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A central limit theorem for normalized functions of the increments of a diffusion process, in the presence of round-off errors

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Let X be a one-dimensional diffusion process. For each $n \geq 1$ we have a round-off level $\alpha_n > 0$ and we consider the rounded-off value $X_t^{(\alpha_n)} = \alpha_n[X_t/\alpha_n]$. We are interested in the asymptotic behaviour of the processes $U(n, \varphi)_t = \frac{1}{2}\Sigma_{1 \leq i \leq [nt]} \varphi(X^{(\alpha_n)}_{(i-1)/n}, \sqrt{n}(X^{(\alpha_n)}_{i/n} - X^{(\alpha_n)}_{(t-1)/n})$ as n goes to $+\infty$: under suitable assumptions on φ, and when the sequence $\alpha_n\sqrt{n}$ goes to a limit $\beta \in [0, \infty)$, we prove the convergence of $U(n, \varphi)$ to a limiting process in probability (for the local uniform topology), and an associated central limit theorem. This is motivated mainly by statistical problems in which one wishes to estimate a parameter occurring in the diffusion coefficient, when the diffusion process is observed at times i/n and is subject to rounding off at some level α_n which is 'small' but not 'very small'.

Keywords: functional limit theorems; round-off errors; stochastic differential equations

1. Introduction

Let us consider a one-dimensional diffusion process X, solution to the equation

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t, \tag{1.1}$$

where W is a standard Brownian motion, and a and σ are smooth enough functions on $\mathbb{R}$. The behaviour of functionals of the form

$$\frac{1}{n}\sum_{i=1}^{[nt]} \varphi(X_{(i-1)/n}, \sqrt{n}(X_{i/n} - X_{(i-1)/n})) \tag{1.2}$$

as $n \to \infty$ is known (see, for example, Jacod 1993), and it is crucial for instance in estimation problems related to diffusion models when one observes the process X at times i/n, $i \geq 1$.

Now, in practical situations not only do we observe the process at 'discrete' times, but also each observation is subject to measurement errors, one of these being the round-off effect: if $\alpha > 0$ is the accuracy of our measurement, we replace the true value X_t by $k\alpha$ when

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$k\alpha \leq X_t < (k+1)\alpha$ with $k \in \mathbb{Z}$. The object of this paper is to study the limiting behaviour of functionals like (1.2) when $X_{i/n}$ is substituted with its rounded-off value.

More precisely, we are given a sequence α_n of positive numbers, where α_n represents the accuracy of measurement when the discretization times are i/n. With each real x we associate its integer part $[x]$ and fractional part $\{x\} = x - [x]$, and for every real x we denote by $x^{(\alpha_n)} = \alpha_n[x/\alpha_n]$ its rounded-off value at level α_n. Instead of (1.2) we consider processes such as

$$U(n, \varphi)_t = \frac{1}{n}\sum_{i=1}^{[nt]} \varphi(X^{(\alpha_n)}_{(i-1)/n}, \sqrt{n}(X^{(\alpha_n)}_{i/n} - X^{(\alpha_n)}_{(i-1)/n})), \tag{1.3}$$

perhaps with φ replaced by a well-behaved sequence φ_n of functions.

In fact, the asymptotic behaviour of (1.3) and of other similar processes will be deduced from the behaviour of the following:

$$V(n,f_n)_t = \frac{1}{n}\sum_{i=1}^{[nt]} f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}(X_{i/n} - X_{(i-1)/n})), \tag{1.4}$$

where f_n are functions on $\mathbb{R} \times [0,1] \times \mathbb{R}$. The interest of (1.4) is that it simultaneously encompasses (1.2) and (1.3), and gives additional results for functions of the fractional parts $\{X_{i/n}/\alpha_n\}$ which may have independent interest (see Section 3).

Throughout this paper we will assume that $\beta_n = \alpha_n\sqrt{n}$ converges to a limit β in $[0, \infty)$.

In Section 2 we state the main results about processes $V(n,f_n)$. They are twofold: first convergence in probability; then an associated central limit theorem for the normalized and compensated processes. In Section 3 we deduce from this the behaviour of processes like (1.3).

In Section 4 we give an example of a statistical application: the process under observation is (1.1) with $a(x) = 0$, $\sigma(x) = \sigma$ and $X_0 = 0$, that is $X_t = \sigma W_t$, and we wish to estimate σ^2 from the observation of the rounded-off values $X^{(\alpha_n)}_{i/n}$ for $i = 1, \ldots, n$. This simple example allows us to exhibit the main features of estimation in the presence of round-off. The statements of Section 4 can be read without the whole arsenal of notation of Sections 2 and 3, and corresponding results concerning general diffusion processes will be developed elsewhere.

The rest of the paper is organized as follows. In Section 5 we prove some (more or less well-known) results about the semigroups of the process X. In Section 6 we introduce the fundamental tool, which is that if a real-valued random variable Y admits a smooth density, then for $\rho > 0$ the variable $\{Y/\rho\}$ is 'almost' independent of Y and uniformly distributed on $[0, 1)$ (the 'almost' being controlled by powers of ρ): this is related to results due to Kosulajeff (1937) and Tukey (1939). In Section 7 we study the functions which occur in the limits of our processes. In Section 8 we introduce a fundamental martingale. This martingale is constructed, approximately, as the martingale used in the proof of the central limit theorem for a triangular array of stationary mixing sequences of random variables, the 'stationary sequence' here being the fractional parts $\{X_{i/n}/\alpha_n\}$. Finally, Section 9 is devoted to proving the main theorems.

The assumption that β_n goes to a finite limit is restrictive, although for statistical purposes it should be a natural assumption.

If $\beta_n \to \infty$ and still $\alpha_n \to 0$, we have seen in Jacod (1996) for the Brownian motion case (i.e. $a = 0$, $\sigma = 1$) that $U(n, \varphi)_t/\beta_n$ converges in probability to $t\sqrt{2/\pi}$ for the function $\varphi(x, y) = y^2$. More generally if φ_n has the form $\varphi_n(x, y) = \psi_n(x)|y|^p$ it is possible to prove convergence in probability of $\beta_n^{1-p} U(n, \varphi_n)$, as well as a corresponding central limit theorem (these results will be developed elsewhere): this implies that for arbitrary functions φ_n the normalizing factors should depend on φ_n in a rather complicated way.

When α_n goes to a limit $\alpha > 0$ (for example, if $\alpha_n = \alpha > 0$ for all n), the situation is quite different: again in the Brownian case and if $\varphi(x, y) = y^2$, then $U(n, \varphi)/\sqrt{n}$ converges in probability to a multiple of the sum $\sum_{k\in\mathbb{Z}} L^{k\alpha}$, where L^a is the local time of X at level a. Presumably a similar result holds here, but the limit is random here and a central limit theorem, if it holds at all, would be of a different nature.

2. Statement of the main results

We first present our assumptions. First, for the process X, we assume the following:

Hypothesis H. *The functions a and σ are of class C^5 and $\sigma > 0$ identically, and for each starting point the process X is non-explosive.*

We denote by P_x the law of the process X starting at $X_0 = x$, on the canonical space $\Omega = C(\mathbb{R}_+, \mathbb{R})$ endowed with the canonical filtration $(\mathscr{F}_t)_{t\geq 0}$.

Next, let $f_n : \mathbb{R} \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a sequence of functions satisfying the following for $r = 1$ or $r = 2$:

Hypothesis K_r. *The functions f_n are C^r in the first variable, and for all $q > 0$ there are constants C_q, r_q such that, for $0 \leq i \leq r, n \geq 1$:*

$$\left|\frac{\partial^i}{\partial x^i} f_n(x, u, y)\right| \leq C_q(1 + |y|^{r_q}) \qquad \text{for } |x| \leq q. \tag{2.1}$$

Furthermore, there is a function $f : \mathbb{R} \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f_n(x, u, y)$ converges $\mathrm{d}u \otimes \mathrm{d}y$*-almost everywhere to $f(x, u, y)$.*

Recall that $\beta_n = \alpha_n\sqrt{n} \to \beta \in [0, \infty)$, and $V(n, f_n)$ is given by (1.4).

For the first theorem, we need some notation. Denote by h_s the density of the normal law $\mathcal{N}(0, s^2)$, and $h = h_1$. For any function f on $\mathbb{R} \times [0, 1] \times \mathbb{R}$ satisfying (2.1) for $i = 0$, we set (σ is as in (1.1)):

$$mf(x, u) = \int h_{\sigma(x)}(y) f(x, u, y)\mathrm{d}y, \qquad Mf(x) = \int_0^1 mf(x, u)\mathrm{d}u. \tag{2.2}$$

Note that Mf is locally bounded.

Theorem 2.1. *Under the hypotheses H and K_1, the processes $V(n, f_n)$ converge in P_x-probability, locally uniformly in time, to the process $\int_0^t Mf(X_s)\mathrm{d}s$.*

We next give a 'central limit theorem' associated with the previous result. Here again we need to introduce a number of functions. Let W be a standard Brownian motion on a space

$(\Omega, \mathscr{G}, P)$, generating the filtration $(\mathscr{G}_t)_{t \geq 0}$. If ψ is a function of polynomial growth on $[0,1] \times \mathbb{R}$, for all $\sigma > 0$, $\rho > 0$, $u \in [0,1]$ we set (for $i \geq 1$):

$$m_\sigma \psi(u) = \mathrm{E}(\psi(u, \sigma W_1)), \qquad M_\sigma \psi = \int_0^1 m_\sigma \psi(u) \mathrm{d}u, \tag{2.3}$$

$$\eta_i\, \psi(\sigma, \rho, u) = \psi(\{u + \sigma W_{i-1}/\rho\}, \sigma(W_i - W_{i-1})) - M_\sigma \psi, \tag{2.4}$$

$$\ell_i\, \psi(\sigma, \rho, u) = \mathrm{E}(\eta_i\, \psi(\sigma, \rho, u)). \tag{2.5}$$

We will prove later (see Section 7) that the series $L\psi = \sum_{i \geq 1} \ell_i\, \psi$ is absolutely convergent, and we can introduce square-integrable random variables by writing (note that $\eta_1 \psi(\sigma, \rho, u)$ does not depend on ρ):

$$\chi\psi(\sigma, \rho, u) = \eta_1 \psi(\sigma, u) + L\psi(\sigma, \rho, \{u + \sigma W_1/\rho\}) - L\psi(\sigma, \rho, u). \tag{2.6}$$

Finally, if φ is another function of the same type as ψ, we set

$$\delta_{\varphi,\psi}(\sigma, \rho, u) = \mathrm{E}(\chi\varphi(\sigma, \rho, u)\chi\psi(\sigma, \rho, u)), \qquad \Delta_{\varphi,\psi}(\sigma, \rho) = \int_0^1 \delta_{\varphi,\psi}(\sigma, \rho, u) \mathrm{d}u. \tag{2.7}$$

Equations (2.4)–(2.7) make no sense when $\rho = 0$. However, we set, for $\rho = 0$:

$$\Delta_{\varphi,\psi}(\sigma, 0) = M_\sigma(\varphi\psi) - M_\sigma \varphi M_\sigma \psi, \tag{2.8}$$

and will prove (again in Section 7) that $\Delta_{\varphi,\psi}$ is continuous on $(0, \infty) \times [0, \infty)$, while for all $\rho \geq 0$:

$$\Delta_{\psi,\psi}(\sigma, \rho) \geq [M_\sigma(\psi\varphi_\sigma)]^2, \tag{2.9}$$

where $\varphi_\sigma(u, y) = y/\sigma$.

The connection between (2.2) and (2.3) is as follows, where $f_x(u, y) = f(x, u, y)$:

$$mf(x, u) = m_{\sigma(x)} f_x(u), \qquad Mf(x) = M_{\sigma(x)} f_x, \tag{2.10}$$

and we introduce in a similar fashion (with $\varphi_\sigma(u, y) = y/\sigma$ again):

$$\Delta(f, g)(x, \rho) = \Delta_{f_x, g_x}(\sigma(x), \rho), \qquad Rf(x) = M_{\sigma(x)}(f_x \varphi_{\sigma(x)}). \tag{2.11}$$

For further reference, we also set:

$$\tilde{f}(x, u, y) = f(x, u, y)\left(y\left(\frac{a(x)}{\sigma(x)^2} - \frac{3\sigma'(x)}{2\sigma(x)}\right) + y^3 \frac{\sigma'(x)}{2\sigma(x)^3}\right). \tag{2.12}$$

where σ' is the first derivative of σ.

