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UNICITY OF ENTIRE FUNCTIONS AND A RELATED PROBLEM

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Abstract

This paper concerns the unicity of entire functions and the growth estimate of entire solutions to certain complex linear non-homogenous differential equations.

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1 Introduction and main result

In this paper, it is assumed that the reader is familiar with the fundamental results in *Nevanlinna's value distribution theory* of meromorphic functions of a single complex variable in the open complex plane $\mathbb{C}$, such as the *First main theorem*, the *Second main theorem*, and the *Lemma of logarithmic derivative* etc., and the basic notations, such as the *characteristic function* $T(r,f)$, the *proximity function* $m(r,f)$, and the *counting function* $N(r,f)$ and the *reduced counting function* $\bar{N}(r,f)$ (of poles) of a non-constant meromorphic function f in $\mathbb{C}$. A meromorphic function $a(\not\equiv\infty)$ is said to be a small function related to f, if $T(r,a)=o(T(r,f))$ holds outside a set of $r\in\mathbb{R}^{+}$ with finite Lebesgue measure. For any two distinct non-constant meromorphic functions f and g, and any common small meromorphic function a related to them, f and g are said to share a *CM* (resp. *IM*) if $f-a$ and

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$g-a$ have the same zeros counting (resp. ignoring) multiplicities. Also, $S(r,f)$ represents any quantity satisfying $S(r,f)=o(T(r,f))$, possibly outside a set of $r\in\mathbb{R}^+$ with finite Lebesgue measure, but not necessarily the same at each occurrence (cf. [13, 16]).

In 1929, R. Nevanlinna proved his well-known *Five-value* and *Four-value theorems*, which started the studies of the unicity problem of meromorphic functions in $\mathbb{C}$. In 1977, when thinking about the unicity problem of an entire function f in regard to its first order derivative f', L.A. Rubel and C.C. Yang (cf. [20]) showed that $f\equiv f'$ if they share two distinct finite values *CM*. Since then, this subject and some other related problems have been further studied, and many results on the unicity of entire and meromorphic functions sharing two or three distinct finite values or small functions with their derivatives or generated linear differential polynomials have been obtained (some new results are [1, 12, 18], corresponding to some previous ones therein the references). (On the other hand, some results of similar type for entire or meromorphic functions of several complex variables defined on $\mathbb{C}^n$ have been obtained by C.A. Bernstein, D.C. Chang and B.Q. Li (cf. [2-4]).)

However, for the unicity problem of entire or meromorphic functions sharing only one finite value or small function with their derivatives or generated linear differential polynomials, not too many results are known. In 1986, G. Jank, E. Mues and L. Volkmann (cf. [15]) studied this problem first, and proved that: If a non-constant meromorphic function f shares a finite, non-zero value a *CM* with f' and f'', then $f\equiv f'$; if f is entire and shares a finite, non-zero value a *IM* with f', and $f''=a$ whenever $f=a$, then also $f\equiv f'$. In 2001, P. Li and C.C. Yang (cf. [17]) generalized the latter case on entire functions as: If a non-constant entire function f shares a finite, non-zero value a *CM* with f' and $f^{(k)}$ $(k\geq 2)$, then $f(z)=ce^{bz}+\frac{a(b-1)}{b}$, where b, c are two non-zero constants with $b^{k-1}=1$.

Recently, J.M. Chang and M.L. Fang (cf. [5-6]) obtained the following generalization and supplement with respect to the above results concerning entire functions.

Theorem 1.1. *Let $f(z)$ be a non-constant entire function, let $a(z)$ be a small meromorphic function related to $f(z)$, and let $k(>2)$ be a positive integer. If $f(z)$, $f'(z)$ share $a(z)$ IM, and $f''(z)=a(z)$ whenever $f(z)=a(z)$ for the case $a(z)\not\equiv a'(z)$, or if $f(z)$, $f'(z)$ and $f^{(k)}(z)$ share $a(z)$ CM for the case $a(z)$ is non-constant, then $f(z)\equiv f'(z)$.*

In 2007, P. Li and W.J. Wang (cf. [19]), and independently, the first author and H.X. Yi (cf. [12]) complemented and extended the above results as below.

