mathbb{B}_N$). Proceeding as in [1] , one can show that $(Af)(z) = f(z)$ if and only if

$$g(s) = N(1-s)^{N+2}\int_0^1 \frac{N+1+ts}{(1-ts)^{N+3}} g(t)(1-t)t^{N-1}dt.$$

□

If B is the Berezin transform on $L^1(\mathbb{D}, dA)$, we have $BA_1 f = A_1 Bf$ for $f \in L^1(\mathbb{D}, dA)$. For details see [1]. If $f \in L^1(\mathbb{D}, dA)$ and $f(z) = g(|z|^2)$, then $g \in L^1[0,1]$. Define for $g \in L^1[0,1]$,

$$(T_1 g)(s) = N(1-s)^{N+2}\int_0^1 \frac{N+1+ts}{(1-ts)^{N+3}} g(t)(1-t)t^{N-1}dt. \tag{2.1}$$

Theorem 2.8. *If T is the integral operator defined on $L^1[0,1]$ as*

$$(Tg)(x) = (1-x)^{N+1} \int_0^1 \frac{N+xs}{(1-xs)^{N+2}} g(s)s^{N-1}ds$$

and T_1 is the integral operator as defined in (2.1) then $TT_1g = T_1Tg$ for all $g \in L^1[0,1]$.

Proof. It is shown in [1] that for $f \in L^1(\mathbb{B}_N, d\nu), BAf = ABf$. Hence if $f(z) = g(|z|^2)$ then $g \in L^1[0,1]$ and $TT_1g = T_1Tg$. □

3 Norm of the Berezin transform

In this section we show that the Berezin transform B defined on $L^2(\mathbb{B}_N, d\eta')$ into itself is a positive operator and has spectral radius less than 1. We also show that $\|B\| = \Phi_N(\frac{N}{2}) < 1$ where $\Phi_N(\gamma) = \frac{\Gamma(\gamma+1)\Gamma(N+1-\gamma)}{\Gamma(N+1)}, \gamma \in \mathbb{N}$.

Further we establish that B is similar to a part of the adjoint of the unilateral shift and $B^n \to 0$ in norm topology. From these results we derive many ergodicity properties of the Berezin transform and the corresponding integral operator T defined on $L^1[0,1]$. Applications of these results are also discussed.

Since the operator B on $L^\infty(\mathbb{B}_N)$ is the adjoint of B on $L^1(\mathbb{B}_N, d\eta')$ and $L^\infty(\mathbb{B}_N) = (L^1(\mathbb{B}_N, d\eta'))^*$, the spectrum of B on $L^\infty(\mathbb{B}_N)$ = spectrum of B on $L^1(\mathbb{B}_N, d\eta')$. The spectrum of B on $L^\infty(\mathbb{B}_N)$ is [1] the set

$$\left\{ \frac{\Gamma(\gamma+1)\Gamma(N+1-\gamma)}{\Gamma(N+1)} : \gamma \in \mathbb{C}, 0 \le \Re\gamma \le N \right\}.$$

Let $\Phi_N(\gamma) = \frac{\Gamma(\gamma+1)\Gamma(N+1-\gamma)}{\Gamma(N+1)} = \frac{\pi\gamma}{\sin(\pi\gamma)} \prod_{j=1}^N (1-\frac{\gamma}{j})$. From Ahern, Flores, Rudin [1], it follows that $|\Phi_N(\gamma)| < 1$ if $0 < \Re\gamma < N$. Further $\Phi_N(0) = \Phi_N(N) = 1$. Thus the spectrum of B on $L^1(\mathbb{B}_N, d\eta')$ and $L^\infty(\mathbb{B}_N)$ contains the point 1 and further since B fixes the constants hence $\|B\| = 1$ and spectral radius of B is 1.

