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Some Jensen Type Inequalities for Square-Convex Functions of Selfadjoint Operators in Hilbert Spaces

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Abstract

Some Jensen type inequalities for square-convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

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1 Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [14, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right)=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

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With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [14] and the references therein. For other results, see [20], [21], [16] and [18].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m,M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{1.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [19] (see also [14, p. 5]):

Theorem 1.1 (Mond- Pečarić, 1993, [19]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m,M]$, then*

$$h(\langle Ax, x \rangle) \leq \langle h(A)x, x \rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1.1 we have the following Hölder-McCarthy inequality:

Theorem 1.2 (Hölder-McCarthy, 1967, [17]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ *for all* $r > 1$;

(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ *for all* $0 < r < 1$;

(iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ *for all* $r < 0$.

For recent results concerning the vectorial Jensen inequality for continuous convex functions of selfadjoint operators (MP) see [5]-[11].

In this paper we introduce the concept of square-convex functions that can be naturally extended to complex-valued functions. We establish here the corresponding Jensen type inequality, provide some simple examples and obtain a number of reverse inequalities of interest.

2 Jensen's Inequality for Square-convex Functions

We introduce the following class of complex valued functions:

Definition 2.1. A function $f : [a,b] \subset \mathbb{R} \to \mathbb{C}$ is called square-convex on $[a,b]$ if the associated function $\varphi : [a,b] \to [0,\infty)$, $\varphi(t) = |f(t)|^2$ is convex on $[a,b]$.

A simple example of such a function is the concave power function $f : [a,b] \subset [0,\infty) \to [0,\infty)$, $f(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$. Also, if $h : [a,b] \to [0,\infty)$ is convex then the complex valued function $f : [a,b] \subset \mathbb{R} \to \mathbb{C}$ given by $f(t) = h^{1/2}(t)\, e^{it}$ is square-convex on $[a,b]$.

The following version of Jensen inequality holds:

Theorem 2.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If $f : [m,M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous square-convex function on $[m,M]$, then for any $x \in H$ with $\|x\| = 1$ we have the inequality*

$$|f(\langle Ax,x\rangle)| \le \|f(A)x\|. \tag{2.1}$$

Proof. We give here two proofs. The first is using the Mond-Pečarić result (MP) and the continuous functional calculus. The second is using the spectral representation (1.1) and the Jensen inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators.

1. Writing the (MP) inequality for $h = |f|^2$ we have

$$|f(\langle Ax,x\rangle)|^2 \le \left\langle |f|^2(A)\, x, x \right\rangle \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$.

However by the continuous functional calculus we have

$$\begin{aligned} \left\langle |f|^2(A)\, x, x \right\rangle &= \left\langle \bar{f}(A)\, f(A)\, x, x \right\rangle = \langle (f(A))^* f(A)\, x, x\rangle \\ &= \langle f(A)\, x, f(A)\, x\rangle = \|f(A)\, x\|^2 \end{aligned} \tag{2.3}$$

for any $x \in H$ with $\|x\| = 1$.

Therefore (2.2) becomes $|f(\langle Ax,x\rangle)|^2 \le \|f(A)\, x\|^2$ which is equivalent with (2.1).

2. If $\{E_t\}_t$ is the spectral family of the operator A, then by the spectral representation (1.1) we have (see for instance [15, p. 257])

$$\|f(A)\, x\|^2 = \int_{m-0}^{M} |f(t)|^2\, d\|E_t x\|^2 = \int_{m-0}^{M} |f(t)|^2\, d(\langle E_t x, x\rangle) \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

The following inequality is the well known Jensen's inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators $u : [a,b] \to \mathbb{R}$

$$\frac{1}{u(b)-u(a)} \int_a^b \Phi(t)\, du(t) \ge \Phi\left(\frac{1}{u(b)-u(a)} \int_a^b t\, du(t)\right), \tag{2.5}$$

provided that Φ is continuous convex on $[a,b]$.

Applying the inequality (2.5) for the functions $\Phi = |f|^2$ and $u = \langle E_{(\cdot)}x, x\rangle$ for a fixed $x \in H$ with $\|x\| = 1$, we have

$$\int_{m-0}^{M} |f(t)|^2 d(\langle E_t x, x\rangle) \geq \left| f\left(\int_{m-0}^{M} t d(\langle E_t x, x\rangle) \right) \right|^2$$

which gives the inequality $|f(\langle Ax, x\rangle)|^2 \leq \|f(A)x\|^2$ for any $x \in H$ with $\|x\| = 1$. □

It is known that for any positive operator B we have the inequality $\left\langle B^2 x, x\right\rangle \geq \langle Bx, x\rangle^2$ for any $x \in H$ with $\|x\| = 1$. Utilising this inequality we have then

$$\|f(A)x\|^2 = \left\langle |f(A)|^2 x, x\right\rangle \geq \langle |f(A)| x, x\rangle^2$$

which gives that

$$\|f(A)x\| \geq \langle |f(A)| x, x\rangle \tag{2.6}$$

for any $x \in H$ with $\|x\| = 1$, where A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and $f : [m, M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous function on $[m, M]$.