After this long list of notation, we also recall that if V_n is a sequence of random variables on $(\Omega, \mathscr{F}, P_x)$, taking values in a Polish space E, we say that V_n converges *stably in law* to a limit V if V is an E-valued random variable defined on an extension $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}_x)$ of the space $(\Omega, \mathscr{F}, P_x)$ and if $\mathrm{E}_x(Yf(V_n)) \to \bar{\mathrm{E}}_x(Yf(V))$ for every bounded random variable Y on $(\Omega, \mathscr{F}, P_x)$ and every bounded continuous function f on E (see Renyi 1963; Aldous and Eagleson 1978; or Jacod and Shiryaev 1987). This is obviously a (slightly) stronger mode of convergence than convergence in law.

We will apply this to processes, so E is the Skorokhod space $\mathbb{D}(\mathbb{R}_+)$. The extension $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}_x)$ is such that it accomodates another standard Brownian motion B independent of W, and we consider the process (recall that $\Delta(f,f)(x,\rho) \geq Rf(x)^2$ by (2.9) and (2.11)):

$$B'_t = \int_0^t (\Delta(f,f)(X_s,\beta) - Rf(X_s)^2)^{1/2} \mathrm{d}B_s. \tag{2.13}$$

Theorem 2.2. *Assume that the hypotheses H and K_2 hold. The processes* $\sqrt{n}(V(n,f_n)_t - \int_0^t Mf_n(X_s)\mathrm{d}s)$ *and* $\sqrt{n}(V(n,f_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]} Mf_n(X_{(i-1)/n}))$ *converge stably in law to the following process* (*with* B' *and* $\tilde{f}$ *given by* (2.13) *and* (2.12)):

$$\int_0^t M\tilde{f}(X_s)\mathrm{d}s + \int_0^t Rf(X_s)\mathrm{d}W_s + B'_t. \tag{2.14}$$

Corollary 2.3. *Assume that the hypotheses H and K_2 hold, and associate $\tilde{f}_n$ with f_n by* (2.12). The two sequences of processes

$$\sqrt{n}\left(V(n,f_n)_t - \int_0^t Mf_n(X_s)\mathrm{d}s - \frac{1}{\sqrt{n}}\int_0^t M\tilde{f}_n(X_s)\mathrm{d}s\right),$$

$$\sqrt{n}\left(V(n,f_n) - \frac{1}{n}\sum_{i=1}^{[nt]} Mf_n(X_{(i-1)/n}) - n^{-3/2}\sum_{i=1}^{[nt]} M\tilde{f}_n(X_{(i-1)/n})\right),$$

converge stably in law to the process $\int_0^t Rf(X_s)\mathrm{d}W_s + B'_t$.

Remark 2.1. Another way of characterizing the process B' is as follows: it is a process on the extension $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}_x)$ such that, conditionally on the σ-field $\mathscr{F}$, it is a continuous Gaussian martingale null at $t = 0$, with (deterministic) bracket

$$\langle B', B' \rangle_t = \int_0^t (\Delta(f,f)(X_s,\beta) - Rf(X_s)^2)\mathrm{d}s. \tag{2.15}$$

Remark 2.2. There is, of course, a version of these results for d-dimensional functions $f_n = (f_n^i)_{1 \leq i \leq d}$ all of whose components satisfy hypothesis K_2. Then the processes $V(n,f_n)$ and functions $M\tilde{f}$ and Rf are d-dimensional as well, as the results are exactly the same as in Theorem 2.2 and Corollary 2.3, provided we describe the d-dimensional process $B' = (B'^i)_{1 \leq i \leq d}$, conditionally on $\mathscr{F}$, as a continuous Gaussian martingale null at $t = 0$, with the following brackets:

$$\langle B'^i, B'^j \rangle_t = \int_0^r (\Delta(f^i,f^j)(X_s,\beta) - Rf^i(X_s)Rf^j(X_s))\mathrm{d}s. \tag{2.16}$$

The proof is exactly the same as for the one-dimensional case. Another description of B' as the stochastic integral with respect to a d-dimensional Brownian motion independent of W is, of course, possible, and involves a square root of the symmetric non-negative matrices $(\Delta(f^i,f^j)(x,\beta) - Rf^i(x)Rf^j(x))_{1 \leq i,\, j \leq d}$.

3. Some applications

We consider here the processes $U(n, \varphi)$ of (1.3). More precisely, let φ_n be a sequence of functions on $\mathbb{R}^2$, satisfying the following assumption (for $r = 1$ or $r = 2$):

Hypothesis L_r**.** *The functions φ_n are C^r in the first variable, continuous in the second variable, and for all $q > 0$ there are constants C_q, r_q such that, for $0 \le i \le r$, $n \ge 1$:*

$$\left|\frac{\partial^i}{\partial x^i}\varphi_n(x, y)\right| \le C_q(1 + |y|^{r_q}) \qquad \text{for } |x| \le q. \tag{3.1}$$

Furthermore, φ_n converges pointwise to a function φ.

Since $X_t^{(\alpha_n)} = X_t - \alpha_n\{X_t/\alpha_n\}$, we have $U(n, \varphi_n) = V(n, f_n)$, where

$$f_n(x, u, y) = \varphi_n(x - \alpha_n u, \beta_n[u + y/\beta_n]). \tag{3.2}$$

Furthermore, we have the following lemma.

Lemma 3.1. *If $\beta_n \to \beta$ the hypothesis L_r implies that the sequence (f_n) defined by* (3.2) *satisfies K_r, with the limiting function f given by*

$$f(x, u, y) = \begin{cases} \varphi(x, \beta[u + y/\beta]) & \text{if } \beta > 0 \\ \varphi(x, y) & \text{if } \beta = 0. \end{cases} \tag{3.3}$$

Proof. Property (2.1) is obvious. Recall that $\alpha_n \to 0$, while $\beta_n[u + y/\beta_n]$ converges to y if $\beta = 0$, and to $\beta[u + y/\beta]$ for $\mathrm{d}u \otimes \mathrm{d}y$ – almost all (u, y) if $\beta > 0$. Hence the continuity of φ_n yields $\varphi_n(x, \beta_n[u + y/\beta_n]) - \varphi_n(x, y) \to 0$ if $\beta = 0$, and $\varphi_n(x - \alpha_n u, \beta_n[u + y/\beta_n]) - \varphi_n(x, \beta[u + y/\beta]) \to 0$ if $\beta > 0$. Since $\varphi_n \to \varphi$ we deduce that $f_n(x, .) \to f(x, .)\mathrm{d}u \otimes \mathrm{d}y$ – almost everywhere. □

In order to translate the results of Section 2 into the present setting, we introduce some more notation. For any function φ on $\mathbb{R}^2$ satisfying (3.1) for $i = 0$, set

$$\Gamma\varphi(x, \rho) = \begin{cases} \int_0^1 \mathrm{d}u \int h(y)\varphi(x, \rho[u + y\sigma(x)/\rho])\mathrm{d}y & \text{if } \rho > 0 \\ \int h(y)\varphi(x, \sigma(x)y)\mathrm{d}y & \text{if } \rho = 0. \end{cases} \tag{3.4}$$

Theorem 3.1. *Under the hypotheses H and L_1 the processes $U(n, \varphi_n)$ converge in P_x-probability, locally uniformly in time, to the process $\int_0^t \Gamma\varphi(X_s, \beta)\mathrm{d}s$.*

Proof. It suffices to observe that $\Gamma\varphi(x, \beta) = Mf(x)$ with f as in (3.3). □

In a similar way to (3.4), we set, for $\rho > 0$:

$$\tilde{\Gamma}\varphi(x, \rho) = \int_0^1 u\mathrm{d}u \int h(y)\varphi(x, \rho[u + y\sigma(x)/\rho])\mathrm{d}y. \tag{3.5}$$

For all φ_n we also write $\varphi_n'(x, y) = \partial\varphi_n(x, y)/\partial x$.

Theorem 3.2. *Assume that the hypotheses H and L_2 hold. The processes*

$$\sqrt{n}\left(U(n,\varphi_n)_t - \int_0^t \Gamma\varphi_n(X_s,\beta_n)\mathrm{d}s + \alpha_n \int_0^t \tilde{\Gamma}\varphi_n'(X_s,\beta_n)\mathrm{d}s\right), \tag{3.6}$$

$$\sqrt{n}\left(U(n,\varphi_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]} \Gamma\varphi_n(X_{(i-1)/n},\beta_n) + \frac{\alpha_n}{n}\sum_{i=1}^{[nt]} \tilde{\Gamma}\varphi_n'(X_{(i-1)/n},\beta_n)\right), \tag{3.7}$$

converge stably in law to the process (2.14), *with f given by* (3.3).

Proof. Set $\gamma_n(x) = Mf_n(x) - \Gamma\varphi_n(x,\beta_n) + \alpha_n\tilde{\Gamma}\varphi_n'(x)$. The processes (3.6) and (3.7) are respectively equal to $\sqrt{n}(V(n,f_n)_t - \int_0^t Mf_n(X_s)\mathrm{d}s) + \sqrt{n}\int_0^t \gamma_n(X_s)\mathrm{d}s$ and $\sqrt{n}(V(n,f_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]} Mf_n(X_{(i-1)/n})) + n^{-1/2}\sum_{i=1}^{[nt]}\gamma_n(X_{(i-1)/n})$. Therefore, the result will follow from Theorem 2.2 if we prove that

$$\sup_{x:|x|\leq A} \sqrt{n}|\gamma_n(x)| \to 0 \qquad \text{for all } A > 0. \tag{3.8}$$

We have

$$\gamma_n(x) = \int_0^1 \mathrm{d}u \int h(y)(\varphi_n(x-\alpha_n u, \beta_n[u+\sigma(x)y/\beta_n]) - \varphi_n(x,\beta_n[u+\sigma(x)y/\beta_n]) + \alpha_n u\varphi_n'(x,\beta_n[u+\sigma(x)y/\beta_n]))\mathrm{d}y.$$

Since $\alpha_n^2\sqrt{n} \to 0$, (3.8) is deduced from hypothesis L_2. □

Remark 3.1. If $\beta = 0$, then $\alpha_n\sqrt{n} \to 0$, while $\tilde{\Gamma}\varphi_n'(x,\beta_n)$ is locally bounded in x, uniformly in n: therefore we can replace (3.6) and (3.7) by the processes

$$\sqrt{n}\left(U(n,\varphi_n)_t - \int_0^t \Gamma\varphi_n(X_s,\beta_n)\mathrm{d}s\right) \qquad \text{and} \qquad \sqrt{n}\left(U(n,\varphi_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\varphi_n(X_{(i-1)/n}\beta_n)\right).$$

Very often in applications, the functions φ_n will be even in the second variable. The results then take a simpler form, as follows.

Corollary 3.3. *Assume that the hypotheses H and L_2 hold, and also that $\varphi(x,y) = \varphi(x,-y)$ identically. The processes* (3.6) *and* (3.7) *converge stably in law to the process* $\int_0^t \Delta(f,f)(X_s,\beta)^{1/2}\mathrm{d}B_s$, *where f is given by* (3.3) *and B is a standard Brownian motion independent of W.*

Proof. It suffices to prove that $M\tilde{f}(x) = Rf(x) = 0$. In view of (2.11) and (2.12), it is enough to prove that $Mg(x) = 0$ if $g(x,u,y) = f(x,u,y)k(x,y)$ where $k(x,y) = A(x)y$ or $k(x,y) = A(x)y^3$ for an arbitrary function A. But (3.3) and the assumption of φ yield that $g(x,u,y) = -g(x,1-u,-y)$ for $\mathrm{d}u\otimes\mathrm{d}y$-almost all (u,y). Since the measure $\mathrm{d}u\otimes h_{\sigma(x)}(y)\mathrm{d}y$ is invariant by the map $(u,y)\to(1-u,-y)$, we deduce $Mg(x) = 0$ from (2.2). □

The processes (3.6) and (3.7) are not fit for statistical applications, since they involve not only the 'observed' values $X^{(\alpha_n)}_{i/n}$, but also the 'non-observed' path $s \to X_s$ in the case of (3.6), or the non-observed values $X_{i/n}$ in the case of (3.7). To circumvent this problem, we can state the following result, the proof of which is postponed until Section 9.