Theorem 3.1. *Let B be the Berezin transform defined on $L^2(\mathbb{B}_N, d\eta')$. Then $B^n \to 0$ in norm topology and B is similar to a part of the adjoint of the unilateral shift.*

Proof. By Lemma 2.2, the operator B is a contraction on $L^2(\mathbb{B}_N, d\eta')$. Further B is a self-adjoint operator on $L^2(\mathbb{B}_N, d\eta')$. Because for $f \in L^2(\mathbb{B}_N, d\eta')$,

$$
\begin{aligned}
\langle Bf, f\rangle &= \int_{\mathbb{B}_N} (Bf)(z)\overline{f(z)}K_{\mathbb{B}_N}(z,z)d\nu(z) \\
&= \int_{\mathbb{B}_N}\left(\int_{\mathbb{B}_N} (f\circ\phi_z)(w)d\nu(w)\right)\overline{f(z)}K_{\mathbb{B}_N}(z,z)d\nu(z) \\
&= \int_{\mathbb{B}_N}\left(\int_{\mathbb{B}_N} f(w)|k_z(w)|^2 d\nu(w)\right)\overline{f(z)}K_{\mathbb{B}_N}(z,z)d\nu(z) \\
&= \int_{\mathbb{B}_N}\int_{\mathbb{B}_N} f(w)|K_{\mathbb{B}_N}(z,w)|^2 d\nu(w)\overline{f(z)}d\nu(z) \\
&= \int_{\mathbb{B}_N} f(w)K_{\mathbb{B}_N}(w,w)d\nu(w)\int_{\mathbb{B}_N}\overline{f(z)}\frac{|K_{\mathbb{B}_N}(z,w)|^2}{K_{\mathbb{B}_N}(w,w)}d\nu(z) \\
&= \int_{\mathbb{B}_N} f(w)d\eta'(w)\overline{\left(\int_{\mathbb{B}_N} f(z)|k_w(z)|^2 d\nu(z)\right)} \\
&= \langle f, Bf\rangle.
\end{aligned}
$$

It is known that in the space $L^2(\mathbb{B}_N, d\eta')$, the Berezin transform is a Fourier multiplier with respect to the Helgason-Fourier transform [13]. Consider the family of conical functions $e_{\lambda,b}$ indexed by $\lambda\in\mathbb{R}$ and $b\in S_N$ given by

$$
e_{\lambda,b}(x) = \left(\frac{1-||x||^2}{||b-x||^N}\right)^{\frac{N}{2}+i\lambda}, x\in\mathbb{B}_N.
$$

On the space $L^2(\mathbb{B}_N, d\eta')$, one defines the Helgason-Fourier transform $\widehat{f}$ of f as

$$
\widehat{f}(\lambda,b) = \int_{\mathbb{B}_N} f(x)e_{\lambda,b}(x)d\eta'(x).
$$

There is also [13] an inversion formula

$$
f(x) = \int_{\mathbb{R}}\int_{S_N}\widehat{f}(\lambda,b)e_{-\lambda,b}(x)|c(\lambda)|^2 dbd\lambda
$$

with some function c on $\mathbb{R}$ (the Harish-chandra c-function) and db the Haar measure on S_N; and a Plancheral isometry

$$
\int_{\mathbb{B}_N} |f(x)|^2 d\eta'(x) = \int_{\mathbb{R}}\int_{S_N} |\widehat{f}(\lambda,b)|^2|c(\lambda)|^2 dbd\lambda,
$$

exists which establishes a unitary isomorphism between $L^2(\mathbb{B}_N, d\eta')$ and a subspace $\mathcal{M}$ of all functions in $L^2(\mathbb{R}\times S_N, |c(\lambda)|^2 dbd\lambda)$ satisfying a certain symmetry condition. Under this isomorphism, an operator on $L^2(\mathbb{B}_N, d\eta')$ commuting with the action of $Aut(\mathbb{B}_N)$ corresponds to the operator on $\mathcal{M}$ of multiplication by a certain function depending only on λ. That is, if B is the Berezin transform on $L^2(\mathbb{B}_N, d\eta')$ then $\widehat{(Bf)}(\lambda,b) = m(\lambda)\widehat{f}(\lambda,b)$ where $m(\lambda) = \Phi_N(N/2+i\lambda) = (\lambda^2+\frac{1}{4})\frac{\pi}{\cosh(\pi\lambda)}\prod_{j=2}^N\left(1-\frac{\frac{1}{2}+i\lambda}{j}\right)$. Thus

$$
\begin{aligned}
\langle Bf, f\rangle &= \langle\widehat{(Bf)}, \widehat{f}\rangle \\
&= \int_{\mathbb{R}}\int_{S_N} m(\lambda)|\widehat{f}(\lambda,b)|^2 dbd\lambda \\
&\geq 0
\end{aligned}
$$

since the multiplier function $m(\lambda) = (\lambda^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi\lambda)} \prod_{j=2}^{N}\left(1 - \frac{\frac{1}{2}+i\lambda}{j}\right)$ is positive. Thus the operator B is positive. This also gives the spectral decomposition of B. Let $E(\beta)$ be the resolution of identity for the self-adjoint operator B. Then $\|B^n f\|^2 = \int_{(0,1)} |\beta^n|^2 d\langle E(\beta) f, f\rangle$. According to the Lebesgue monotone convergence theorem, this tends to $\|(I - E(1-)) f\|^2 = \|P_{\ker(B-I)} f\|^2$. Now $\ker(I - B) = \{0\}$ since 1 is not in the spectrum of B, so $\|B^n f\|$ tends to zero.