We can provide the following refinement of (2.6):

Corollary 2.3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $f : [m, M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous square-convex function on $[m, M]$ and f is concave in absolute value, i.e. $|f|$ is concave, then for any $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\|f(A)x\| \geq |f(\langle Ax, x\rangle)| \geq \langle |f(A)| x, x\rangle \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

The proof is obvious since the second inequality in (2.7) follows by (MP) applied for the concave function $h = |f|$.

Remark 2.4. We notice that the function $f(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$ is concave and square-convex on $[0, \infty)$. Therefore, for any positive operator we have the inequalities

$$\left\|A^r x\right\| \geq \langle Ax, x\rangle^r \geq \langle A^r x, x\rangle \tag{2.8}$$

for any $x \in H$ with $\|x\| = 1$ and $r \in \left[\frac{1}{2}, 1\right]$.

Consider the function $f(t) = \ln(t+1)$. We observe that it is concave and positive on $(0, \infty)$ and if define $\varphi(t) = [\ln(t+1)]^2$, then we have that

$$\varphi''(t) = \frac{2}{(t+1)^2}[1 - \ln(t+1)], \ t > -1,$$

showing that f is square-convex on the interval $[0, e-1]$. Therefore, for any selfadjoint operator A with $Sp(A) \subseteq [0, e-1]$ we have the inequality

$$\|\ln(A + 1_H)x\| \geq \ln(\langle Ax, x\rangle + 1) \geq \langle \ln(A + 1_H)x, x\rangle \tag{2.9}$$

for any $x \in H$ with $\|x\| = 1$.

Another example for trigonometric functions is for instance $f(t) = \cos t, t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The function $\varphi(t) = \cos^2 t$ has the second derivative $\varphi''(t) = -2\cos(2t)$ which is positive for $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Therefore, for any selfadjoint operator A with $Sp(A) \subseteq \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ we have the inequality

$$\|\cos Ax\| \geq |\cos\langle Ax, x\rangle| \geq \langle \cos Ax, x\rangle \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

The following reverse of Jensen's inequality holds:

Theorem 2.5. *With the assumptions of Theorem 2.2 we have*

$$\|f(A)x\| \leq \left\langle \frac{(M1_H - A)|f(m)|^2 + (A - m1_H)|f(M)|^2}{M-m} x, x \right\rangle^{1/2} \tag{2.11}$$

$$\leq \begin{cases} \left[\frac{1}{2} + \left\langle \left| \frac{A - \frac{m+M}{2} 1_H}{M-m} \right| x, x \right\rangle \right]^{1/2} \left[|f(m)|^2 + |f(M)|^2\right]^{1/2}; \\ \left\langle \left[\left(\frac{M1_H - A}{M-m}\right)^q + \left(\frac{A - m1_H}{M-m}\right)^q\right]^{1/q} x, x \right\rangle^{1/2} \\ \times \left[|f(m)|^{2p} + |f(M)|^{2p}\right]^{\frac{1}{2p}}, p > 1, 1/p + 1/q = 1; \\ \max\{|f(m)|, |f(M)|\}; \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Utilising the convexity of the function $|f|^2$ we have

$$\begin{aligned} |f(t)|^2 &= \left| f\left(\frac{(M-t)m + (t-m)M}{M-m}\right)\right|^2 \\ &\leq \frac{(M-t)|f(m)|^2 + (t-m)|f(M)|^2}{M-m} \\ &\leq \frac{1}{M-m} \begin{cases} \left[\frac{M-m}{2} + \left|t - \frac{m+M}{2}\right|\right]\left[|f(m)|^2 + |f(M)|^2\right] \\ \left[(M-t)^q + (t-m)^q\right]^{1/q} \\ \times \left[|f(m)|^{2p} + |f(M)|^{2p}\right]^{1/p}, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \max\left\{|f(m)|^2, |f(M)|^2\right\}(M-m) \end{cases} \end{aligned} \tag{2.12}$$

for any $t \in [m, M]$. For the last inequality we used the Hölder inequality for two positive numbers.