Theorem 1.2. *Let $f(z)$ be a non-constant entire function, let $a(z)$ be a small meromorphic function related to $f(z)$ and of finite order, and let $k(\geq 2)$ be a positive integer. If $f(z)$, $f'(z)$ share $a(z)$ CM, and $f^{(k)}(z)=a(z)$ whenever $f(z)=a(z)$, then $f(z)$ assumes one of the following three forms:*

(i) $f(z)=a(z)+c\exp\{\int e^{\alpha(z)}dz\}$ and $a(z)\equiv a'(z)$, where c is a non-zero constant and $\alpha(z)$ is an entire function (Note that if $a(z)$ is non-constant, so is $\alpha(z)$.);

(ii) $f(z)=ce^{bz}+\frac{a(b-1)}{b}$ and $a(z)$ is a non-zero constant, where b, c are two non-zero constants such that $b^{k-1}=1$;

(iii) $f(z)\equiv f'(z)$ and $a(z)$ is non-constat such that $a(z)\not\equiv a'(z)$,
where the order of $a(z)$ is defined as $\rho(a):=\limsup\limits_{r\to+\infty}\frac{\log T(r,a)}{\log r}$.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function, and let*

$$L[f](z) := a_2(z)f''(z) + a_3(z)f'''(z) + \cdots + a_n(z)f^{(n)}(z) \quad (a_n(z) \not\equiv 0)$$

be a linear differential polynomial in $f(z)$ with rational coefficients $a_k(z)$ for $k = 2, 3, \ldots, n$. If $f(z)$, $f'(z)$ share a non-zero polynomial $a(z)$ of degree p CM, and $L[f](z) = a(z)$ whenever $f(z) = a(z)$, then $f(z) = ce^{bz} + (b-1)\Phi(z)$, where b, c are two non-zero constants and $\Phi(z) := \sum_{j=1}^{p+1} \frac{a^{(j-1)}(z)}{b^j}$ such that, for $\Psi(z) := \sum_{k=2}^{n} a_k(z)b^k$, we have

$$b = \frac{a(z) - (b-1)\Phi'(z)}{a(z) - (b-1)\Phi(z)} \quad \text{and} \quad \Psi(z) \equiv \frac{a(z) - (b-1)L[\Phi](z)}{a(z) - (b-1)\Phi(z)}.$$

Remark 1.4. From (4.4), the equation immediately below (4.5) for $k = 1$ and (4.9) in [12, pp 67-68], the identity $b = \frac{a(z)-(b-1)\Phi'(z)}{a(z)-(b-1)\Phi(z)}$ follows immediately.

It is easy to see, from the assumptions and conclusions of Theorem 1.2, that, except for the case (i), the small meromorphic function a related to f must satisfy $\rho(a) < 1$ there. Inspired by this observation, we will prove the main result of this paper below.

Theorem 1.5. *Let $f(z)$ be a non-constant entire function, let $a(z)(\not\equiv 0)$ be a small meromorphic function related to $f(z)$ such that $\rho(a) < 1$, and let $L[f](z)$ be a linear differential polynomial in $f(z)$ defined in the statement of Theorem 1.3 yet with small meromorphic coefficients $a_k(z)$ related to $f(z)$ such that $\rho(a_k) < 1$ for $k = 2, 3, \ldots, n$. If $f(z)$, $f'(z)$ share $a(z)$ CM, and $L[f](z) = a(z)$ whenever $f(z) = a(z)$, then $f(z)$ assumes the form*

$$f(z) = ce^{bz} + (1-b)e^{bz}\int a(z)e^{-bz}dz, \tag{1.1}$$

where b, c are two non-zero constants. Furthermore, $a(z)$ is an entire solution to the following complex linear homogenous differential equation about $\omega(z)$'s

$$\sum_{k=2}^{n} a_k(z) \sum_{j=0}^{k-1} b^j \left(\omega^{(k-1-j)}(z) - \omega^{(k-j)}(z)\right) + \sum_{k=2}^{n} a_k(z)\omega^{(k-1)}(z) - \omega(z) = 0. \tag{1.2}$$

Remark 1.6. Under the assumptions of Theorem 1.3, we could show that $b = \frac{a(z)-(b-1)\Phi'(z)}{a(z)-(b-1)\Phi(z)}$ and $\Psi(z) \equiv \frac{a(z)-(b-1)L[\Phi](z)}{a(z)-(b-1)\Phi(z)}$ are sufficient for condition (1.2).