Theorem 3.4. *Assume that the hypotheses H and L_2 hold.*

(a) *The processes*

$$\sqrt{n}\left(U(n,\varphi_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\varphi_n\left(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \beta_n\right) + \frac{\alpha_n}{n}\sum_{i=1}^{[nt]}\tilde{\Gamma}\varphi'_n\left(X^{(\alpha_n)}_{(i-1)/n}, \beta_n\right)\right) \tag{3.9}$$

converge stably in law to the process (2.14), *with f given by* (3.3).

(b) *If, further, $\varphi(x,y) = \varphi(x,-y)$ identically, then the processes*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\left(\varphi_n\left(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \sqrt{n}\left(X^{(\alpha_n)}_{i/n} - X^{(\alpha_n)}_{(i-1)/n}\right)\right) - \Gamma\varphi_n\left(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \beta_n\right)\right) \tag{3.10}$$

converge stably in law to the process $\int_0^t \Delta(f,f)(X_s,\beta)^{1/2}\mathrm{d}B_s$, where f is given by (3.3) *and B is a standard Brownian motion independent of W.*

Remark 3.2. As for Theorem 3.2, if $\beta = 0$ we can replace the process (3.9) by $\sqrt{n}(U(n,\varphi_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\varphi_n(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \beta_n))$, and even by $\sqrt{n}(U(n,\varphi_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\varphi_n(X^{(\alpha_n)}_{(i-1)/n}, \beta_n))$ because $|\Gamma\varphi_n(x+\alpha_n/2,\beta_n) - \Gamma\varphi_n(x,\beta_n)| \leq g(x)\alpha_n \leq g(x)\beta_n/\sqrt{n}$ for some locally bounded function g.

Remark 3.3. Other versions of (3.9) are possible: for example, we can replace $\Gamma\varphi_n(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \beta_n)$ by $\Gamma_n\varphi_n(X^{(\alpha_n)}_{(i-1)/n}, \beta_n)$, where

$$\Gamma_n\varphi_n(x) = \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \int h(y)\varphi_n(x + \alpha_n v, \beta_n[u + y\sigma(x)/\beta_n])\mathrm{d}y.$$

We can also replace $\tilde{\Gamma}\varphi'_n(X^{(\alpha_n)}_{(i-1)/n}, \beta_n)$ by $\tilde{\Gamma}\varphi'_n(X^{(\alpha_n)}_{(i-1)/n} + \frac{\alpha_n}{2}, \beta_n)$.

Remark 3.4. As in Corollary 3.3, if φ is even in the second variable, the limit in Theorem 3.4 is $\int_0^t \Delta(f,f)(X_s,\beta)^{1/2}\mathrm{d}B_s$.

Remark 3.5. As in Section 2, these results admit a multidimensional version, when each φ_n takes values in $\mathbb{R}^d$. We leave the details to the reader.

Finally we give some very simple applications to the processes

$$U^n_t(p) = \frac{1}{n}\sum_{i=1}^{[nt]}\{X_{i/n}/\alpha_n\}^p. \tag{3.11}$$

where $p \in \mathbb{R}_+$.

Theorem 3.5. *Assume that the hypothesis H holds. Then the processes $U^n_t(p)$ converge locally uniformly in time, in $\mathbb{L}^q(P_x)$ for all q, to the function $t/(p+1)$. Furthermore, the processes*

$\sqrt{n}(U_t^n(p) - t/(p+1))$ converge stably in law to $\int_0^t \Delta(f,f)(X_s,\beta)^{1/2}\mathrm{d}B_s$, where $f(x,u,y) = u^p$ and B is a standard Brownian motion independent of W.

Note that if $\beta = 0$, then $\Delta(f,f)(x,0) = 1/(p^2+1) - (1/(p+1))^2$, so the limit above is again a homogeneous Brownian motion, independent of W. If $\beta > 0$, then $\Delta(f,f)(x,\beta)$ depends on x and the limit in not independent of W.

Proof. We only have to notice that $U_t^n(p) = V(n,f)_t + \{X_{[nt]/n}/\alpha_n\}^p/n$, where f is as above: we have the hypothesis K_2 for $f_n = f$, and we can apply Theorems 2.1 and 2.2, and check that $Rf(x) = M\tilde{f}(x) = 0$ and that $Mf(x) = 1/(p+1)$. □

4. A simple statistical application

In this section we consider the following statistical problem: the process X is $X = \sigma W$, where W is a standard Brownian motion, and $\sigma > 0$ is unknown. We wish to estimate $\vartheta = \sigma^2$, from the observation of $X_{i/n}^{(\alpha_n)}$ for $i = 1, \ldots, n$. The estimation will be based on the discretized quadratic variation, calculated from these rounded-off values, i.e. the variables

$$\tilde{V}^n = \sum_{i=1}^n \left(X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)} \right)^2, \tag{4.1}$$

since it is well known that without round-off error (i.e. $\alpha_n = 0$), $\tilde{V}^n$ is (in all possible senses) the best estimator of ϑ, and that $\sqrt{n}(\tilde{V}^n - \vartheta)$ converges in law to $\mathcal{N}(0, 2\vartheta^2)$ if the true value of the parameter is ϑ.

First, the following result, easily deduced from Theorem 3.1, has already been proved in Jacod (1996). Below, P^ϑ denotes the law of X for the value ϑ of the parameter.

Theorem 4.1. *The variables $\tilde{V}^n$ converge in P^ϑ-probability to the number*

$$\gamma(\beta,\vartheta) = \begin{cases} \int_0^1 \mathrm{d}u \int h(y)\beta^2 \left[u + \dfrac{y\sqrt{\vartheta}}{\beta} \right]^2 \mathrm{d}y & \text{if } \beta > 0 \\ \vartheta & \text{if } \beta = 0. \end{cases} \tag{4.2}$$

Proof. Setting $\varphi(x,y) = y^2$, it is enough to observe first that $\tilde{V}^n = U(n,\varphi)$, and second that $\gamma(\beta,\vartheta) = \Gamma\varphi(x,\beta)$ with the notation of (3.4) since $\sigma(x) = \sqrt{\vartheta}$. □

It can be shown that $\gamma(\beta,\vartheta) > \vartheta$ if $\beta > 0$: hence the estimators $\tilde{V}^n$ are consistent if $\beta = 0$, but are *not* consistent if $\beta > 0$.

Furthermore, the function $\beta \to \gamma(\beta,\vartheta)$ is twice differentiable, and we can prove that $\partial\gamma(0,\vartheta)/\partial\beta = 0$ and $\partial^2\gamma(\beta,\vartheta)/\partial\beta^2 = \frac{1}{3}$. Then when $\beta = 0$, it follows from Theorem 3.2 (applied to $\varphi_n(x,y) = y^2$, so that $\Gamma\varphi_n'(x,\beta_n) = 0$) that $\sqrt{n}(\tilde{V}^n - \vartheta)$ converges in law to $\mathcal{N}(0, 2\vartheta^2)$ if $\sqrt{n}\beta_n^2 \to 0$, whereas it explodes when $\sqrt{n}\beta_n^2 \to \infty$, and it converges to a non-centred normal variable if $\sqrt{n}\beta_n^2$ converges to a limit in $(0,\infty)$: this means that, unless α_n goes to 0 very fast (i.e. $n^{3/4}\alpha_n \to 0$), then $\tilde{V}^n$ does not go to ϑ at the rate $1/\sqrt{n}$.

So there is a need for better estimators. In fact, the function $\vartheta \to \gamma(\beta, \vartheta)$ is an increasing bijection from $\mathbb{R}_+$ into $\mathbb{R}_+$, whose inverse is denoted by $\gamma^{-1}(\beta, \vartheta)$. We then have the following result.

Theorem 4.2. *The estimators $\hat{\vartheta}_n$, defined by $\hat{\vartheta}_n = \gamma^{-1}(\beta_n, \tilde{V}^n)$, are consistent, and $\sqrt{n}(\hat{\vartheta}_n - \vartheta)$ converges in law under P^ϑ to $\mathcal{N}(0, \Sigma(\beta, \vartheta))$, for some $\Sigma(\beta, \vartheta)$ satisfying $\Sigma(0, \vartheta) = 2\vartheta^2$.*

This implies that if $\beta = 0$, then the $\hat{\vartheta}_n$s are efficient since they achieve the same bound as if the true values $X_{i/n}$ were observed. When $\beta > 0$ they achieve at least the best rate $1/\sqrt{n}$ (we do not know whether they are efficient in this case, relative to the observed σ-fields).

Proof. The continuity of the function γ and Theorem 4.1 yield that $\gamma^{-1}(\beta_n, \tilde{V}^n) \to \gamma^{-1}(\beta, \gamma(\beta, \vartheta)) = \vartheta$ in P^ϑ-probability, hence the consistency.

Let $\Delta(\beta, \vartheta)$ be the quantity $\Delta(f, f)(x, \beta)$ with f associated with $\varphi(x, y) = y^2$ by (3.3) and $\sigma(x) = \sqrt{\vartheta}$ (clearly this does not depend on x).

By construction $\gamma(\beta_n, \hat{\vartheta}_n) = \tilde{V}^n$, so Corollary 3.3 yields that the variables $\sqrt{n}(\gamma(\beta_n, \hat{\vartheta}_n) - \gamma(\beta_n, \vartheta))$ converge in law to $\mathcal{N}(0, \Delta(\beta, \vartheta))$ (recall that here $\tilde{\Gamma}\varphi = 0$). Using the fact that $\vartheta \to \gamma(\beta, \vartheta)$ is continuously differentiable with a positive derivative, the consistency and Taylor's formula yield that $\sqrt{n}(\hat{\vartheta}_n - \vartheta)$ converges in law to $\mathcal{N}(0, \Delta(\beta, \vartheta)/(\partial\gamma(\beta, \vartheta)/\partial\vartheta)^2)$. Finally (4.2) gives $\partial\gamma(0, \vartheta)/\partial\vartheta = 1$, while (2.8) yields $\Delta(0, \vartheta) = 2\vartheta^2$, hence the final result. □

5. Preliminaries

The first aim of this section is to prove that we can replace the hypotheses H and K_r by the following:

Hypothesis H'. *a and σ are C_b^5 functions, and* $\inf_x \sigma(x) > 0$.

Hypothesis K_r'. *f and f_n are as in hypothesis K_r, and there are constants $p \in \mathbb{N}$, $K > 0$, such that for $0 \le i \le r$ and all n, x, y, u:*

$$\left| \frac{\partial^i}{\partial x^i} f_n(x, u, y) \right| + |f(x, u, y)| \le K(1 + |y|^p). \tag{5.1}$$

Assume that the hypotheses K and K_r hold, and suppose for a moment that the process X is defined on the canonical space of the Brownian motion W and starts at $X_0 = x_0$. Also, let $A = \sup \alpha_n$.

For all $q \ge |x_0|$ there are functions (a_q, σ_q) satisfying H', such that $a_q(x) = a(x)$ and $\sigma_q(x) = \sigma(x)$ if $|x| \le q + A$. There are also functions (f_n^q, f^q) satisfying K_r' and such that $f_n^q(x, u, y) = f_n(x, u, y)$ and $f^q(x, u, y) = f(x, u, y)$ if $|x|, |y| \le q + A$.

Denote by X^q the solution of (1.1) with the coefficients a_q, σ_q, and set $T_q = \inf(t : |X_t| \ge q + A)$. Obviously $X^q = X$ and $X^{q(\alpha_n)} = X^{(\alpha_n)}$ on $[0, T_q]$, so all processes associated with (X, f_n, f) or with (X^q, f_n^q, f^q) as in Section 2 coincide on $[0, T_q]$. Since $T_q \to \infty$ almost surely because X is non-explosive, it is clearly enough to prove all results for all triples (X^q, f_n^q, f^q), $q \ge |x_0|$.