Applying the property (P) to the inequality (2.12) we have

$$\begin{aligned}
&\left\langle |f(A)|^2 x, x\right\rangle && (2.13)\\
&\leq \left\langle \frac{|f(m)|^2 (M1_H - A) + |f(M)|^2 (A - m1_H)}{M-m} x, x\right\rangle \\
&\leq \begin{cases}
\left[\frac{1}{2} + \left\langle \left|\frac{A - \frac{m+M}{2} 1_H}{M-m}\right| x, x\right\rangle\right]\left[|f(m)|^2 + |f(M)|^2\right] \\
\\
\left\langle \left[\left(\frac{M1_H - A}{M-m}\right)^q + \left(\frac{A - m1_H}{M-m}\right)^q\right]^{1/q} x, x\right\rangle \\
\times\left[|f(m)|^{2p} + |f(M)|^{2p}\right]^{1/p}, p>1, \frac{1}{p}+\frac{1}{q}=1 \\
\\
\max\left\{|f(m)|^2, |f(M)|^2\right\}
\end{cases}
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Since $\left\langle |f(A)|^2 x, x\right\rangle = \|f(A)x\|$, then by taking the square root in (2.13) we deduce the desired result (2.11). □

Remark 2.6. If we consider a selfadjoint operator A with $Sp(A) \subseteq [0, e-1]$, then by (2.11) we get

$$\|\ln(A + 1_H)x\| \leq \frac{1}{\sqrt{e-1}} \langle Ax, x\rangle^{1/2}$$

for any $x \in H$ with $\|x\| = 1$. In particular, for any selfadjoint operator P with $0 \leq P \leq 1_H$ we have from (2.11) that

$$\|\ln(P + 1_H)x\| \leq \langle Px, x\rangle^{1/2} \ln 2$$

for any $x \in H$ with $\|x\| = 1$.

3 General Reverses

In this section some upper bounds for the positive quantity

$$0 \leq \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2$$

for $x \in H$ with $\|x\| = 1$, where $f : [m, M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous square-convex function on $[m, M]$ and A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ are obtained.

Theorem 3.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $f : [m, M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous* *square-convex function on* $[m,M]$*, then for any* $x\in H$ *with* $\|x\|=1$ *we have the inequality*

$$\begin{aligned}0 &\le \|f(A)x\|^2-|f(\langle Ax,x\rangle)|^2 \qquad (3.1)\\ &\le 2\max\left\{\frac{M-\langle Ax,x\rangle}{M-m},\frac{\langle Ax,x\rangle-m}{M-m}\right\}\\ &\times\left[\frac{|f(m)|^2+|f(M)|^2}{2}-\left|f\left(\frac{M+m}{2}\right)\right|^2\right]\\ &\le 2\left[\frac{|f(m)|^2+|f(M)|^2}{2}-\left|f\left(\frac{M+m}{2}\right)\right|^2\right].\end{aligned}$$

Proof. First of all, we recall the following result obtained by the author in [12] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}& n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \qquad (3.2)\\ &\le \frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)\\ &\le n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right],\end{aligned}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n:=\sum_{i=1}^n p_i>0$.

For $n=2$ we deduce from (3.2) that

$$\begin{aligned}& 2\min\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \qquad (3.3)\\ &\le t\Phi(x)+(1-t)\Phi(y)-\Phi(tx+(1-t)y)\\ &\le 2\max\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in C$ and $t\in[0,1]$.

Since $|f|^2$ is convex, then we have

$$\begin{aligned}& |f(t)|^2-|f(\langle Ax,x\rangle)|^2 \qquad (3.4)\\ &=\left|f\left(\frac{(M-t)m+(t-m)M}{M-m}\right)\right|^2-|f(\langle Ax,x\rangle)|^2\\ &\le\frac{(M-t)|f(m)|^2+(t-m)|f(M)|^2}{M-m}\\ &-\left|f\left(\frac{(M-\langle Ax,x\rangle)m+(\langle Ax,x\rangle-m)M}{M-m}\right)\right|^2\end{aligned}$$

for any $t\in[m,M]$ and any $x\in H$ with $\|x\|=1$.

Fix $x \in H$ with $\|x\| = 1$ and apply the inequality (P) to get in the operator order the following inequality

$$\begin{aligned} & |f(A)|^2 - |f(\langle Ax,x\rangle)|^2 1_H \\ & \le \frac{|f(m)|^2 (M1_H - A) + |f(M)|^2 (A - m1_H)}{M-m} \\ & - \left| f\left(\frac{(M - \langle Ax,x\rangle) m + (\langle Ax,x\rangle - m) M}{M-m} \right) \right|^2 1_H. \end{aligned} \tag{3.5}$$

We notice that (3.5) implies the following vectorial inequality

$$\begin{aligned} & \left\langle |f(A)|^2 x,x \right\rangle - |f(\langle Ax,x\rangle)|^2 \\ & \le \frac{|f(m)|^2 (M - \langle Ax,x\rangle) + |f(M)|^2 (\langle Ax,x\rangle - m1_H)}{M-m} \\ & - \left| f\left(\frac{(M - \langle Ax,x\rangle) m + (\langle Ax,x\rangle - m) M}{M-m} \right) \right|^2 \end{aligned} \tag{3.6}$$

for any $x \in H$ with $\|x\| = 1$. This inequality is also of interest in itself.