2 A lemma

Lemma 2.1. *(cf. [19, Lemma 4]). Suppose that $h(z)$ is a non-constant meromorphic function such that $N(r,h) + \bar{N}\left(r, \frac{1}{h}\right) = S(r,h)$, and suppose that*

$$R(h)(z) = \frac{a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z)}{b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z)} \tag{2.1}$$

is an irreducible rational function in $h(z)$ with small meromorphic coefficients $a_0(z)$, $a_1(z)$, $\ldots$, $a_p(z)$ and $b_0(z)$, $b_1(z)$, $\ldots$, $b_q(z)$ related to $h(z)$ such that $a_0(z)b_0(z) \not\equiv 0$. If $N(r, R(h)) = S(r,h)$, then we would have $b_1(z) = b_2(z) = \cdots = b_q(z) = 0$.

3 Proof of Theorem 1.5

We first define the following crucial auxiliary function φ as

$$\varphi(z) := \frac{(a(z)-a'(z))\,(L[f](z)-L[a](z))}{f(z)-a(z)} - \frac{(a(z)-L[a](z))\,(f'(z)-a'(z))}{f(z)-a(z)}. \tag{3.1}$$

The *Lemma of logarithmic derivative* yields $m(r,\varphi)=S(r,f)$. On the other hand, it is easy to see that the poles of φ arise from the zeros of $a-a'$, since f and f' share a *CM*, and the poles of a_k for $k=2,3,\ldots,n$. Then, by assumption, we also have $N(r,\varphi)=S(r,f)$ so that, by definition, $T(r,\varphi)=m(r,\varphi)+N(r,\varphi)=S(r,f)$.

Since f and f' share a *CM*, there exists an entire function α such that

$$\frac{f'(z)-a(z)}{f(z)-a(z)} = e^{\alpha(z)}. \tag{3.2}$$

Rewriting (3.2) as

$$f'(z) = e^{\alpha(z)} f(z) + \left(1-e^{\alpha(z)}\right) a(z), \tag{3.3}$$

and applying similar discussions as those in [12, pp 70-71], we obtain

$$f^{(k)}(z) = \left(e^{k\alpha(z)} + P_{k-1}\left[e^{\alpha(z)}\right]\right) f(z) - a(z)e^{k\alpha(z)} + P^*_{k-1}\left[e^{\alpha(z)}\right] \tag{3.4}$$

for $k=1,2,\ldots$, such that

$$\begin{aligned} &a(z)\left(e^{(k+1)\alpha(z)} + P_k\left[e^{\alpha(z)}\right]\right) - a(z)e^{(k+1)\alpha(z)} + P^*_k\left[e^{\alpha(z)}\right] \\ &= a(z)P_k\left[e^{\alpha(z)}\right] + P^*_k\left[e^{\alpha(z)}\right] = (a(z)-a'(z))e^{k\alpha(z)} + P^{**}_{k-1}\left[e^{\alpha(z)}\right], \end{aligned} \tag{3.5}$$

where $P_{k-1}[e^{\alpha}]$, $P^*_{k-1}[e^{\alpha}]$ and $P^{**}_{k-1}[e^{\alpha}]$ are differential polynomials in e^{α} with small meromorphic coefficients (related to e^{α}) written in terms of a, α, their derivatives and their combinations such that $\max\{\gamma_{P_{k-1}}, \gamma_{P^*_{k-1}}, \gamma_{P^{**}_{k-1}}\} \le k-1$. Here, $\gamma_{P_{k-1}}$ denotes the degree of $P_{k-1}[e^{\alpha}]$, and $\gamma_{P^*_{k-1}}$, $\gamma_{P^{**}_{k-1}}$ are similarly defined (cf. [12, 16]).

Then, it follows that

$$L[f](z) = \sum_{k=2}^{n} a_k(z)\left(e^{k\alpha(z)} + P_{k-1}\left[e^{\alpha(z)}\right]\right) f(z) + \sum_{k=2}^{n} a_k(z)\left(-a(z)e^{k\alpha(z)} + P^*_{k-1}\left[e^{\alpha(z)}\right]\right). \tag{3.6}$$