Hence we can and will assume throughout the rest of this paper that H' and K'_r are in force.

Since all results are 'local' in time, we will also fix an arbitrary time interval $[0, T]$, with $T \in \mathbb{N}$. All constants below may depend on the coefficients (a, σ), on T, and on the constants (K, p) of (5.1), and also on the sequence (α_n), but they do not depend otherwise on f_n, f, or on n or ω.

Now we come back to the canonical space $(\Omega, \mathscr{F}, P_x)$ with the canonical process X. We construct a standard Brownian motion W, simultaneously for all measures P_x, by the formula

$$W_t = \int_0^t \frac{1}{\sigma(X_s)} \mathrm{d}X_s - \int_0^t \frac{a(X_s)}{\sigma(X_s)} \mathrm{d}s.$$

Let $(\mathscr{F}_t)_{t \geq 0}$ be the filtration generated by X, or equivalently by W.

Now we recall some results concerning the densities $(p_t(x, y) : x, y \in \mathbb{R})_{t>0}$ of the transition semigroup of the process X, under H'. Some of these are more or less well known, some seem to be new.

First, we recall an 'explicit' form of p_t in terms of a standard Brownian bridge denoted in this section by $B = (B_t)_{t \in [0,1]}$. Set

$$\begin{aligned} S(x) &= \int_0^x \frac{1}{\sigma(y)} \mathrm{d}y, \qquad b = a/\sigma^2 - \sigma'/2\sigma, \\ H(x) &= \int_0^x b(y) \mathrm{d}y, \qquad c = -\frac{1}{2}(\sigma^2 b^2 + \sigma\sigma' b + \sigma^2 b') \circ S^{-1}(x), \\ V_t(x, y) &= t \int_0^1 c((1-u)S(x) + uS(y) + \sqrt{t} B_u) \mathrm{d}u, \qquad r_t(x, y) = \mathrm{E}(\mathrm{e}^{V_t(x,y)}). \end{aligned}$$

Then (see, for example, Dacunha-Castelle and Florens-Zmirou 1986):

$$p_t(x, y) = \frac{1}{\sigma(y)\sqrt{2\pi t}} r_t(x, y) \exp\left\{ H(y) - H(x) - \frac{(S(y) - S(x))^2}{2t} \right\}. \tag{5.2}$$

We also set $q_t(x, y) = p_t(x, x + y)$, so that $y \to q_t(x, y)$ is the density of $X_t - X_0$ under P_x. Recall that h_s is the density of the law $\mathcal{N}(0, s^2)$ and $h = h_1$, and we set

$$g(x, y) = y\left(\frac{a(x)}{\sigma(x)^2} - \frac{3\sigma'(x)}{2\sigma(x)}\right) + y^3 \frac{\sigma'(x)}{2\sigma(x)^3}. \tag{5.3}$$

We also recall that $t \leq T$ (the constants below may depend on T).

Lemma 5.1. *There are constants $C, L > 0$ such that (with g as in (5.3)):*

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial y^j} p_t(x, y) \right| \leq C h_{L\sqrt{t}}(y - x) \left(1 + \left| \frac{y - x}{Lt} \right|^{i+j} + t^{-(i+j)/2} \right) \qquad \text{if } i + j \leq 3, \tag{5.4}$$

$$\left|\frac{\partial^i}{\partial x^i} q_t(x,y)\right| \leq C h_{L\sqrt{t}}(y)(1 + (y^2/Lt)^i) \qquad \textit{if } i \leq 3, \tag{5.5}$$

$$|y| \leq t^{1/3} \Rightarrow |q_t(x,y) - (1+\sqrt{t}g(x,y/\sqrt{t}))h_{\sigma(x)\sqrt{t}}(y)| \leq Ct(1 + (y/\sqrt{t})^8)h_{\sigma(x)\sqrt{t}}(y). \tag{5.6}$$

Proof. H and S are C^3 functions, with all derivatives of order $1, 2, 3$ bounded. Next, $V_t(x, y, \omega)$ are C^3_b functions of (x, y), with bounds on the functions and their partial derivatives independent of ω, hence r_t are C^3_b functions and $1/r_t \leq C$. Elementary calculations show that

$$\left|\frac{\partial^{i+j}}{\partial x^i \partial y^j} p_t(x,y)\right| \leq C p_t(x,y)\left[1 + \left|\frac{y-x}{t}\right|^{i+j} + t^{-(i+j)/2}\right] \qquad \text{if } i+j \leq 3.$$

Since H and S are Lipschitz and $\inf_{x \neq y} |\frac{S(x)-S(y)}{x-y}| > 0$, another simple computation shows the existence of $L > 0$ with $p_t(x,y) \leq C h_{L\sqrt{t}}(y - x)$, hence (5.4). A third calculation shows that

$$\left|\frac{\partial^i}{\partial x^i}\, q_t(x,y)\right| \leq C q_t(x,y)[1 + (y^2/t)^i] \qquad \text{if } i \leq 3,$$

while $q_t(x,y) \leq C h_{Lt}(y)$: so we have (5.5).

Write

$$\Delta(x,y) = H(x+y) - H(x) - \frac{1}{2t}\left((S(x+y) - S(x))^2 - \frac{y^2}{\sigma(x)^2}\right),$$

so that (5.2) yields

$$q_t(x,y) = h_{\sigma(x)\sqrt{t}}(y)\frac{\sigma(x)}{\sigma(x+y)} r_t(x, x+y)\mathrm{e}^{\Delta(x,y)}.$$

We have $|S(x+y) - S(x) - y/\sigma(x) + y^2\sigma'(x)/2\sigma(x)^2| \leq Cy^3$ and $|H(x+y) - H(x) - yb(x)| \leq Cy^2$, hence

$$\left|\Delta(x,y) - yb(x) - y^3\frac{\sigma'(x)}{2t\sigma(x)^3}\right| \leq C(y^2 + y^4/t).$$

So if $|y| \leq t^{1/3}$ it follows that

$$\left|\mathrm{e}^{\Delta(x,y)} - 1 - yb(x) - y^3\frac{\sigma'(x)}{2t\sigma(x)^3}\right| \leq C(y^2 + y^6/t^2).$$

Next, $|V_t| \leq C$ yields $|r_t(x, x+y) - 1| \leq Ct$. Finally $|\sigma(x+y) - \sigma(x) - y\sigma'(x)| \leq Cy^2$, while $\inf_x \sigma(x) > 0$, hence

$$\left|\frac{\sigma(x)}{\sigma(x+y)} - 1 + y\frac{\sigma'(x)}{\sigma(x)}\right| \leq Cy^2.$$

Putting all these results together immediately yields (5.6). □

Since $\int h_{L\sqrt{t}}(y)|y|^q \mathrm{d}y \le C_q t^{q/2}$, we easily deduce from (5.4) and (5.5) that

$$\int \left| \frac{\partial^{i+j}}{\partial x^i \partial y^j} p_t(x,y) \right| \mathrm{d}y \le C t^{-(i+j)/2} \qquad \text{if } i+j \le 3, \tag{5.7}$$

$$\int \left| \frac{\partial^{i}}{\partial x^i} q_t(x,y) \right| |y|^q \mathrm{d}y \le C_q t^{q/2} \qquad \text{if } i \le 3. \tag{5.8}$$

Recall the following well-known upper bounds, under H':

$$\mathrm{E}_x(|X_t - X_0|^p) \le C_p t^{p/2}, \qquad \mathrm{E}_x(|X_t - X_0 - \sigma(X_0)W_t|^p) \le C_p t^p. \tag{5.9}$$

Lemma 5.2. *There are constants C_r such that, for all $t > 0$ and all functions f having $|f(x)| \le M(1 + |x/\sqrt{t}|^r)$, we have*

$$|\mathrm{E}_x(f(X_t - x)) - \mathrm{E}_x(f(\sigma(x)W_t))| \le C_r M \sqrt{t}, \tag{5.10}$$

$$|\mathrm{E}_x(f(X_t - x)) - \mathrm{E}_x(f(\sigma(x)W_t)(1 + \sqrt{t} g(x, \sigma(x)W_t/\sqrt{t}))| \le C_r M t. \tag{5.11}$$

Proof. We first prove (5.11). Denote the left-hand side of (5.11) by $A = |\int (q_t(x,y) - h_{\sigma(x)\sqrt{t}}(y))(1 + \sqrt{t} g(x, y/\sqrt{t})) f(y) \mathrm{d}y|$. We have $A \le B + B'$, where

$$B = \left| \int_{|y| \le t^{1/3}} (q_t(x,y) - h_{\sigma(x)\sqrt{t}}(y)(1 + \sqrt{t} g(x, y/\sqrt{t})) f(y) \mathrm{d}y \right|$$

$$B' = \left| \int_{|y| > t^{1/3}} (q_t(x,y) - h_{\sigma(x)\sqrt{t}}(y)(1 + \sqrt{t} g(x, y/\sqrt{t})) f(y) \mathrm{d}y \right|.$$

First, (5.6) yields

$$B \le C_r M t \int h_{\sigma(x)\sqrt{t}}(y)(1 + |y/\sqrt{t}|^{8+r}) \mathrm{d}y \le C_r M t.$$

Second, by (5.5) and the hypothesis H' we have $h_{\sigma(x)\sqrt{t}}(y) \le C h_{Lt}(y)$ and $q_t(x,y) \le C h_{Lt}(y)(1 + y^2/Lt)$ for some $L > 0$. Further, in view of (5.3) and H', we also have $|\sqrt{t} g(x, y/\sqrt{t})| \le C|y|(1 + y^2/t)$; thus

$$B' \le MC \int_{|y| > t^{1/3}} h_{L\sqrt{t}}(y)(1 + |y/\sqrt{t}|^r)(1 + |y|(1 + y^2/t)) \mathrm{d}y \le C_r M t.$$

These two majorations yield (5.11).

Now let A' be the left-hand side of (5.10). We have $A' \le A + A''$, where

$$A'' = M \int h_{\sigma(x)\sqrt{t}}(y)(1 + |y/\sqrt{t}|^r)|y|(1 + y^2/t) \le C_r M \sqrt{t}. \qquad \square$$

Finally, we give a simple result on Riemann approximations.

Lemma 5.3. *Let $A_t^n = \frac{1}{n} \sum_{i=1}^{[nt]} f(X_{(i-1)/n}) - \int_0^t f(X_s) \mathrm{d}s$, where f is a function on $\mathbb{R}$.*

(a) *If f is differentiable and $M = \sup_x(|f(x)| + f'(x)|)$, then*

$$\mathrm{E}_x(\sup_{t \leq T} |A_t^n|^2) \to 0. \tag{5.12}$$

(b) *If f is twice differentiable and $M = \sup_x(|f(x)| + |f'(x)| + |f''(x)|)$,*

$$\mathrm{E}_x \sup_{t \leq T} |A_t^n|^2) \leq CM^2/n^2. \tag{5.13}$$

Proof. (a) Set $\xi_i^n = \int_{(i-1)/n}^{i/n}(f(X_s) - f(X_{(i-1)/n})\mathrm{d}s$ and $\kappa_t^n = -\int_{[nt]/n}^{t} f(X_s)\mathrm{d}s$. Then $A_t^n = \kappa_t^n - \sum_{i=1}^{[nt]} \xi_i^n$. Furthermore, $|\kappa_t^n| \leq M/n$, and if $w_T(\vartheta)$ denotes the modulus of continuity of $t \to X_t$ on $[0, T]$ we have $|\xi_i^n| \leq Mw(1/n)/n$. Thus $\sup_{t \leq T} |A_t^n| \leq M(1/n + w_T(1/n))$, and $\mathrm{E}_x(w_T(1/n)^2) \to 0$ as $n \to \infty$ (because $w_T(1/n) \to 0$ and $w_T(1/n) \leq 2 \sup_{t \leq T} |X_t| \in \mathbb{L}^2(P_x)$ under H'), and we get (5.12).