It is well known [7] that the spectrum of a multiplication operator is the essential range of its symbol. In the case of the Berezin transform the multiplier function is m and the range of m is the set $\{m(\lambda) : \lambda \in \mathbb{R}\} = \left\{\Phi_N(\lambda) : \Re\lambda = \frac{N}{2}\right\}$. Thus in view of the spectral decomposition of B on $L^2(\mathbb{B}_N, d\eta')$ given by the Helgason-Fourier transform, the spectrum of B on $L^2(\mathbb{B}_N, d\eta')$ consists of

$$\left\{\Phi_N(\gamma) : \Re\gamma = \frac{N}{2}\right\}.$$

From the properties of the Gamma function [1] it follows that for $\gamma = \frac{N}{2} + it$, t real, $\Phi_N(\gamma)$ decreases to 0 as t tends to infinity, and has maximum at $t = 0$. Hence $\|B\| = r(B) = \Phi_N(\frac{N}{2})$ and thus is < 1 by the sub-multiplicativity (log-convexity) of the [1] Gamma function. Thus $B^n \to 0$ in norm as $\|B\| < 1$ and it follows from [10] that B is similar to a part of the adjoint of the unilateral shift. □

Corollary 3.2. *Let B be the Berezin transform defined from $L^2(\mathbb{B}_N, d\eta')$ into itself. The following assertions hold.*

(i) $\|B^n\| \leq \beta\alpha^n$ for every $n \geq 0$, for some $\beta \geq 1$ and $0 < \alpha < 1$.

(ii) $\sum_{n=0}^{\infty} \|B^n\|^k < \infty$ for an arbitrary $k > 0$.

(iii) $\sum_{n=0}^{\infty} \|B^n f\|^k < \infty$ for all $f \in L^2(\mathbb{B}_N, d\eta')$ and for an arbitrary $k > 0$.

(iv) $\sum_{n=0}^{\infty} |\langle B^n f, g\rangle|^k < \infty$ for all $f, g \in L^2(\mathbb{B}_N, d\eta')$, for an arbitrary $k \geq 1$.

(v) The space RangeB is the set of all $g \in L^2(\mathbb{B}_N, d\eta')$ for which the series $\sum_{k=0}^{\infty} (I - B)^k g$ converges with respect to the norm of $L^2(\mathbb{B}_N,, d\eta')$. In this case if $f = \sum_{k=0}^{\infty} (I - B)^k g$ then $f \in (\ker B)^{\perp}$ and $Bf = g$.

(vi) The function $g \in$ rangeB if and only if $\sum_{k=0}^{\infty} \|(I - B^2)^{\frac{k}{2}} g\|^2 < \infty$. Further, the series $\sum_{k=0}^{\infty} (I - B^2)^k Bg$ converges and if $\sum_{k=0}^{\infty} (I - B^2)^k Bg = e$ then $g = Be$.

Proof. The proof follows from [20] and [6].

□

Corollary 3.3. *Suppose $N \in \mathbb{Z}_+$. Consider the integral operator T on $L^2([0,1], \frac{t^{N-1}dt}{(1-t)^{N+1}})$. Then $T^n \to 0$ in norm.*

Proof. It follows from Theorem 3.1 that $\|T\| < 1$ and the Corollary follows. □

Corollary 3.4. *The following is true for the Berezin transform B as an operator on $L^2(\mathbb{B}_N, d\eta)$: $\frac{1}{n}\sum_{k=0}^{n-1} B^k \to 0$ in the strong operator topology as $n \to \infty$.*

Proof. The result follows from Theorem 3.1 and [17]. □

Corollary 3.5. *The following is true for the integral operator T as an operator on $L^2([0,1], \frac{t^{N-1}dt}{(1-t)^{N+1}})$: $\frac{1}{n}\sum_{k=0}^{n-1} T^k \to 0$ in the strong operator topology as $n \to \infty$.*

Proof. The proof follows from corollary 3.3 and [17]. □

A continuous real-valued function u is subharmonic in $\mathbb{D}$ if and only if it satisfies the inequality

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

for every disk $|z - z_0| \leq r$ contained in $\mathbb{D}$. For a more detailed discussion on subharmonic functions see [11].