Since a is non-zero and $\rho(a)<1$, it follows that $a \not\equiv a'$. Combining (3.1), (3.3), the second equality in (3.5), and (3.6) yields

$$\begin{aligned} &a_n(z)e^{n\alpha(z)} + Q_{n-1}\left[e^{\alpha(z)}\right] - \frac{a(z)-L[a](z)}{a(z)-a'(z)}e^{\alpha(z)} - \frac{\varphi(z)}{a(z)-a'(z)} \\ &= \frac{a_n(z)(a'(z)-a(z))e^{(n-1)\alpha(z)} - Q^*_{n-2}\left[e^{\alpha(z)}\right] + a(z)}{f(z)-a(z)}, \end{aligned} \tag{3.7}$$

where $Q_{n-1}[e^{\alpha}]$ and $Q^*_{n-2}[e^{\alpha}]$ are differential polynomials in e^{α} with small meromorphic coefficients (related to e^{α}) written in terms of a, α, their derivatives, a_k $(k=2,3,\ldots,n)$, and their combinations such that $\gamma_{Q_{n-1}} \le n-1$ and $\gamma_{Q^*_{n-2}} \le n-2$.

If $a_ne^{n\alpha}+Q_{n-1}[e^{\alpha}]-\frac{a-L[a]}{a-a'}e^{\alpha}-\frac{\varphi}{a-a'}\equiv 0$, then by (3.7) we have

$$a_n(z)(a(z)-a'(z))e^{(n-1)\alpha(z)}\equiv a-Q^*_{n-2}\left[e^{\alpha(z)}\right]. \tag{3.8}$$

Considering the well-known *Clunie's Lemma* (cf. [12, Lemma 1]), and noting that $\rho(a)<1$, $\rho(a_k)<1$ for $k=2$, 3, …, n, it follows immediately that e^{α} must be a non-zero constant, say, b. Then, integrating (3.2) yields equation (1.1). On the other hand, replacing e^{α} by the constant b in (3.3)-(3.6) leads to the following identities

$$f^{(k)}(z)=b^kf+(1-b)\sum_{j=0}^{k-1}b^ja^{(k-1-j)}(z) \tag{3.9}$$

for $k=1$, 2, …, and

$$L[f](z)=\sum_{k=2}^{n}a_k(z)b^kf(z)+(1-b)\sum_{k=2}^{n}a_k(z)\left(\sum_{j=0}^{k-1}b^ja^{(k-1-j)}(z)\right). \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.1), and noting (3.8), together with the same calculation as that for the derivation of $Q^*_{n-2}[e^{\alpha}]$, would yield equation (1.2).

If $a_ne^{n\alpha}+Q_{n-1}[e^{\alpha}]-\frac{a-L[a]}{a-a'}e^{\alpha}-\frac{\varphi}{a-a'}\not\equiv 0$, then it follows that

$$f(z)-a(z)=\frac{a_n(z)(a'(z)-a(z))e^{(n-1)\alpha(z)}-Q^*_{n-2}\left[e^{\alpha(z)}\right]+a(z)}{a_n(z)e^{n\alpha(z)}+Q_{n-1}\left[e^{\alpha(z)}\right]-\frac{a(z)-L[a](z)}{a(z)-a'(z)}e^{\alpha(z)}-\frac{\varphi(z)}{a(z)-a'(z)}}, \tag{3.11}$$

where, without lose of generality, we may suppose that the right hand side rational function of e^{α} in (3.11) is irreducible (Otherwise, noting that the degrees in the leading terms of the numerator and denominator of the right hand side rational function of e^{α} in (3.11) are $n-1$ and n, respectively, after killing the common factors, we arrive at a new rational function of e^{α} of a similar form with the degrees in the leading terms of the numerator and denominator being $s-1$ and s, respectively, which only affects the following proof by switching n to s, where $1\le s\le n$.). Noting that $N(r,f-a)=N(r,a)\le T(r,a)=S(r,e^{\alpha})$ since $T(r,f)=nT(r,e^{\alpha})$, applying the conclusions of Lemma 2.1 to (3.11) yields

$$f(z)-a(z)=\frac{Q^{**}_{n-1}\left[e^{\alpha(z)}\right]}{a_n(z)e^{n\alpha(z)}}, \tag{3.12}$$

where $Q^{**}_{n-1}[e^{\alpha}]$ is a differential polynomial in e^{α} of degree $\gamma_{Q^{**}_{n-1}}\le n-1$ with small meromorphic coefficients (related to e^{α}). Also, the "constant term" of $Q^{**}_{n-1}[e^{\alpha}]$, say, β, is not identically zero since we suppose the rational function of e^{α} in (3.11) is irreducible. Then,

$$f'(z)=\frac{a_n^2(z)e^{n\alpha(z)}a'(z)+a_n(z)\left(Q^{**}_{n-1}\left[e^{\alpha(z)}\right]\right)'-(a_n'(z)+n\alpha'(z)a_n(z))Q^{**}_{n-1}\left[e^{\alpha(z)}\right]}{a_n^2(z)e^{n\alpha(z)}}. \tag{3.13}$$