(b) If f is twice differentiable, Itô's formula yields $\xi_i^n = \eta_i^n + \zeta_i^n$, where

$$\eta_i^n = \int_{(i-1)/n}^{i/n} \mathrm{d}s \int_{(i-1)/n}^{s} (f'\sigma)(X_r)\mathrm{d}W_r\,,$$

$$\zeta_i^n = \int_{(i-1)/n}^{i/n} \mathrm{d}s \int_{(i-1)/n}^{s} (f'a + \frac{1}{2} f''\sigma^2)(X_r)\mathrm{d}r.$$

We have $|\kappa_t^n| \leq M/n$ and $|\zeta_i^n| \leq CMn^{-2}$. Thus in order to obtain (5.13) it suffices to prove that, if $B_i^n = \sum_{j=1}^{i}, \eta_j^n$, we have $\mathrm{E}_x(\sup_{i \leq nT}(B_i^n)^2) \leq CM^2/n^2$. But $(B_i^n)_{i \in \mathbb{N}}$ is a martingale relative to the discrete-time filtration $(\mathscr{F}_{i/n})_{i \in \mathbb{N}}$, so by Doob's inequality it suffices to prove that $\mathrm{E}_x(\sum_{j=1}^{nT}(\eta_j^n)^2) \leq CM^2/n^2$, or even that $\mathrm{E}((\eta^n)^2) \leq CM^2/n^3$. But, by the Cauchy–Schwarz inequality, we obtain

$$\mathrm{E}_x((\eta_i^n)^2) \leq \frac{1}{n} \int_{(i-1)/n}^{i/n} \mathrm{d}s\, \mathrm{E}_x\left(\int_{(i-1)/n}^{s} (f'\sigma)^2(X_r)\mathrm{d}r\right) \leq CM/n^3. \qquad \square$$

6. The fractional part of a random variable

We begin with a fundamental result.

Lemma 6.1. *There are universal constants C_N such that for all $\rho > 0$, and all Borel functions k on $\mathbb{R}$ and f on $\mathbb{R} \times [0, 1)$ such that $x \to g(x, y) := k(x)f(x, y)$ is of class C^N $(N \geq 1)$, we have:*

$$\left|\int_{\mathbb{R}} k(x)f\left(x, \left\{\frac{x}{\rho}\right\}\right)\mathrm{d}x - \int_{\mathbb{R}} k(x)\mathrm{d}x \int_0^1 f(x, u)\mathrm{d}u\right| \leq C_N \rho^N \int_{\mathbb{R}} \mathrm{d}x \int_0^1 \left|\frac{\partial^N}{\partial x^N} g(x, u)\right| \mathrm{d}u. \tag{6.1}$$

When k is the density of a random variable Y, the left-hand side of (6.1) is $|\mathrm{E}(f(Y, \{\frac{Y}{\rho}\})) - E(\int_0^1 f(Y, u)\mathrm{d}u)|$: we thus refine some old results of Kosulajeff (1937) and Tukey (1939).

Proof. First, let φ be a C^N function on $[a, a+\rho)$. Taylor's formula yields, for $k \leq N-1$ and $z \in [a, a+\rho)$:

$$\varphi(z) = \sum_{k=0}^{N-1} \varphi^{(k)}(a) \frac{(z-a)^k}{k!} + \int_a^z \varphi^{(N)}(v) \frac{(z-v)^{N-1}}{(N-1)!} \mathrm{d}v,$$

$$\int_a^{a+\rho} \varphi^{(k)}(u) \mathrm{d}u = \sum_{\ell=k}^{N-1} \varphi^{(\ell)}(a) \frac{\rho^{\ell+1-k}}{(\ell+1-k)!} + \int_a^{a+\rho} \varphi^{(N)}(z) \frac{(a+\rho-z)^{N-k}}{(N-k)!} \mathrm{d}z.$$

Introduce the polynomials P_k given by

$$(i+1)x^i = \sum_{k=0}^{i} \frac{(i+1)!}{(i+1-k)!} P_k(x).$$

(Then $P_0(x) = 1$ and P_k is of degree k.) We obtain

$$\rho\varphi(a+\rho y) - \sum_{k=1}^{N-1} P_k(y)\rho^k \int_a^{a+\rho} \varphi^{(k)}(u)\mathrm{d}u = A + B,$$

where

$$A = \sum_{k=0}^{N-1} \left(\varphi^{(k)}(a) \frac{\rho^{k+1} y^k}{k!} - \sum_{\ell=k}^{N-1} P_k(y) \frac{\rho^{\ell+1}}{(\ell+1-k)!} \varphi^{(\ell)}(a) \right),$$

$$B = \rho \int_a^{a+\rho y} \varphi^{(N)}(v) \frac{(a+\rho y - v)^{N-1}}{(N-1)!} \mathrm{d}v - \sum_{k=0}^{N-1} P_k(y)\rho^k \int_a^{a+p} \varphi^{(N)}(z) \frac{(a+\rho-z)^{N-k}}{(N-k)!} \mathrm{d}z,$$

while the definition of P_k yields $A = 0$. The existence of a universal constant C_N such that the following holds for all $y \in [0,1)$ is obvious:

$$\left| \rho\varphi(a+\rho y) - \sum_{k=0}^{N-1} P_k(y)\rho^k \int_a^{a+\rho} \varphi^{(k)}(u)\mathrm{d}u \right| \leq C_N \rho^N \int_a^{a+\rho} \left| \varphi^{(N)}(v) \right| \mathrm{d}v. \tag{6.2}$$

Now set $A = \int k(x) f(x, \{\frac{x}{\rho}\})\mathrm{d}x$. We have:

$$A = \sum_{j \in \mathbb{Z}} \int_{j\rho}^{(j+1)\rho} k(u) f(u, u/\rho - j)\mathrm{d}u = \sum_{j \in \mathbb{Z}} \int_0^1 \rho g(\rho j + \rho y, y)\mathrm{d}y. \tag{6.3}$$

with $g(x,y) = k(x)f(x,y)$. Also set $g^{(\ell)}(x,y) = \partial^\ell g(x,y)/\partial x^\ell$, $G_i^\ell(x) = \int_0^1 g^{(\ell)}(x,y) y^i \mathrm{d}y$ and $\gamma_\ell = \int_{\mathbb{R}} \mathrm{d}x \int_0^1 |g^{(\ell)}(x,y)|\mathrm{d}y$. Clearly, $\int_{\mathbb{R}} |G_i^\ell(x)|\mathrm{d}x \leq \gamma_\ell$, and we assume $\gamma_N < \infty$, otherwise there is nothing to prove. If $u_\ell = \sum_{j \in \mathbb{Z}} \int_{j\rho}^{(j+1)\rho} \mathrm{d}x \int_0^1 P_\ell(y) g^{(\ell)}(x,y)\mathrm{d}y$ we obtain, by (6.2) and (6.3):

$$\left| A - \sum_{0 \leq \ell \leq N-1} \rho^\ell u_\ell \right| \leq C_N \rho^N \gamma_N.$$

Since $P_0 = 1$ we have $u_0 = \int_{\mathbb{R}} k(x)\mathrm{d}x \int_0^1 f(x,y)\mathrm{d}y$. If $\ell \geq 1$, u_ℓ is a linear combination of the numbers $\int_{\mathbb{R}} G_i^\ell(x)\mathrm{d}x$ for $0 \leq i \leq \ell$. Now, G_i^ℓ and $G_i^{\ell-1}$ are integrable, and $G_i^\ell = \partial G_i^{\ell-1}/\partial x$, hence $\int_{\mathbb{R}} G_i^\ell(x)\mathrm{d}x = 0$ and therefore $u_\ell = 0$ if $\ell \geq 1$: we thus deduce the result. $\square$

As a particular case, there is a constant C such that, for all $\rho > 0$, all Borel sets I in $[0, 1]$ of Lebesgue measure $\ell(I)$ and all random variables Y with C^1 density k, we have (apply (6.1) to $f(x, y) = 1_I(y)$):

$$P\left(\left\{\frac{Y}{\rho}\right\} \in I\right) \leq \ell(I)\left(1 + C\rho\int_{\mathbb{R}} |k'(x)|\mathrm{d}x\right). \tag{6.4}$$

7. The function Δ

The aim of this section is to study the functions $\Delta_{\psi,\psi}$ defined in (2.7), and also to prove (2.9) and the following estimate on the functions of (2.5):

$$|\ell_i\psi(\sigma, \rho, u)| \leq \begin{cases} C & \text{if } i = 1 \\ C(\rho/\sigma)^3(i-1)^{-3/2} & \text{if } i \geq 2. \end{cases} \tag{7.2}$$

Below we consider functions ψ on $[0, 1] \times \mathbb{R}$, satisfying (as in (5.1)):

$$|\psi(u, y)| \leq K(1 + |y|^p). \tag{7.2}$$

We also assume that $1/K' \leq \sigma \leq K'$ and $\rho \leq K'$ for some $K' < \infty$. When the function $\sigma(x)$ is used, it is assumed to satisfy H'. The constants C below will depend only on p, K, K' and on the constants occurring in H'.

The basic relation relates ℓ_{i+1} with ℓ_1 and is as follows for $i \geq 1$:

$$\ell_{i+1}\psi(\sigma, \rho, u) = \mathrm{E}(\ell_1\psi(\sigma, \{u + \sigma W_i/\rho\})) \tag{7.3}$$

(note that $\ell_1\psi(\sigma, u) = m_\sigma\psi(u) - M_\sigma\psi$ does not depend on ρ). Observe that under (7.2) we have $|\ell_1\psi| \leq C$ and $\int_0^1 \ell_1\psi(\sigma, u)\mathrm{d}u = 0$, so (7.3) and (6.1) with $N = 3$, along with $k(x) = h(y - \rho u/\sigma)$ and $f(x, y) = \ell_1\psi(\sigma, y)$, readily yield (7.1). If we set $L\psi(\sigma, 0, u) = \ell_1\psi(\sigma, u))$, and since $\sigma \geq 1/K'$, we obtain, for all $\rho \geq 0$ (by integration of (7.3), and Fubini's theorem for (7.5) below):

$$|L\psi(\sigma, \rho, u)| \leq C, \qquad |L\psi(\sigma, \rho, u) - L\psi(\sigma, 0, u)| \leq C\rho^3, \tag{7.4}$$

$$\int_0^1 L\psi(\sigma, \rho, u)\mathrm{d}u = 0. \tag{7.5}$$

Using (2.7), (2.8) and the fact that $\mathrm{E}(|\eta_1\psi(\sigma, u)|^2) \leq C$, we deduce:

$$|\delta_{\psi,\psi}(\sigma, \rho, u)| \leq C, \qquad |\Delta_{\psi,\psi}(\sigma, \rho)| \leq C. \tag{7.6}$$

Lemma 7.1. *We have* (2.9), *and the following* (*with* $\varphi_\sigma(u, y) = y/\sigma$):

$$L\varphi_\sigma(\sigma, \rho, u) = m_\sigma\varphi_\sigma(u) = M_\sigma\varphi_\sigma = 0, \qquad \Delta_{\varphi_\sigma,\varphi_\sigma}(\sigma, \rho) = 1, \tag{7.7}$$

$$\Delta_{\psi,\varphi_\sigma}(\sigma, \rho) = M_\sigma(\psi\varphi_\sigma). \tag{7.8}$$

Proof. That $m_\sigma\varphi_\sigma(u) = M_\sigma\varphi_\sigma = 0$ is obvious, so $\eta_i\varphi_\sigma(\sigma,\rho,u) = W_i - W_{i-1}$ and thus $L\varphi_\sigma(\sigma,\rho,u) = 0$ for all $\rho \geq 0$. Then $\chi\varphi_\sigma(\sigma,\rho,u) = W_1$ and the last part of (7.7) is also obvious. Equation (7.8) is obvious if $\rho = 0$. If $\rho > 0$ we have

$$\delta_{\psi,\varphi_\sigma}(\sigma,\rho,u) = \mathrm{E}(\psi(u,\sigma W_1)\varphi_\sigma(\sigma W_1)) + \mathrm{E}(W_1 L\psi(\sigma,\rho,\{u+\sigma W_1/\rho\})),$$

and thus (7.8) follows from (7.5).