Now, on applying the second inequality in (3.3) we have

$$\begin{aligned} & \frac{|f(m)|^2 (M - \langle Ax,x\rangle) + |f(M)|^2 (\langle Ax,x\rangle - m1_H)}{M-m} \\ & - \left| f\left(\frac{(M - \langle Ax,x\rangle) m + (\langle Ax,x\rangle - m) M}{M-m} \right) \right|^2 \\ & \le 2 \max \left\{ \frac{M - \langle Ax,x\rangle}{M-m}, \frac{\langle Ax,x\rangle - m}{M-m} \right\} \\ & \times \left[\frac{|f(m)|^2 + |f(M)|^2}{2} - \left| f\left(\frac{M+m}{2} \right) \right|^2 \right] \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

The last part is obvious since

$$\frac{M - \langle Ax,x\rangle}{M-m}, \frac{\langle Ax,x\rangle - m}{M-m} \le 1$$

for any $x \in H$ with $\|x\| = 1$. □

Remark 3.2. Utilising the elementary inequality $0 \le \sqrt{a} - \sqrt{b} \le \sqrt{a-b}$ provided $0 \le b \le a$ we get from (3.1) the simpler, however the coarser inequality

$$\begin{aligned} & 0 \le \|f(A)x\| - |f(\langle Ax,x\rangle)| \\ & \le \sqrt{2} \max \left\{ \left(\frac{M - \langle Ax,x\rangle}{M-m} \right)^{1/2}, \left(\frac{\langle Ax,x\rangle - m}{M-m} \right)^{1/2} \right\} \\ & \times \left[\frac{|f(m)|^2 + |f(M)|^2}{2} - \left| f\left(\frac{M+m}{2} \right) \right|^2 \right]^{1/2} \\ & \le \sqrt{2} \left[\frac{|f(m)|^2 + |f(M)|^2}{2} - \left| f\left(\frac{M+m}{2} \right) \right|^2 \right]^{1/2}, \end{aligned} \tag{3.7}$$

for any $x \in H$ with $\|x\| = 1$.

Example 3.3. If we apply the inequality (3.1) for the square-convex function $f(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$ on $[m, M]$ with $0 \le m \le M$, then we get:

$$\begin{aligned} 0 &\le \left\|A^r x\right\|^2 - \langle Ax, x\rangle^{2r} \\ &\le 2\max\left\{\frac{M - \langle Ax, x\rangle}{M-m}, \frac{\langle Ax, x\rangle - m}{M-m}\right\} \\ &\times\left[\frac{m^{2r} + M^{2r}}{2} - \left(\frac{M+m}{2}\right)^{2r}\right] \\ &\le 2\left[\frac{m^{2r} + M^{2r}}{2} - \left(\frac{M+m}{2}\right)^{2r}\right], \end{aligned} \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

Theorem 3.4. *With the assumptions of Theorem 3.1 we have for any $x \in H$ with $\|x\| = 1$ that*

$$\begin{aligned} 0 &\le \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2 \\ &\le \begin{cases} \frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)}{M-m} \sup_{t\in(m,M)} \Phi_f(t; m, M) \\ \\ \frac{1}{4}(M-m)\Phi_f(\langle Ax, x\rangle; m, M), \ \langle Ax, x\rangle \ne m, M, \end{cases} \end{aligned} \tag{3.9}$$

where

$$\Phi_f(t; m, M) = \frac{|f(M)|^2 - |f(t)|^2}{M-t} - \frac{|f(t)|^2 - |f(m)|^2}{t-m}. \tag{3.10}$$

Proof. By denoting

$$\Lambda_f(t; m, M) := \frac{(t-m)|f(M)|^2 + (M-t)|f(m)|^2}{M-m} - |f(t)|^2, \quad t \in [m, M]$$

we have

$$\begin{aligned} &\Lambda_f(t; m, M) \\ &= \frac{(t-m)|f(M)|^2 + (M-t)|f(m)|^2 - (M-m)|f(t)|^2}{M-m} \\ &= \frac{(t-m)|f(M)|^2 + (M-t)|f(m)|^2 - (M-t+t-m)|f(t)|^2}{M-m} \\ &= \frac{(t-m)\left[|f(M)|^2 - |f(t)|^2\right] - (M-t)\left[|f(t)|^2 - |f(m)|^2\right]}{M-m} \\ &= \frac{(M-t)(t-m)}{M-m}\Phi_f(t; m, M) \end{aligned} \tag{3.11}$$

for any $t \in (m, M)$.