Definition 3.6. Suppose $f \in L^1(\mathbb{D}, dA)$ is a real-valued subharmonic function on $\mathbb{D}$. We say f admits an integrable harmonic majorant if there exists a function $v \in L^1(\mathbb{D}, dA)$ harmonic on $\mathbb{D}$ and such that $v(x) \geq f(x)$ for all $x \in \mathbb{D}$.

Corollary 3.7. *Assume that $f \in L^1(\mathbb{D}, dA)$ is a real-valued, radial, subharmonic function on $\mathbb{D}$ which is twice continuously differentiable and admits an integrable harmonic majorant u. Let $f(z) = g(|z|^2)$. Then $T^m g \to c$, as $m \to \infty$, where c is a fixed constant and T is the integral operator defined in (1.7).*

Proof. From [9], it follows that $B^m f \to u$, the least harmonic majorant of f. The function f is radial and belong to $L^1(\mathbb{D}, dA)$. This implies $Bf = Tg$. We have already seen that if f is radial, then Bf is radial. Thus $B^2 f = B(Bf) = T(Bf) = T(Tg) = T^2 g$. By induction, we can show that $B^m f = T^m g$. Since $B^m f \to u$, the sequence $T^m g \to v$, a radial harmonic function. Hence v is a constant c. That is, $T^m g \to c$. □

Theorem 3.8. *Assume $f \in L^1(\mathbb{D}, dA)$ is real-valued subharmonic function on $\mathbb{D}$ which admits an integrable harmonic majorant v. Then the following hold:*

(i) The functions $B^n f$ are subharmonic for all $n \in \mathbb{N}$. Further, if f is radial, $f(z) = g(|z|^2)$, then the functions $T^m g$ are subharmonic for all $m \in \mathbb{N}$.

(ii)If $f \in V(\mathbb{D}) = \{f \in L^\infty(\mathbb{D}) : ess\ lim_{|z|\to 1} f(z) = 0\}$ then $B^n f$ converges uniformly to 0. Moreover, if $f \in V(\mathbb{D})$ is radial and $f(z) = g(|z|^2)$, then $T^m g$ converges to 0 uniformly.

(iii) If $f \in C(\overline{\mathbb{D}})$ then $\{B^n f\}$ converges uniformly to h, the harmonic function whose boundary values coincide with $f|_{\mathbb{T}}$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. Suppose $f \in C(\overline{\mathbb{D}})$ is radial. Let $f(z) = g(|z|^2)$ for all $z \in \mathbb{D}$. Then $T^m g$ converges to a constant.

Proof. The theorem follows from [9] and the fact that if $f(z) = g(|z|^2)$ then $Bf = Tg$. □

Corollary 3.9. *Let $S = \frac{I+B}{2}$. Then $S^n \to 0$ in norm in the space $\mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$.*

Proof. Notice that $S^n f = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}B^j f$, $f \in L^2(\mathbb{B}_N, d\eta')$ and hence

$$\begin{aligned} BS^n f &= \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}B^{j+1} f \\ &= \frac{1}{2^n}\sum_{j=1}^{n+1}\binom{n}{j-1}B^j f,\ f \in L^2(\mathbb{B}_N, d\eta'). \end{aligned}$$

We may assume that $n = 2k$ is even, the case when n is odd being similar. Since $\binom{n}{j} = \binom{n}{n-j}$ and $SB = BS$, we obtain

$$\begin{aligned} S^n(I-B)f &= S^n f - S^n B f = \frac{1}{2^n}\left\{[f - B^{n+1}f] + \textstyle\sum_{j=1}^{n}[\binom{n}{j} - \binom{n}{j-1}]B^j f\right\} \\ &= \frac{1}{2^n}\left\{[f - B^{n+1}f] + \textstyle\sum_{j=1}^{k}[\binom{n}{j} - \binom{n}{j-1}](B^j f - B^{n-j+1}f)\right\}. \end{aligned}$$