Since the "constant term", $a_n\beta'-(a_n'+n\alpha'a_n)\beta$, of the numerator of the right hand side rational function of e^{α} in (3.13) is not identically zero (Otherwise, it yields $e^{n\alpha}=\frac{\beta}{a_n}$, and by (3.12), we derive that $T(r,f)=T(r,e^{n\alpha})=T\left(r,\frac{\beta}{a_n}\right)=S(r,f)$, a contradiction.), (3.13) implies that $T(r,f')=nT(r,e^{\alpha})$. However, combining (3.3) and (3.12) reads

$$f'(z)=\frac{Q^{**}_{n-1}\left[e^{\alpha(z)}\right]-a(z)a_n(z)e^{(n-1)\alpha(z)}}{a_n(z)e^{(n-1)\alpha(z)}}, \tag{3.14}$$

which further implies that $T(r,f') = (n-1)T(r,e^{\alpha})$, and hence $T(r,e^{\alpha}) = S(r,f)$. A contradiction follows immediately, since, by (3.12) and the foregoing discussions, we would have $T(r,f) = nT(r,e^{\alpha}) = S(r,f)$.

4 A related problem

In [12, Lemma 3], the first author showed that any entire solution of the following complex linear non-homogenous differential equation

$$f^{(k)}(z) - e^{Q(z)}f(z) = P(z) \tag{4.1}$$

would be of infinite order, where $k(\geq 1)$ is an integer, and $P(\not\equiv 0)$, Q are two polynomials with Q non-constant.

For the following complex linear differential equation

$$f^{(n)}(z) + p_{n-1}(z)f^{(n-1)}(z) + \cdots + p_0(z)f_0(z) = H(z), \tag{4.2}$$

a lot of beautiful results on the growth estimate of its entire solutions could be found in [9-11, 16, 21], where $p_{n-1}, \ldots, p_1, p_0(\not\equiv 0)$ are polynomials and H is an entire function. (If $H \not\equiv 0$, then equation (4.2) is non-homogenous.) Roughly speaking, except for polynomial solutions, all the other entire solutions to equation (4.2) would be of order no less than one or even of infinite order. (See also the fine paper [7] for basic tools.)

On the other hand, for the following complex linear differential equation

$$f^{(n)}(z) + A_{n-1}(z)f^{(n-1)}(z) + \cdots + A_0(z)f_0(z) = H(z), \tag{4.3}$$

G.G. Gundersen and E.M. Steinbart (cf. [8]), and independently, S. Hellerstein, J. Miles and J. Rossi (cf. [14]) obtained the result below.

Theorem 4.1. *If there exists one and only one index $k \in \{0, 1, \ldots, n-1\}$ such that*

$$\max\left\{\rho(H), \max_{0\leq j\leq n-1,\ j\neq k} \rho(A_j)\right\} < \rho(A_k) \leq \frac{1}{2}, \tag{4.4}$$

then any non-polynomial entire solution to equation (4.3) is of infinite order.

The above results make us wonder whether equation (1.2) admits non-polynomial finite order entire solutions (Notice that its coefficients are of order less than one.), which were also asked by S. Hellerstein, J. Miles and J. Rossi (cf. [14]). If the answer is negative, then the only possible entire functions satisfying the hypothesis in our main result would be of exponential type, and the shared small function would be rational.

On the other hand, it is natural to ask whether all the entire solutions to equation (4.1) are of infinite order, provided that $f^{(k)}$ is replaced by a linear differential polynomial in f'. Also, is there any relationship between the hyper-order of the entire solutions to equation (4.1) and the order of e^Q? A counterexample below destroys our dream for the first question, yet fortunately, we could give a definite answer to the second one.

Example 4.2. Let $f := e^{-z}(z^2 - 2z + 1)$ be an entire function of finite order. However, we have $f^{(4)} + 3f''' + 3f'' + f' + e^z f = z^2 - 2z + 1$.