Since

$$\begin{aligned} &\frac{|f(m)|^2(M - \langle Ax, x\rangle) + |f(M)|^2(\langle Ax, x\rangle - m1_H)}{M-m} \\ &-\left|f\left(\frac{(M-\langle Ax, x\rangle)m + (\langle Ax, x\rangle - m)M}{M-m}\right)\right|^2 \\ &= \Lambda_f(\langle Ax, x\rangle; m, M) \end{aligned} \tag{3.12}$$

for any $x \in H$ with $\|x\| = 1$, then by (3.6) and (3.11) we have the following inequality

$$
\begin{aligned}
0 &\le \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2 \\
&\le \frac{(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)}{M - m} \Phi_f(\langle Ax, x\rangle; m, M) \\
&\le \begin{cases} \frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)}{M-m} \sup_{t\in(m,M)} \Phi_f(t; m, M) \\ \\ \frac{1}{4}(M-m)\Phi_f(\langle Ax, x\rangle; m, M). \end{cases}
\end{aligned} \tag{3.13}
$$

The first branch holds for any $x \in H$ with $\|x\| = 1$. The second branch holds if $\langle Ax, x\rangle \ne m, M$, $x \in H$ with $\|x\| = 1$. □

Example 3.5. If we apply the second inequality from (3.9) for the square-convex function $f(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$ on $[m, M]$ with $0 \le m \le M$, then for any selfadjoint operator A with $Sp(A) \subseteq [m, M]$ we get:

$$
\begin{aligned}
0 &\le \left\|A^r x\right\|^2 - \langle Ax, x\rangle^{2r} \\
&\le \frac{1}{4}(M - m)\left[\frac{M^{2r} - \langle Ax, x\rangle^{2r}}{M - \langle Ax, x\rangle} - \frac{\langle Ax, x\rangle^{2r} - m^{2r}}{\langle Ax, x\rangle - m}\right]
\end{aligned} \tag{3.14}
$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \ne m, M$.

4 More Reverses for Differentiable Functions

In order to prove another reverse of the Jensen's inequality, we need the following Grüss type result obtained in [3]. For the sake of completeness, we give here a simple proof.

Lemma 4.1. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $h, g : [m, M] \longrightarrow \mathbb{R}$ are continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$
\begin{aligned}
&|\langle h(A) g(A) x, x\rangle - \langle h(A) x, x\rangle \langle g(A) x, x\rangle| \\
&\le \frac{1}{2}(\Delta - \delta)\langle |h(A) - \langle h(A) x, x\rangle \cdot 1_H| x, x\rangle \\
&\le \frac{1}{2}(\Delta - \delta)\left[\|h(A)x\|^2 - \langle h(A) x, x\rangle^2\right]^{1/2},
\end{aligned} \tag{4.1}
$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we have

$$
\left| g(t) - \frac{\Delta + \delta}{2} \right| \le \frac{1}{2}(\Delta - \delta), \tag{4.2}
$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (4.2) with $|h(t) - \langle h(A) x, x\rangle|$ we get

$$
\begin{aligned}
&\left| h(t) g(t) - \langle h(A) x, x\rangle g(t) - \frac{\Delta + \delta}{2} h(t) + \frac{\Delta + \delta}{2}\langle h(A) x, x\rangle \right| \\
&\le \frac{1}{2}(\Delta - \delta)|h(t) - \langle h(A) x, x\rangle|,
\end{aligned} \tag{4.3}
$$

for any $t \in [m,M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (4.3) and a selfadjoint operator B with $Sp(B) \subset [m,M]$, then we get the following inequality of interest in itself:

$$\begin{aligned}
&|\langle h(B)g(B)y,y\rangle - \langle h(A)x,x\rangle\langle g(B)y,y\rangle \\
&-\frac{\Delta+\delta}{2}\langle h(B)y,y\rangle + \frac{\Delta+\delta}{2}\langle h(A)x,x\rangle\Big| \\
&\le \frac{1}{2}(\Delta-\delta)\langle |h(B)-\langle h(A)x,x\rangle\cdot 1_H|y,y\rangle,
\end{aligned} \tag{4.4}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

If we choose in (4.4) $y = x$ and $B = A$, then we deduce the first inequality in (4.1).