Let $r = \sup\{\|B^i f - B^j f\| : i, j \geq 0\}$. Since $\binom{n}{j} - \binom{n}{j-1} > 0$ for $1 \leq j \leq k$, we obtain by Stirling's formula ($\frac{\sqrt{n}(2n)!}{(2^n n!)^2} \approx \frac{1}{\sqrt{\pi}}$ as $n \to \infty$)

$$\begin{aligned} \|S^n(I-B)f\| &\leq \frac{r}{2^n}\left\{1 + \textstyle\sum_{j=1}^{k}[\binom{n}{j} - \binom{n}{j-1}]\right\} \\ &= \frac{r}{2^n}\binom{n}{k} \\ &= \frac{r}{2^n}\frac{n!}{(k!)^2} \\ &\approx \frac{r}{\sqrt{\pi}\sqrt{k}} \\ &= \frac{\sqrt{2}r}{\sqrt{\pi}\sqrt{n}} \to 0 \end{aligned}$$

as $n \to \infty$. Since Range$(I - B) = L^2(\mathbb{B}_N, d\eta')$, hence $S^n \to 0$ strongly. We now show that $S^n \to 0$ in norm.
From [14], it follow that $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} \subseteq \{1\}$. Now Range$(I - B) = L^2(\mathbb{B}_N, d\eta')$ if and only if $1 \notin \sigma(B)$, the spectrum of B. This is true if and only if $1 \notin \sigma(S)$. That is, if and only if $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$. Hence $\|S^n\| \to 0$ as $n \to \infty$. □

Corollary 3.10. *If V is a linear power bounded operator from $L^2(\mathbb{B}_N, d\eta')$ into itself, $V^n \to 0$ strongly, $VB = BV$ and $S = (\frac{V+B}{2})$ then $S^n \to 0$ strongly in $\mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$.*

Proof. We have already verified that $\|B\| < 1$, hence $\|B^n\| < 1$ for all $n \geq 1$. Further $SB = BS$. Since Range$(I - B) = L^2(\mathbb{B}_N, d\eta')$, it is sufficient to establish that $\lim_{n\to\infty}\|S^n(I - B)f\| = \lim_{n\to\infty}\|S^n f - BS^n f\| = 0$ for all $f \in L^2(\mathbb{B}_N, d\eta')$. Notice that

$$\begin{aligned} \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}V^{n-j}B^j f &= \left(\frac{V+B}{2}\right)^n f \\ &= S^n f \end{aligned}$$

and

$$\begin{aligned} BS^n f &= \tfrac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}V^{n-j}B^{j+1}f \\ &= \tfrac{1}{2^n}\sum_{j=1}^{n+1}\binom{n}{j-1}V^{n-j+1}B^{j}f. \end{aligned}$$

Hence

$$\begin{aligned} S^n(I-B)f &= (I-B)S^n f \\ &= \tfrac{1}{2^n}\left(V^n f - B^{n+1}f\right) + \tfrac{1}{2^n}\sum_{j=1}^{n}\left[\binom{n}{j}V^{n-j} - \binom{n}{j-1}V^{n-j+1}\right]B^j f \\ &= \tfrac{1}{2^n}\left(V^n f - B^{n+1}f\right) + \tfrac{1}{2^n}\sum_{j=1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &+ \tfrac{1}{2^n}\sum_{j=1}^{n}\binom{n}{j-1}B^j\left(V^{n-j}f - V^{n-j+1}f\right) \\ &= C_n + D_n + E_n. \end{aligned}$$

Let $s = \sup\{\|V^iB^jf - V^kB^lf\| : i,j,k,l \geq 0\}$. Notice that s is finite since V and B are power bounded. It is not difficult to see that $\|C_n\| \leq \frac{s}{2^n} \to 0$ as $n \to \infty$.