Theorem 4.3. *The hyper-order of any entire solution $f(z)$ to equation (4.1) is equal to the order of $e^{Q(z)}$, i.e., the degree of $Q(z)$, where the hyper-order of $f(z)$ is defined as $\rho_2(f) := \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}$.*

Proof. Without loss of generality, we suppose that $k=1$. The classical *Wiman-Valiron estimate* (cf. [16]) states that, for any transcendental entire function f, and for any sufficiently small $\tau > 0$, if we let z with $|z| = r$ be such that $|f(z)| > M(r,f)\big(\nu(r,f)\big)^{-\tau}$, then there exists a set $\mathbb{E} \subset \mathbb{R}(1,\infty)$ with finite logarithmic measure such that

$$\frac{f^{(s)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^s (1+o(1)) \tag{4.5}$$

holds for all $s \in \mathbb{N}$ and all $r \notin \mathbb{E}\cup\mathbb{R}[0,1]$, where $M(r,f)$ denotes the *maximal module* of f at $|z| = r$, and $\nu(r,f)$ denotes the *central index* of f at $|z| = r$.

Suppose $\deg P = l$. A routine calculation of equation (4.1) leads to

$$\frac{f^{(l+2)}(z)}{f(z)} = e^{Q(z)}\left(\frac{f^{(l+1)}(z)}{f(z)} + Q_l(z)\frac{f^{(l)}(z)}{f(z)} + \cdots + Q_0(z)\right), \tag{4.6}$$

where Q_j's are polynomials written in terms of Q, its derivatives and their combinations for $j = 0, 1, \ldots, l$.

Suppose $\deg Q = \mu$. Noting that e^Q is entire, we have $\mu = \limsup_{r\to+\infty} \frac{\log\log M\left(r,e^Q\right)}{\log r}$ (cf. [16]). So, for any sufficiently small $\varepsilon > 0$ and some properly chosen $\gamma > 1$, we have

$$\left|Q^*(z)e^{Q(z)}\right|_{|z|=r} \le M\left(r, Q^* e^Q\right) \le e^{\gamma r^{\mu+\varepsilon}}, \tag{4.7}$$

where Q^* is any non-zero polynomial.

Now, since f is of infinite order, applying (4.5) and (4.7) to (4.6) yields

$$\left(\frac{\nu(r,f)}{r}\right)^{l+2}(1+o(1)) \le e^{\gamma r^{\mu+\varepsilon}}\left(\frac{\nu(r,f)}{r}\right)^{l+1}(1+o(1)), \tag{4.8}$$

for $|z| = r(\notin \mathbb{E}\cup\mathbb{R}[0,1])$ with r sufficiently large. Combining (4.8) and the defining formula $\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log\nu(r,f)}{\log r}$ leads to the inequality below

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log\nu(r,f)}{\log r} \le \limsup_{r\to+\infty} \frac{\log\log e^{\gamma r^{\mu+2\varepsilon}}}{\log r} = \mu + 2\varepsilon, \tag{4.9}$$

which further implies that $\rho_2(f) \le \mu$ since ε is arbitrary.

On the other hand, we rewrite (4.6) as

$$e^{Q(z)} = \frac{\frac{f^{(l+2)}(z)}{f(z)}}{\frac{f^{(l+1)}(z)}{f(z)} + Q_l(z)\frac{f^{(l)}(z)}{f(z)} + \cdots + Q_0(z)}. \tag{4.10}$$

Noting that $\rho(f) = \limsup_{r\to+\infty} \frac{\log\nu(r,f)}{\log r} = +\infty$, we have, for sufficiently large r outside the set $\mathbb{E}\cup\mathbb{R}[0,1]$ with $|z| = r$,

$$\left|e^{Q(z)}\right|_{|z|=r} \le \frac{\left(\frac{\nu(r,f)}{r}\right)^{l+2}(1+o(1))}{\left(\frac{\nu(r,f)}{r}\right)^{l+1}(1-o(1))}, \tag{4.11}$$

which implies that

$$T\left(r,e^{Q}\right)=\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}\left|e^{Q(re^{i\theta})}\right|d\theta+O(1)\leq\log^{+}\nu(r,f)+O(\log r). \tag{4.12}$$

Therefore, (4.12) leads to the inequality below

$$\mu=\limsup_{r\to+\infty}\frac{\log T(r,e^{Q})}{\log r}\leq\limsup_{r\to+\infty}\frac{\log\log\nu(r,f)+2\log\log r}{\log r}=\rho_2(f), \tag{4.13}$$

which together with the conclusion of (4.9) yields

$$\rho(e^{Q})=\mu=\rho_2(f) \tag{4.14}$$

and terminates our proof. □

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