Now, by the Schwarz inequality in H we have

$$\begin{aligned}
\langle |h(A)-\langle h(A)x,x\rangle\cdot 1_H|x,x\rangle &\le \||h(A)-\langle h(A)x,x\rangle\cdot 1_H|x\| \\
&= \|h(A)x - \langle h(A)x,x\rangle\cdot x\| \\
&= \left[\|h(A)x\|^2 - \langle h(A)x,x\rangle^2\right]^{1/2}
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, and the second part of (4.1) is also proved. □

For other related results see [1] and [2].

Before we state the next result, we say that the function $f : I \to \mathbb{C}$ is *square differentiable* on I if the function $|f|^2$ is differentiable on I. It is clear that, if f is differentiable on I then it is square differentiable on I and

$$\begin{aligned}
\frac{d\left(|f|^2\right)}{dt} &= 2\operatorname{Re}\left(\bar{f}\cdot\frac{df}{dt}\right) = 2\operatorname{Re}\left(f\cdot\frac{d\bar{f}}{dt}\right) \\
&= 2\left[\operatorname{Re}(f)\operatorname{Re}\left(\frac{df}{dt}\right) + \operatorname{Im}(f)\operatorname{Im}\left(\frac{df}{dt}\right)\right].
\end{aligned}$$

For a real function $g : [m,M] \to \mathbb{R}$ and two distinct points $\alpha,\beta \in [m,M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha,\beta;g] := \frac{g(\beta)-g(\alpha)}{\beta-\alpha}.$$

With these preparations we can state and prove another reverse of the Jensen inequality.

Theorem 4.2. *Let I be an interval and $f : I \to \mathbb{C}$ be a square-convex, square differentiable function on $\mathring{I}$ (the interior of I). If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m,M] \subset \mathring{I}$, then*

$$\begin{aligned}
0 &\le \|f(A)x\|^2 - |f(\langle Ax,x\rangle)|^2 \\
&\le \frac{1}{2}\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)\langle |A-\langle Ax,x\rangle\cdot 1_H|x,x\rangle \\
&\le \frac{1}{2}\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \\
&\le \frac{1}{4}(M-m)\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)
\end{aligned} \tag{4.5}$$

for any $x \in H$ *with* $\|x\| = 1$ *and* $\langle Ax, x\rangle \neq m, M$.

We also have

$$0 \le \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2 \tag{4.6}$$
$$\le \frac{1}{2}\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)\langle |A - \langle Ax,x\rangle \cdot 1_H| x, x\rangle$$
$$\le \frac{1}{2}\left(\frac{d\left(|f|^2\right)(M)}{dt} - \frac{d\left(|f|^2\right)(m)}{dt}\right)\langle |A - \langle Ax,x\rangle \cdot 1_H| x, x\rangle$$
$$\le \frac{1}{2}\left(\frac{d\left(|f|^2\right)(M)}{dt} - \frac{d\left(|f|^2\right)(m)}{dt}\right)\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2}$$
$$\le \frac{1}{4}(M-m)\left(\frac{d\left(|f|^2\right)(M)}{dt} - \frac{d\left(|f|^2\right)(m)}{dt}\right)$$

for any $x \in H$ *with* $\|x\| = 1$ *and* $\langle Ax, x\rangle \neq m, M$.

Proof. Let $x \in H$ with $\|x\| = 1$ and define the function $\delta_x : [m, M] \to \mathbb{R}$ by

$$\delta_x(t) := \begin{cases} \frac{|f|^2(t) - |f|^2(\langle Ax,x\rangle)}{t - \langle Ax,x\rangle} & t \neq \langle Ax,x\rangle \\ \\ \frac{d\left(|f|^2\right)(\langle Ax,x\rangle)}{dt} & t = \langle Ax,x\rangle. \end{cases}$$

Since the function f is a square-convex, square differentiable function on $\mathring{I}$, then the function is continuous and monotonic on $[m, M]$.

Therefore we have that

$$\frac{d\left(|f|^2\right)(m)}{dt} \le \frac{|f|^2(m) - |f|^2(\langle Ax,x\rangle)}{m - \langle Ax,x\rangle} \le \delta_x(t) \tag{4.7}$$
$$\le \frac{|f|^2(M) - |f|^2(\langle Ax,x\rangle)}{M - \langle Ax,x\rangle} \le \frac{d\left(|f|^2\right)(M)}{dt}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \neq m, M$.