Without loss of generality we may assume that $n = 2k$ is an even integer, the case of an odd integer n being similar. Using again the fact that $\binom{n}{j} = \binom{n}{n-j}$, we have

$$\begin{aligned} \sum_{j=1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f &= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &+ \sum_{j=k+1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &+ \sum_{j=1}^{k}\left[\binom{n}{k+j} - \binom{n}{k+j-1}\right]V^{n-k-j}B^{k+j} f \\ &= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &+ \sum_{j=1}^{k}\left[\binom{n}{k-j} - \binom{n}{k-j+1}\right]V^{n-k-j}B^{k+j} f \\ &= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]V^{n-j}B^j f \\ &+ \sum_{j=1}^{k}\left[\binom{n}{j-1} - \binom{n}{j}\right]V^{j-1}B^{n-j+1} f \\ &= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]\left(V^{n-j}B^j f - V^{j-1}B^{n-j+1}f\right). \end{aligned}$$

Hence

$$\begin{aligned} \|D_n\| &\leq \tfrac{s}{2^n}\sum_{j=1}^{k}\left[\binom{n}{j}-\binom{n}{j-1}\right] \\ &\leq \tfrac{s}{2^n}\tbinom{n}{k} \\ &\approx \tfrac{s}{\sqrt{\pi k}} \\ &= \tfrac{\sqrt{2}s}{\sqrt{\pi n}} \to 0 \end{aligned}$$

as $n \to \infty$ by Stirling's formula.

To prove that $\|E_n\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \|E_n\| &\leq \tfrac{1}{2^n}\sum_{j=1}^{n}\binom{n}{j-1}\|V^{n-j}f - V^{n-j+1}f\| \\ &= \tfrac{1}{2^n}\sum_{j=0}^{n-1}\binom{n}{n-j-1}\|V^{j}f - V^{j+1}f\| \\ &= \tfrac{1}{2^n}\sum_{j=0}^{n-1}\binom{n}{j+1}\|V^{j}f - V^{j+1}f\|. \end{aligned}$$

Now since $V^n \to 0$ strongly, we obtain $\|V^n f - V^{n+1}f\| \to 0$ as $n \to \infty$ and hence for any given $\epsilon > 0$, there is an integer $n_0 > 0$ such that $\|V^j f - V^{j+1}f\| < \epsilon$ for all $j \geq n_0$. It follows that, for $n > n_0$,

$$\begin{aligned} \|E_n\| &\leq \tfrac{1}{2^n}\left\{\epsilon\sum_{j=n_0}^{n-1}\binom{n}{j+1} + s\sum_{j=0}^{n_0-1}\binom{n}{j+1}\right\} \\ &< \epsilon + s\sum_{j=0}^{n_0-1}\frac{1}{2^n}\binom{n}{j+1}. \end{aligned}$$

Since $\lim_{n\to\infty}\frac{1}{2^n}\binom{n}{j} = 0$, hence for every fixed integer $j \geq 0$, we have $\lim_{n\to\infty}\|E_n\| = 0$. Thus $\lim_{n\to\infty}\|S^n(I-B)f\| = \lim_{n\to\infty}\|S^n f - BS^n f\| = 0$ for all $f \in L^2(\mathbb{B}_N, d\eta')$. □

Corollary 3.11. *If the operator B_λ is a convex combination of B and I in $\mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$ then $B_\lambda^n \to 0$ strongly.*

Proof. Let $0 < \lambda < 1$ and $B_\lambda = (1-\lambda)I + \lambda B$. We claim $B_\lambda^n \to 0$ strongly in $\mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$. First we consider the case $0 < \lambda < \frac{1}{2}$. Let $\mu = 2\lambda$ and $B_\mu = (1-\mu)I + \mu B$. The operator B_μ is a power bounded operator since $\|B\| < 1$ implies $\|B_\mu\| = \|(1-\mu)I + \mu B\| \leq (1-\mu) + \mu\|B\| < (1-\mu) + \mu = 1$. Proceeding similarly as in Corollary 3.9, we have $B_\lambda^n \to 0$ strongly since $B_\lambda = (1-\lambda)I + \lambda B = \frac{1}{2}(I + B_\mu)$. For $\lambda = \frac{1}{2}$, the Corollary follows from Corollary 3.9.