Let us define $\bar\Omega = \Omega \times [0,1]$, $\bar{\mathscr{G}} = \mathscr{G} \otimes \mathscr{B}([0,1])$, $\bar P(\mathrm{d}\omega,\mathrm{d}u) = P(\mathrm{d}\omega)\mathrm{d}u$. If we set $(\chi\psi)_{\sigma,\rho}(\omega,u) = \chi\psi(\sigma,\rho,u)(\omega)$ if $\rho > 0$ and $(\chi\psi)_{\sigma,0}(\omega,u) = \eta_1\psi(\sigma,u)(\omega)$, it follows from (2.7) and (2.8) that $\Delta_{\psi,\psi}(\sigma,\rho) = \bar{\mathrm{E}}(|(\chi\psi)_{\sigma,\rho}|^2)$ for all $\rho \geq 0$. Thus (7.7) yields $\Delta_{\psi,\psi}(\sigma,\rho)^{1/2} \geq \bar{\mathrm{E}}((\chi\psi)_{\sigma,\rho}(\chi\varphi_\sigma)_{\sigma,\rho}) = \int_0^1 \mathrm{E}(\chi\psi(\sigma,\rho,u)W_1)\mathrm{d}u$ by the Cauchy–Schwarz inequality. But (2.6) and (7.5) give

$$\int_0^1 \mathrm{E}(\chi\psi(\sigma,\rho,u)W_1)\mathrm{d}u = \int_0^1 \mathrm{E}((\psi(u,\sigma W_1) - M_\sigma\psi)W_1)\mathrm{d}u = \int_0^1 \mathrm{E}((\psi\varphi_\sigma)(u,\sigma W_1))\mathrm{d}u$$

which equals $M_\sigma(\psi\varphi_\sigma)$, and (2.9) is proved. □

In the next lemma we are given a family $(\psi_x)_{x\in\mathbb{R}}$ of functions satisfying (7.2), such that $x \to \psi_x(u,y)$ is differentiable and each $\partial\psi_x(u,y)/\partial x$ also satisfies (7.2).

Lemma 7.2. *Under the above assumptions, $x \to \delta_{\psi_x,\psi_x}(\sigma(x),\rho,u)$ is differentiable and, for $0 < \rho \leq K'$:*

$$\left|\frac{\partial}{\partial x}\delta_{\psi_x,\psi_x}(\sigma(x),\rho,u)\right| \leq C. \tag{7.9}$$

Proof. (a) Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be differentiable in the first variable, with $f(x,.)$ and $\partial f(x,.)/\partial x$ satisfying (7.2), and $F(x) = \mathrm{E}(f(x,\sigma(x)W_1)) = \int \frac{1}{\sigma(x)} h(\frac{z}{\sigma(x)})f(x,z)\mathrm{d}z$. Since $h'(z) = -zh(z)$, we obtain by Lebesgue's theorem:

$$F'(x) = \int h(z)\left(\frac{\partial}{\partial x}f(x,\sigma(x)z) + \frac{\sigma'(x)}{\sigma(x)}(z^2-1)f(x,\sigma(x)z)\right)\mathrm{d}z.$$

Therefore $|F(x)| + |F'(x)| \leq C$ (recall H').

(b) Applying this to $f(x,y) = \psi_x(u,y)$ gives that $x \to m_{\sigma(x)}\psi_x(u)$ and thus $x \to M_{\sigma(x)}\psi_x$ are bounded with bounded derivatives. Hence $g(x,u) := \ell_1\psi_x(\sigma(x),u)$ also satisfies $|g(x,u)| \leq C$ and $|\partial g(x,u)/\partial x| \leq C$.

By (7.3),

$$\ell_{i+1}\psi_x(\sigma(x),\rho,u) = \int \frac{\rho}{\sigma(x)\sqrt{i}} h\left(\frac{\rho z}{\sigma(x)\sqrt{i}}\right) g(x,\{u+z\})\mathrm{d}z.$$

Differentiate again under the integral sign to obtain

$$\frac{\partial}{\partial x}\ell_{i+1}\psi_x(\sigma(x),\rho,u) = \int h\left(z - \frac{\rho u}{\sigma(x)\sqrt{i}}\right)\frac{\partial}{\partial x}g(x,\{z\})\mathrm{d}z$$
$$+ \int h\left(z - \frac{\rho u}{\sigma(x)\sqrt{i}}\right)\left(\left(z - \frac{\rho u}{\sigma(x)\sqrt{i}}\right)^2 - 1\right)\frac{\sigma'(x)}{\sigma(x)} g(x,\{z\})\mathrm{d}z.$$

Then we can apply (6.1) twice with $N = 3$, taking into account the fact that $\int_0^1 g(x,u)\mathrm{d}u = 0$ and thus $\int_0^1 \frac{\partial}{\partial x} g(x,u)\mathrm{d}u = 0$, and obtain $|\frac{\partial}{\partial x}\ell_{i+1}\psi_x(\sigma(x),\rho,u)| \leq Ci^{-3/2}$ (recall that $\rho \leq K'$ here). Hence $|\frac{\partial}{\partial x}L\psi_x(\sigma(x),\rho,u)| \leq C$.

Now (2.6) yields $\chi\psi_x(\sigma(x),\rho,u) = f(x,\sigma(x)W_1)$ if we set

$$f(x,y) = \psi_x(u,y) - M_{\sigma(x)}\psi_x + L\psi_x(\sigma(x),\rho,\{u+y/\rho\}) - L\psi_x(\sigma(x),\rho,u).$$

What precedes shows that the function f (hence f^2 as well) satisfies the requirements of (a). Since $\delta_{\psi_x,\psi_x}(\sigma(x),\rho,u) = \mathrm{E}(f^2(x,\sigma(x)W_1))$, the result follows from (a). □

Now we consider a sequence ψ_n of functions satisfying (7.2), and a sequence ρ_n of positive numbers. We assume that

$$\psi_n \to \psi \,\mathrm{d}u \otimes \mathrm{d}y\text{-almost surely}, \qquad \rho_n \to \rho \in [0,\infty),$$

where ψ is another function (satisfying (7.2) as well, of course).

Lemma 7.3. *Under the previous hypotheses,* $\Delta_{\psi_n,\psi_n}(\sigma,\rho_n) \to \Delta_{\psi,\psi}(\sigma,\rho)$.

Note that by Lemmas 7.2 and 7.3, $(\sigma,\rho) \to \Delta_{\psi,\psi}(\sigma,\rho)$ is continuous on $(0,\infty)\times[0,\infty)$. By the bilinearity of $(\varphi,\psi) \to \Delta_{\varphi,\psi}(\sigma,\rho)$ and the polarization principle, $\Delta_{\varphi,\psi}$ is also continuous on $(0,\infty)\times[0,\infty)$ if φ and ψ satisfy (7.2).

Proof. (a) Consider $(\bar{\Omega},\bar{\mathscr{G}},\bar{P})$ as defined in the proof of Lemma 7.1, and $\chi_n(\omega,u) = \chi\psi_n(\sigma,\rho_n,u)(\omega)$. We have seen that $\Delta_{\psi_n,\psi_n}(\sigma,\rho_n) = \bar{\mathrm{E}}(\chi_n^2)$. By (2.6), we have $\chi_n = f_n + k_n$, where

$$\begin{aligned} f_n(\omega,u) &= \psi_n(u,\sigma W_1(\omega)) - M_\sigma\psi_n - L\psi_n(\sigma,\rho_n,u) \\ &\quad + L\psi_n(\sigma,\rho_n,\{u+\sigma W_1(\omega)/\rho_n\}) - L\psi(\sigma,\rho,\{u+\sigma W_1(\omega)/\rho_n\}), \\ k_n(\omega,u) &= L\psi(\sigma,\rho,\{u+\sigma W_1(\omega)/\rho_n\}). \end{aligned}$$

(b) From (2.3) we clearly have that $m_\sigma\psi_n \to m_\sigma\psi$ du-almost surely, hence $M_\sigma\psi_n \to M_\sigma\psi$ and $\ell_1\psi_n(\sigma,.) \to \ell_1\psi(\sigma,.)$ du-almost surely. Then (7.3) yields, for $i \geq 1$:

$$\ell_{i+1}\psi_n(\sigma,\rho_n,u) = \int \frac{\rho_n}{\sigma\sqrt{i}} h\left(\frac{z\rho_n}{\sigma\sqrt{i}}\right)\ell_1\psi_n(\{u+z\})\mathrm{d}z.$$

If $\rho > 0$ and if u is fixed, then $\ell_1\psi_n(\{u+z\}) \to \ell_1\psi(\{u+z\})$ for dz-almost all z, hence $\ell_{i+1}\psi_n(\sigma,\rho_n,u) \to \ell_{i+1}\psi(\sigma,\rho,u)$. Using (7.1) and Lebesgue's theorem, we deduce that $L\psi_n(\sigma,\rho_n,u) \to L\psi(\sigma,\rho,u)$ for all u if $\rho > 0$, and also for $\rho = 0$ since $L\psi(\sigma,0,u) = \ell_1\psi(\sigma,u)$.

By Egoroff's theorem, for all $\varepsilon > 0$ there is a Borel set A_ε in $[0,1]$ such that $\int_0^1 1_{A_\varepsilon}(u)\mathrm{d}u \leq \varepsilon$ and $\eta_n := \sup_{u\notin A_\varepsilon}|L\psi_n(\sigma,\rho_n,u) - L\psi(\sigma,\rho,u)| \to 0$. Then if

$$f(\omega,u) = \psi(u,\sigma W_1(\omega)) - M_\sigma\psi - L\psi(\sigma,\rho,u), \tag{7.10}$$

for all u we have $\limsup_n |f_n(\omega,u) - f(\omega,u)| 1_{\{\{u+\sigma W_1(\omega)/\rho_n\}\notin A_\varepsilon\}} = 0$ P-almost surely. Since (6.4) yields $P(\{u+\sigma W_1/\rho_n\}\notin A_\varepsilon) \leq C\varepsilon$ and since $|f_n(\omega,u)| \leq C(1+|W_1(\omega)|^p)$, and since $\varepsilon > 0$ is arbitrary, it follows that

$$f_n \to f \qquad \text{in } \mathbb{L}^2(\bar{P}). \tag{7.11}$$

(c) Now we suppose that $\rho > 0$. We have $\Delta_{\psi,\psi}(\sigma,\rho) = \bar{\mathrm{E}}(\chi^2)$, where $\chi(\omega,u) := \chi\psi(\sigma,\rho,u)(\omega)$, and $\chi = f + k$, where $k(\omega,u) = L\psi(\sigma,\rho,\{u+\sigma W_1(\omega)/\rho\})$ (use (2.6)). In view of (7.11) and $|k_n| \le C$, the result will follow if we prove

$$\bar{\mathrm{E}}(k_n^2) \to \bar{\mathrm{E}}(k^2), \qquad \bar{\mathrm{E}}(k_n f) \to \bar{\mathrm{E}}(kf). \tag{7.12}$$

For the first property above, observe that

$$\bar{\mathrm{E}}(k_n^2) = \int_0^1 \mathrm{d}u \int \frac{\rho_n}{\sigma} h\Big(\frac{z\rho_n}{\sigma}\Big) L\psi(\sigma,\rho,\{u+z\})^2 \mathrm{d}z,$$

which clearly converges to $\bar{\mathrm{E}}(k^2)$. Similarly $\mathrm{E}(L\psi(\sigma,\rho,\{u+\sigma W_1/\rho_n\})) \to \mathrm{E}(L\psi(\sigma,\rho,\{u+\sigma W_1/\rho\}))$, so in view of (7.10), in order to prove the second property in (7.12) it is enough to prove that for all u:

$$\mathrm{E}(\psi(u,\sigma W_1)L\psi(\sigma,\rho,\{u+\sigma W_1/\rho_n\})) \to \mathrm{E}(\psi(u,\sigma W_1)L\psi(\sigma,\rho,\{u+\sigma W_1/\rho\})). \tag{7.13}$$

For all $\varepsilon > 0$ there is a C_b^1 function φ_ε on $\mathbb{R}$ such that $\mathrm{E}(|\psi(u,\sigma W_1) - \varphi_\varepsilon(\sigma W_1)|) \le \varepsilon$. We also have

$$\mathrm{E}(\varphi_\varepsilon(\sigma W_1)L\psi(\sigma,\rho,\{u+\sigma W_1/\rho_n\})) = \int \frac{\rho_n}{\sigma} h\Big(\frac{z\rho_n}{\sigma}\Big)\varphi_\varepsilon(z\rho_n)L\psi(\sigma,\rho,\{u+z\})\mathrm{d}z,$$

which converges to $\mathrm{E}(\varphi_\varepsilon(\sigma W_1)L\psi(\sigma,\rho,\{u+\sigma W_1/\rho\}))$ because φ_ε is continuous and bounded and $L\psi$ is bounded. Since $\varepsilon > 0$ is arbitrary, we deduce (7.13), hence (7.12) and the lemma is proved when $\rho > 0$.