Applying the Grüss type result (4.1) for the functions $h_x(t) = t - \langle Ax, x\rangle$ and $g_x(t) = \delta_x(t), t \in [m, M]$ we have that

$$|\langle h_x(A) g_x(A) x, x\rangle - \langle h_x(A)x, x\rangle \langle g_x(A)x, x\rangle| \tag{4.8}$$
$$\le \frac{1}{2}\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)$$
$$\times \langle |h_x(A) - \langle h_x(A)x,x\rangle \cdot 1_H| x, x\rangle$$
$$\le \frac{1}{2}\left(\left[\langle Ax,x\rangle, M, |f|^2\right] - \left[m, \langle Ax,x\rangle, |f|^2\right]\right)$$
$$\times \left[\|h_x(A)x\|^2 - \langle h_x(A)x, x\rangle^2\right]^{1/2},$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \neq m, M$.

Since

$$\langle h_x(A) g_x(A) x, x\rangle = \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2,$$

$$\langle h_x(A) x, x\rangle = 0,\ h_x(A) - \langle h_x(A) x, x\rangle \cdot 1_H = A - \langle Ax, x\rangle \cdot 1_H$$

and

$$\|h_x(A)x\|^2 = \|Ax\|^2 - \langle Ax, x\rangle^2$$

then by (4.8) we deduce the second and the third inequality in (4.5).

The last part follows from the fact that

$$\|Ax\|^2 - \langle Ax, x\rangle^2 \le \frac{1}{4}(M-m)^2,$$

for any $x \in H$ with $\|x\| = 1$.

The inequality follows by (4.7) and the theorem is proved. □

Example 4.3. If we apply the second inequality from (3.9) for the square-convex function $f(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$ on $[m, M]$ with $0 \le m \le M$, then for any selfadjoint operator A with $Sp(A) \subseteq [m, M]$ we get the following refinement of (3.14):

$$\begin{aligned}
0 &\le \left\|A^r x\right\|^2 - \langle Ax, x\rangle^{2r} \\
&\le \frac{1}{2}\langle |A - \langle Ax, x\rangle \cdot 1_H| x, x\rangle \\
&\quad\times\left[\frac{M^{2r} - \langle Ax, x\rangle^{2r}}{M - \langle Ax, x\rangle} - \frac{\langle Ax, x\rangle^{2r} - m^{2r}}{\langle Ax, x\rangle - m}\right] \\
&\le \frac{1}{2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{1/2} \\
&\quad\times\left[\frac{M^{2r} - \langle Ax, x\rangle^{2r}}{M - \langle Ax, x\rangle} - \frac{\langle Ax, x\rangle^{2r} - m^{2r}}{\langle Ax, x\rangle - m}\right] \\
&\le \frac{1}{4}(M-m)\left[\frac{M^{2r} - \langle Ax, x\rangle^{2r}}{M - \langle Ax, x\rangle} - \frac{\langle Ax, x\rangle^{2r} - m^{2r}}{\langle Ax, x\rangle - m}\right],
\end{aligned} \tag{4.9}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \ne m, M$.

From (4.6) we also have:

$$\begin{aligned}
0 &\le \left\|A^r x\right\|^2 - \langle Ax, x\rangle^{2r} \\
&\le \frac{1}{2}\langle |A - \langle Ax, x\rangle \cdot 1_H| x, x\rangle \\
&\quad\times\left[\frac{M^{2r} - \langle Ax, x\rangle^{2r}}{M - \langle Ax, x\rangle} - \frac{\langle Ax, x\rangle^{2r} - m^{2r}}{\langle Ax, x\rangle - m}\right] \\
&\le r\left(M^{2r-1} - m^{2r-1}\right)\langle |A - \langle Ax, x\rangle \cdot 1_H| x, x\rangle \\
&\le r\left(M^{2r-1} - m^{2r-1}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\
&\le \frac{1}{2} r(M-m)\left(M^{2r-1} - m^{2r-1}\right),
\end{aligned} \tag{4.10}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \ne m, M$.

In the recent paper [4] we have obtained the following reverse of the Jensen inequality:

Lemma 4.4. *Let I be an interval and $h : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \le) \langle h(A)x, x\rangle - h(\langle Ax, x\rangle) \le \langle h'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle h'(A)x, x\rangle \tag{4.11}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising this result we are able to provide a different reverse for the Jensen inequality (2.1).