Now suppose V is as given in Corollary 3.10 and $0 < \lambda < \frac{1}{2}$. Let $\mu = 2\lambda < 1$ and $B_\mu^V = (1-\mu)V + \mu B$ and $S_\mu = \frac{V + B_\mu^V}{2}$. Then by Corollary 3.10,

$$S_\mu^n \to 0 \tag{3.1}$$

strongly where $S_\mu = \frac{V+B_\mu^V}{2} = \frac{V+(1-\mu)V+\mu B}{2} = (1-\lambda)V + \lambda B$. Now we prove the rest of the claim in the corollary. Notice that the set of points of the form $\frac{k}{2^m}$, where $m \geq 1$ and $k = 1, 2, \cdots, 2^m - 1$, is dense in $(0,1)$. Hence we see that for every $\lambda \in (0,1)$, $B_\lambda = (1-\lambda)I + \lambda B = (1-\beta)B_\mu + \beta B$, where $0 < \beta < \frac{1}{2}$ and $\mu = \frac{k}{2^m} < \rho$ (but close enough to ρ) for some $1 \leq k \leq 2^m - 1$ and $m \geq 1$. Since B_μ is power bounded and $0 < \beta < \frac{1}{2}$, $B_\mu^n \to 0$ strongly and $B_\mu B = BB_\mu$, it follows from (3.1) that $B_\lambda^n \to 0$ strongly in $\mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$.

□

Corollary 3.12. *If B is the Berezin transform defined from $L^2(\mathbb{B}_N, d\eta')$ into itself then (i)* $\ker(I-B) = \ker(I-B)^2 = \{0\}$ *and (ii)Range$(I-B)$ = Range$(I-B)^2 = L^2(\mathbb{B}_N, d\eta')$.*

Proof. (i) The operator $I-B$ is invertible since $\|B\| < 1$. Thus $\ker(I-B) \cap \text{Range}(I-B) = \{0\}$. Let $f \in \ker(I-B)^2$. Then $g = (I-B)f$ is in the intersection of the spaces $\ker(I-B)$ and Range$(I-B)$ which is trivial. That is, $g = (I-B)f = 0$. Thus $f \in \ker(I-B)$. Hence $\ker(I-B)^2 \subseteq \ker(I-B)$. The other inclusion is always true. (ii) By [16] is enough to prove that Range$(I-B) + \ker(I-B)$ is closed. Now Range$(I-B) = L^2(\mathbb{B}_N, d\eta')$ and $\ker(I-B) = \{0\}$. Thus from [16], it follows that Range$(I-B)^2$ is closed and Range$(I-B)$ = Range$(I-B)^2$ = $L^2(\mathbb{B}_N, d\eta')$. □

Corollary 3.13. *Let $U \in \mathcal{L}(L^2(\mathbb{B}_N, d\eta'))$ be unitary and B be the Berezin transform defined on $L^2(\mathbb{B}_N, d\eta')$. Then*

$$1 - \Phi_N\left(\frac{N}{2}\right) \leq \|U - B\| \leq 1 + \Phi_N\left(\frac{N}{2}\right).$$

Proof. Let $f \in L^2(\mathbb{B}_N, d\eta')$ be such that $\|f\| = 1$. Then

$$\|(U-B)f\|^2 = \langle (I + B^2 - U^*B - BU)f, f\rangle \geq 1 + \|Bf\|^2 - 2\|Bf\| = (1 - \|Bf\|)^2.$$

But since B is positive,

$$\inf_{\|f\|=1} \|Bf\| = \inf_{\|f\|=1} \langle Bf, f\rangle$$

and by Theorem 3.1,

$$\sup_{\|f\|=1} \|Bf\| = \sup_{\|f\|=1} \langle Bf, f\rangle.$$

Hence

$$\begin{aligned} \|(U-B)\| &\geq \sup_{\|f\|=1} |1 - \|Bf\|| \\ &= \sup_{\|f\|=1} |1 - \langle Bf, f\rangle| \\ &= \sup_{\|f\|=1} |\langle (I-B)f, f\rangle| \\ &= \|I-B\| \geq \|I\| - \|B\| = 1 - \Phi_N\left(\tfrac{N}{2}\right). \end{aligned}$$

This proves the left inequality. Again by Theorem 3.1,

$$\begin{aligned} \|(U-B)\| &= \sup_{\|f\|=1} \|Uf - Bf\| \\ &\leq \sup_{\|f\|=1} (1 + \|Bf\|) \\ &= \sup_{\|f\|=1} \langle (I+B)f, f\rangle \\ &= \|I+B\| \leq \|I\| + \|B\| = 1 + \Phi_N\left(\tfrac{N}{2}\right). \end{aligned}$$

Thus we obtain

$$1-\Phi_N\left(\frac{N}{2}\right)=\|U\|-\|B\|\leq\|(U-B)\|\leq\|U\|+\|B\|=1+\Phi_N\left(\frac{N}{2}\right)$$

and the result follows. □

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and insightful comments.

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