(d) All that then remains is to consider the case $\rho = 0$. Recall that $L\psi(\sigma,0,u) = m_\sigma\psi(u) - M_\sigma\psi$, hence $f(\omega,u) = \psi(u,\sigma W_1(\omega)) - m_\sigma\psi(u)$ by (7.10), and a simple computation shows that $\bar{\mathrm{E}}(f^2) = M_\sigma(\psi^2) - \int_0^1 m_\sigma\psi(u)^2\mathrm{d}u$. Using (6.1) for $N = 1$ and for the functions $k(x) = h(x - u\rho_n/\sigma)$ and $f(x,y) = \varphi(x - u\rho_n/\sigma)L\psi(\sigma,0,y)^i$ (where $\varphi \in C_\mathrm{b}^1$ and $i = 1, 2$) yields

$$\left|\mathrm{E}(\varphi(\sigma W_1)L\psi(\sigma,0,\{u+\sigma W_1/\rho_n\})^i) - \mathrm{E}(\varphi(\sigma W_1))\int_0^1 L\psi(\sigma,0,y)^i\mathrm{d}y\right| \le C\rho_n \to 0. \tag{7.14}$$

Since $\int_0^1 L\psi(\sigma,0,y)^2\mathrm{d}y = \int_0^1 m_\sigma\psi(u)^2\mathrm{d}u - (M_\sigma\psi)^2$, we deduce that $\bar{\mathrm{E}}(k_n^2) \to \int_0^1 m_\sigma\psi(u)^2\mathrm{d}u - (M_\sigma\psi)^2$. In view of (2.8) and (7.11), it remains to prove that $\bar{\mathrm{E}}(k_n f) \to 0$. Because of (7.14) for $i = 1$ and $\varphi = 1$ and from (7.5) (valid also for $\rho = 0$), it remains to prove that $\mathrm{E}(\psi(u,\sigma W_1)L\psi(\sigma,0,\{u+\sigma W_1/\rho_n\})) \to 0$. Exactly as in (c), we can replace $\psi(u,.)$ by a C_b^1 function φ_ε, and (7.14) for $i = 1$ and $\varphi = \varphi_\varepsilon$ and (7.5) give the result. □

8. Some auxiliary results

We assume below that the hypotheses H' and K'_r hold for $r = 1$ or $r = 2$. In addition to

(2.2) and (2.3), for all functions φ satisfying (5.1) for $i = 1$ we set

$$\left.\begin{aligned} m_n\varphi(x,u) &= \int q_{1/n}(x,y)\varphi(x,u,y\sqrt{n})\mathrm{d}y, \qquad M_n\varphi(x) = \int_0^1 m_n\varphi(x,u)\mathrm{d}u, \\ \bar{m}_n\varphi(x) &= m_n\varphi(x,\{x/\alpha_n\}) - M_n\varphi(x), \qquad \bar{m}\varphi(x) = m\varphi(x,\{x/\alpha_n\}) - M\varphi(x). \end{aligned}\right\} \tag{8.1}$$

In the following all constants, denoted by C, may depend on T, on K and p in (5.1), on the coefficients a, σ and on the sequence (α_n).

Lemma 8.1. *Under K_r' we have the upper bounds*

$$\left|\frac{\partial^i}{\partial x^i} m_n f_n\right| + \left|\frac{\partial^i}{\partial x^i} m f_n\right| + |m_n f| + |mf| \leq C \qquad \textit{for } 0 \leq i \leq r \tag{8.2}$$

$$|m_n f_n - m f_n| + |\bar{m}_n f_n - \bar{m} f_n| \leq C/\sqrt{n} \tag{8.3}$$

$$|m_n f_n - m f_n - m\tilde{f}_n/\sqrt{n}| \leq C/n, \tag{8.4}$$

where $\tilde{f}_n$ is given by (2.12).

Proof. Property (8.2) readily follows from K_r' and (5.8). Observing that $mf_n(x,u) = \int h_{\sigma\sigma(x)/n}(y) f_n(x,u,y\sqrt{n})\mathrm{d}y$, (8.3) and (8.4) follow from (5.10) and (5.11) applied to the function $f(y) = f_n(x,u,y\sqrt{n})$. □

Next we set for $i, n, k \in \mathbb{N}^*$:

$$\eta_i^n = f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}(X_{i/n} - X_{(i-1)/n}) - M_n f_n(X_{(i-1)/n}) \tag{8.5}$$

$$\mu_i^n(k) = \sum_{j=i}^{i+k-1} (\mathrm{E}_x(\eta_j^n|\mathscr{F}_{i/n}) - \mathrm{E}_x(\eta_j^n|\mathscr{F}_{(i-1)/n})) \tag{8.6}$$

$$M_t^n(k) = n^{-1/2} \sum_{i=1}^{[nt]} \mu_i^n(k). \tag{8.7}$$

Due to K_r', along with (5.9) and (8.2), every $\mu_i^n(k)$ is square-integrable, hence $M^n(k)$ is a locally square-integrable martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_{[nt]/n})_{t\geq 0}, P_x)$.

For further reference, we also deduce from (8.6) and (8.7) that

$$\begin{aligned} \mu_i^n(k) = \eta_i^n + \bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n}) - \int p_{(k-1)/n}(X_{(i-1)/n}, y)\bar{m}_n f_n(y)\mathrm{d}y \\ + \sum_{j=1}^{k-2} \int (p_{j/n}(X_{i/n}, y) - p_{j/n}(X_{(i-1)/n}, y))\bar{m}_n f_n(y)\mathrm{d}y, \end{aligned} \tag{8.9}$$

$$\begin{aligned} M_t^n(k) = n^{-1/2} \sum_{i=1}^{[nt]} \eta_i^n + n^{-1/2} \Bigg(\bar{m}_n f_n(X_{[nt]/n}) - \bar{m}_n f_n(X_0) + \sum_{i=1}^{k-2} \int (p_{i/n}(X_{[nt]/n}, y) \\ - p_{i/n}(X_0, y))\bar{m}_n f_n(y)\mathrm{d}y - \sum_{i=0}^{[nt]-1} \int p_{(k-1)/n}(X_{i/n}, y)\bar{m}_n f_n(y)\mathrm{d}y \Bigg). \end{aligned} \tag{8.10}$$

We presently give some estimates of $\mu_i^n(k)$ and $M_t^n(k)$. We first set

$$\delta^n(k,x) = \mathrm{E}_x(|\mu_1^n(k)|^2), \tag{8.11}$$

$$H_t^n(k) = M_t^n(k) - n^{-1/2}\sum_{i=1}^{[nt]} \eta_i^n. \tag{8.12}$$

Lemma 8.2. *We have, for $j \le nT$:*

$$\int p_{j/n}(x,y)\bar{m}_n f_n(y)\mathrm{d}y \le \begin{cases} C/\sqrt{j} & \textit{under } K_1' \\ C/j & \textit{under } K_2' \end{cases} \tag{8.13}$$

$$\int (p_{j/n}(x,y) - p_{j/n}(x',y))\bar{m}_n f_n(y)\mathrm{d}y \le C|x-x'|\frac{\sqrt{n}}{j^{3/2}} \qquad \textit{under } K_2'. \tag{8.14}$$

Proof. For (8.13) it is enough to apply (6.1) to $k(y) = p_{j/n}(x,y)$ and $f(y,u) = m_n f_n(y,u) - M_n f_n(y)$ with $N=1$ ($N=2$) and $\rho = \alpha_n$, and to use (5.7) and (8.2) and the facts that $\sup(\alpha_n/\sqrt{n}) < \infty$ and $j \le nT$. Observing that

$$\int (p_{j/n}(x,y) - p_{j/n}(x',y))\bar{m}_n f_n(y)\mathrm{d}y = \int_x^{x'} \mathrm{d}z \int \frac{\partial}{\partial z} p_{j/n}(z,y)\bar{m}_n f_n(y)\mathrm{d}y,$$

we similarly deduce (8.14) from (6.1) with $k(y) = \frac{\partial}{\partial z} p_{j/n}(z,y)$ and f as above and $N=2$, by using (5.7) and (8.2) again. □

It follows from (8.2), (5.9), (8.9) and Lemma 8.2 that

$$2 \le k \le nT \Rightarrow \mathrm{E}_x(|\mu_1^n(k)|^4) \le \begin{cases} Ck^2 & \text{under } K_1' \\ C & \text{under } K_2'. \end{cases} \tag{8.15}$$

By (5.9), (8.9) and Lemma 8.2 we also have, under K_2' and for $2 \le k' \le k \le nT$, that

$$\mathrm{E}_x(|\mu_1^n(k) - \mu_1^n(k')|^2) \le C(k^{-2} + k'^{-2} + k'^{-1}) \le C/k',$$

and this, together with (8.13) and the Cauchy–Schwarz inequality, gives

$$2 \le k' \le k \le nT \text{ and } K_2' \Rightarrow |\delta^n(k,x) - \delta^n(k',x)| \le C/\sqrt{k'}. \tag{8.16}$$

Similarly, (8.10), (8.2) and (8.13) yield

$$2 \le k \le nT \Rightarrow |H_t^n(k)| \le \begin{cases} C\sqrt{n/k} & \text{under } K_1' \\ C(\sqrt{n}/k + (\log k)/\sqrt{n}) & \text{under } K_2'. \end{cases} \tag{8.17}$$

Finally, recalling (2.7), we prove the following lemma.

Lemma 8.3. *Under K_2' and if $f_{n,x}(u,y) = f_n(x,u,y)$, we have, for $16 \le k \le nT$:*

$$|\delta^n(k,x) - \delta_{f_{n,x'}f_{n,x}}(\sigma(x), \beta_n, \{x/\alpha_n\})| \le Ck^{-1/8}. \tag{8.18}$$

Proof. Recall the notation used in (8.1) and (2.3), and also set

$$\bar{m}'f_n(x,x') := mf_n(x,\{x'/\alpha_n\}) - Mf_n(x) = m_{\sigma(x)}f_{n,x}(\{x'/\alpha_n\}) - M_{\sigma(x)}f_{n,x}.$$

Note that $\bar{m}f_n(x) = \bar{m}'f_n(x,x)$. From the proof of Lemma 7.2, $x \to \bar{m}'f_n(x,x')$ has a bounded derivative, hence by (8.3):

$$|\bar{m}'f_n(x,x') - \bar{m}_nf_n(x')| \le C(n^{-1/2} + |x-x'|). \tag{8.19}$$

Let us set $k' = [k^{1/4}]$, hence $2 \le k' \le k \le nT$. We also set

$$b^n_{k'}(x) = \bar{m}_nf_n(x) + \sum_{j=1}^{k'-2} \int p_{j/n}(x,y)\bar{m}_nf_n(y)\mathrm{d}y,$$

$$c^n_{k'}(x,x') = \bar{m}'f_n(x,x') + \sum_{j=1}^{k'-2} \int h_{\sigma(x)\sqrt{j/n}}(y-x')\bar{m}'f_n(x,y)\mathrm{d}y.$$

Then (8.9) can be written as

$$\mu^n_1(k') = \eta^n_1 + b^n_{k'}(X_{1/n}) - b^n_{k'+1}(X_0). \tag{8.20}$$

Since $\bar{m}'f_n$ is bounded, we deduce from H' that

$$\left|\int h_{\sigma(x)\sqrt{j/n}}(y-x')\bar{m}'f_n(x,y)\mathrm{d}y - \int h_{\sigma(x')\sqrt{j/n}}(y-x')\bar{m}'f_n(x,y)\mathrm{d}y\right| \le C|x-x'|.$$