Theorem 4.5. *Let I be an interval and $f : I \to \mathbb{C}$ be a square-convex, square differentiable function on $\mathring{I}$ and with the derivative continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned}
0 &\le \|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2 \qquad\qquad (4.12)\\
&\le \left\langle A\frac{d\left(|f|^2\right)(A)}{dt}x, x\right\rangle - \langle Ax, x\rangle \cdot \left\langle \frac{d\left(|f|^2\right)(A)}{dt}x, x\right\rangle\\
&\le \begin{cases} \frac{1}{2}\left(\frac{d(|f|^2)(M)}{dt} - \frac{d(|f|^2)(m)}{dt}\right)\langle |A - \langle Ax, x\rangle \cdot 1_H| x, x\rangle \\[2ex] \frac{1}{2}(M - m)\left\langle \left|\frac{d(|f|^2)(A)}{dt} - \left\langle \frac{d(|f|^2)(A)}{dt}x, x\right\rangle \cdot 1_H\right| x, x\right\rangle \end{cases}\\
&\le \begin{cases} \frac{1}{2}\left(\frac{d(|f|^2)(M)}{dt} - \frac{d(|f|^2)(m)}{dt}\right)\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{1/2} \\[2ex] \frac{1}{2}(M - m)\left[\left\|\frac{d(|f|^2)(A)}{dt}x\right\|^2 - \left\langle \frac{d(|f|^2)(A)}{dt}x, x\right\rangle^2\right]^{1/2} \end{cases}\\
&\le \frac{1}{4}(M - m)\left(\frac{d\left(|f|^2\right)(M)}{dt} - \frac{d\left(|f|^2\right)(m)}{dt}\right),
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. If we write the inequality (4.11) for $h = |f|^2$ we get

$$\begin{aligned}
(0 \le) &\|f(A)x\|^2 - |f(\langle Ax, x\rangle)|^2 \qquad\qquad (4.13)\\
&\le \left\langle A\frac{d\left(|f|^2\right)(A)}{dt}x, x\right\rangle - \langle Ax, x\rangle \cdot \left\langle \frac{d\left(|f|^2\right)(A)}{dt}x, x\right\rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Further, on making use of the Gruss' type inequality (4.1) we also have

$$\begin{aligned}
&\left\langle A\frac{d\left(|f|^2\right)(A)}{dt}x,x\right\rangle-\langle Ax,x\rangle\cdot\left\langle\frac{d\left(|f|^2\right)(A)}{dt}x,x\right\rangle && (4.14)\\
&\le\begin{cases}\frac{1}{2}\left(\frac{d\left(|f|^2\right)(M)}{dt}-\frac{d\left(|f|^2\right)(m)}{dt}\right)\langle|A-\langle Ax,x\rangle\cdot 1_H|x,x\rangle\\ \\ \frac{1}{2}(M-m)\left\langle\left|\frac{d\left(|f|^2\right)(A)}{dt}-\left\langle\frac{d\left(|f|^2\right)(A)}{dt}x,x\right\rangle\cdot 1_H\right|x,x\right\rangle\end{cases}\\
&\le\begin{cases}\frac{1}{2}\left(\frac{d\left(|f|^2\right)(M)}{dt}-\frac{d\left(|f|^2\right)(m)}{dt}\right)\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\left\|\frac{d\left(|f|^2\right)(A)}{dt}x\right\|^2-\left\langle\frac{d\left(|f|^2\right)(A)}{dt}x,x\right\rangle^2\right]^{1/2}\end{cases}\\
&\le\frac{1}{4}(M-m)\left(\frac{d\left(|f|^2\right)(M)}{dt}-\frac{d\left(|f|^2\right)(m)}{dt}\right)
\end{aligned}$$

and the proof is completed. □

Example 4.6. If we apply the second inequality from (4.12) for the square-convex function $f(t)=t^r$ with $r\in\left[\frac{1}{2},1\right]$ on $[m,M]$ with $0\le m\le M$, then for any selfadjoint operator with $Sp(A)\subseteq[m,M]$ we get

$$\begin{aligned}
0&\le\left\|A^rx\right\|^2-\langle Ax,x\rangle^{2r} && (4.15)\\
&\le 2r\left[\left\langle A^{2r}x,x\right\rangle-\langle Ax,x\rangle\cdot\left\langle A^{2r-1}x,x\right\rangle\right]\\
&\le r\begin{cases}\left(M^{2r-1}-m^{2r-1}\right)\langle|A-\langle Ax,x\rangle\cdot 1_H|x,x\rangle\\ \\ (M-m)\left\langle\left|A^{2r-1}-\left\langle A^{2r-1}x,x\right\rangle\cdot 1_H\right|x,x\right\rangle\end{cases}\\
&\le r\begin{cases}\left(M^{2r-1}-m^{2r-1}\right)\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{1/2}\\ \\ (M-m)\left[\left\|A^{2r-1}x\right\|^2-\left\langle A^{2r-1}x,x\right\rangle^2\right]^{1/2}\end{cases}\\
&\le\frac{1}{2}r(M-m)\left(M^{2r-1}-m^{2r-1}\right),
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Finally, we observe that the interested reader may obtain other similar results by considering the square-convex, square-differentiable functions $\varphi(t)=\ln(t+1), t\in[0,e-1]$ and $\varphi(t)=\cos t, t\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$. The details are omitted.

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