Next, (5.10) and (8.2) yield

$$\left|\int p_{j/n}(x',y)\bar{m}_nf_n(y)\mathrm{d}y - \int h_{\sigma(x')\sqrt{j/n}}(y-x')\bar{m}_nf_n(y)\mathrm{d}y\right| \le C\sqrt{j/n}.$$

Finally, $\int h_{\sigma(x')\sqrt{j/n}}(y-x')|y-x|\mathrm{d}y \le |x-x'| + C\sqrt{j/n}$, hence (8.19) yields

$$\int h_{\sigma(x')\sqrt{j/n}}(y-x')|\bar{m}_nf_n(y) - \bar{m}'f_n(x,y)|\mathrm{d}y \le C(\sqrt{j/n} + |x-x'|).$$

Putting all these upper bounds together, and using (8.19) once more, we obtain

$$|b^n_{k'}(x') - c^n_{k'}(x,x')| \le C(k'^{3/2}n^{-1/2} + k'|x-x'|). \tag{8.21}$$

We also set $\bar{\eta}^n = f_n(X_0,\{X_0/\alpha_n\},\sqrt{n}(X_{1/n}-X_0)) - Mf_n(X_0)$, so that, in view of (8.3) and (8.5), we have $|\eta^n_1 - \bar{\eta}^n| \le C/\sqrt{n}$. Therefore, if

$$\bar{\mu}^n(k') = \bar{\eta}^n + c^n_{k'}(X_0,X_{1/n}) - c^n_{k'+1}(X_0,X_0), \tag{8.22}$$

we deduce from (5.9), (8.20) and (8.21) that $\mathrm{E}_x(|\mu^n_1(k') - \bar{\mu}^n(k')|^2) \le C(k'^3/n + k'^2/n) \le Ck'^3/n \le Cn^{-1/4}$, because $k' \le Cn^{1/4}$. This, the Cauchy–Schwarz inequality and the second part of (8.15) yield

$$|\mathrm{E}_x(|\mu^n_1(k')|^2) - \mathrm{E}_x(|\bar{\mu}^n(k')|^2)| \le Cn^{-1/8}. \tag{8.23}$$

We now consider a function ψ on $[0,1]\times\mathbb{R}$ satisfying (7.2). Using the notation (2.4) and (2.5), we set $L_{k'}\psi=\sum_{i=1}^{k'}\ell_i\psi$ and

$$\mu\psi(k')(\sigma,\rho,u)=\eta_1\psi(\sigma,\rho,u)+L_{k'-1}\psi(\sigma,\rho,\{u+\sigma W_1/\rho\})-L_{k'}\psi(\sigma,\rho,u). \tag{8.24}$$

Since $|L\psi(\sigma,\rho,u)-L_{k'}\psi(\sigma,\rho,u)|\le C(1+(\rho/\sigma)^3)k'^{-1/2}$ by (7.1), we obtain

$$|\chi\psi(\sigma,\rho,u)|\le|\eta\psi(\sigma,\rho,u)|+C(1+(\rho/\sigma)^3),$$

$$|\chi\psi(\sigma,\rho,u)-\mu\psi(k')(\sigma,\rho,u)|\le C(1+(\rho/\sigma)^3)k'^{-1/2}.$$

In particular,

$$|\delta_{\psi,\psi}(\sigma,\rho,u)-\mathrm{E}(|\mu\psi(k')(\sigma,\rho,u)|^2)|\le C(1+(\rho/\sigma)^3)k'^{-1/2}. \tag{8.25}$$

We now fix n and x, and set $\psi(u,y)=f_n(x,u,y)$, $\sigma=\sigma(x)$, $\rho=\beta_n$. Note that $\ell_1\psi(\sigma,\rho,u)=\bar{m}'f_n(x,\alpha_n u)$ and $\ell_{i+1}\psi(\sigma,\rho,u)=\mathrm{E}(\ell_1\psi(\sigma,\rho,\{u+\sigma W_i/\rho\}))=\int h_{\sigma(x)\sqrt{i/n}}(z-\alpha_n u)\bar{m}'f_n(x,z)\mathrm{d}z$. Hence $c^n_{k'}(x,x')=L_{k'-1}\psi(\sigma,\rho,\{x'/\alpha_n\})$ and (8.22) yields that, P_x-almost surely,

$$\bar{\mu}^n(k')=\psi(\{x/\alpha_n\},\sqrt{n}(X_{1/n}-x))+L_{k'-1}\psi(\sigma,\rho,\{X_{1/n}/\alpha_n\})-L_{k'}\psi(\sigma,\rho,\{x/\alpha_n\}).$$

In other words, $\bar{\mu}^n(k')=\varphi_n(X_{1/n})$ for a function φ_n satisfying $|\varphi_n(y)|\le C(1+(y\sqrt{n})^p)$ and (5.10) shows that if $\bar{\mu}'^n(k')=\varphi_n(x+\sigma(x)W_{1/n})$ we have

$$|\mathrm{E}(|\bar{\mu}^n(k')|^2)-\mathrm{E}(|\bar{\mu}'^n(k')|^2)|\le C/\sqrt{n}. \tag{8.26}$$

But by (8.24), the variables $\mu\psi(k')(\sigma,\rho,\{x/\alpha_n\})$ under P and $\bar{\mu}'^n(k')$ under P_x have the same distribution: then a combination of (8.23), (8.25) and (8.26) gives

$$|\delta^n(k',x)-\delta_{f_{n,x},f_{n,x}}(\sigma(x),\beta_n,\{x/\beta_n\})|\le C(k'^{-1/2}+n^{-1/8})$$

Using (8.16), along with $k'=[k^{1/4}]$ and $k\le nT$, gives the result. □

9. Proofs of the main theorems

In this section we prove the theorems of Section 2 and Theorem 3.4. As said in Section 5, we can and will assume that the hypotheses H' and K'_r are in force. We also use the notation of Section 8: η^n_i, $\mu^n_i(k)$ and $M^n_t(k)$ of (8.5)–(8.7) and $H^n_t(k)$ of (8.12). We set

$$U^n_t=\frac{1}{n}\sum_{i=1}^{[nt]}Mf_n(X_{(i-1)/n}),\qquad \tilde{U}^n_t=\frac{1}{n}\sum_{i=1}^{[nt]}M\tilde{f}_n(X_{(i-1)/n}),$$

$$\bar{U}^n_t=\frac{1}{n}\sum_{i=1}^{[nt]}M_nf_n(X_{(i-1)/n}),$$

so that we have, for all k:

$$\begin{aligned} V(n,f_n)-U^n&=M^n(k)/\sqrt{n}+(\bar{U}^n-U^n)-H^n(k)/\sqrt{n}\\ \sqrt{n}(V(n,f_n)-U^n)&=M^n(k)+\tilde{U}^n+\sqrt{n}(\bar{U}^n-U^n-\tilde{U}^n/\sqrt{n})-H^n(k)\end{aligned} \tag{9.1}$$

Proof of Theorem 2.1. We assume K_1' and take $k_n = [n^{1/3}]$.

Since $M^n(k_n)$ is a square-integrable martingale, we have by Doob's inequality and expressions (8.7) and (8.15):

$$\mathrm{E}_x(\sup_{t\le T}|M_t^n(k_n)|^2) \le 4\mathrm{E}_x(|M_T^n(k_n)|^2) = \frac{4}{n}\sum_{i=1}^{nT}\mathrm{E}_x(|\mu_i^n(k_n)|^2) \le Cn^{1/3}.$$

Expression (8.17) yields $|H_t^n(k_n)\sqrt{n}| \le Cn^{-1/6}$, and (8.3) yields $\sup_{t\le T}|U_t^n - \bar{U}_t^n| \le C/\sqrt{n}$, so that by (9.1) we obtain

$$\sup_{t\le T}|V(n,f_n)_t - U_t^n| \to 0 \qquad \text{in } \mathbb{L}^2(P_x). \tag{9.2}$$

Now, (8.2) and (5.12) imply that $\sup_{t\le T}|U_t^n - \int_0^t Mf_n(X_s)\mathrm{d}s| \to 0$ in $\mathbb{L}^2(P_x)$. We can easily check from (2.2) (using K_1' again) that $Mf_n \to Mf$ pointwise, and $|Mf_n| \le C$, hence we also have $\sup_{t\le T}|U_t^n - \int_0^t Mf(X_s)\mathrm{d}s| \to 0$ in $\mathbb{L}^2(P_x)$. This and (9.2) yield the result. □

Remark 9.1. Supose that K_1' holds, except that the sequence f_n does not converge to a limit f. The previous proof for (9.2) remains valid.

Proof of Theorem 2.2. We assume K_2' and take $k_n = [n^{3/4}]$.

(a) In view of (8.2) and (5.13), the processes $\sqrt{n}(U_t^n - \int_0^t Mf_n(X_s)\mathrm{d}s)$ converge in law to 0, so it is enough to prove the stable convergence in law of $\sqrt{n}(V(n,f_n) - U^n)$. By (8.4), $|\sqrt{n}(\bar{U}_t^n - U_t^n - \tilde{U}_t^n/\sqrt{n}| \le C/\sqrt{n}$, while by (8.24) we have $|H_t^n(k_n)| \le Cn^{-1/4}$. By (5.14), $\sup_{t\le T}|\tilde{U}_t^n - \int_0^t M\tilde{f}_n(X_s)\mathrm{d}s| \to 0$ in $\mathbb{L}^2(P_x)$, and we deduce that $\sup_{t\le T}|\tilde{U}_t^n - \int_0^t M\tilde{f}(X_s)\mathrm{d}s| \to 0$ in $\mathbb{L}^2(P_x)$ exactly as in the previous proof. Therefore,

$$\sup_{t\le T}|\tilde{U}_t^n + \sqrt{n}(\bar{U}_t^n - U_t^n - \tilde{U}_t^n/\sqrt{n}) + H_t^n(k_n) - \int_0^t M\tilde{f}(X_s)\mathrm{d}s| \to 0 \qquad \text{in } \mathbb{L}^2(P_x).$$

It is known that if a sequence of processes Z^n converges stably in law to some limit Z and if another sequence of processes Y^n converges locally uniformly in probability to Y, then the sums $Y^n + Z^n$ converge stably in law to $Y + Z$. Thus, in view of (9.1), it remains to prove that (with the notation of (2.13))

$$M^n(k_n) \to U := \int_0^{\cdot} Rf(X_s)\mathrm{d}W_s + B' \qquad \text{stably in law.} \tag{9.3}$$

(b) The process U of (9.3) is a martingale on an extended space, which is characterized by its brackets

$$B_t := \langle U, W\rangle_t = \int_0^t Rf(X_s)\mathrm{d}s, \qquad C_t := \langle U, U\rangle_t = \int_0^t \Delta(f,f)(X_s,\beta)\mathrm{d}s \tag{9.4}$$

(use (2.13)). On the other hand, if $W_t^n = W_{[nt]/n}$, both processes W^n and $M^n(k_n)$ are square-integrable martingales with respect to the filtration $(\mathscr{F}_{[nt]/n})_{t\geq 0}$, with brackets

$$B_t^n := \langle M^n(k_n), W^n\rangle_t = \frac{1}{n}\sum_{i=1}^{[nt]} \mathrm{E}_{X_{(i-1)/n}}(\mu_1^n(k_n)\sqrt{n}W_{1/n}) \tag{9.5}$$

$$C_t^n := \langle M^n(k_n), M^n(k_n)\rangle_t = \frac{1}{n}\sum_{i=1}^{[nt]} \mathrm{E}_{X_{(i-1)/n}}(\mu_1^n(k_n)^2). \tag{9.6}$$

Now, following Genon-Catalot and Jacod (1993, Section 5.c), as soon as the following convergences in P_x-probability (for all t) hold:

$$B_t^n \to B_t, \qquad C_t^n \to C_t, \qquad n^{-2}\sum_{i=1}^{[nt]} \mathrm{E}_{X_{(i-1)/n}}(\mu_1^n(k_n)^4) \to 0, \tag{9.7}$$

we have convergence in law under P_x of the pair $(M^n(k_n), W^n)$ to the pair (U, W), where U is as in (9.3). Since W^n converges locally uniformly in time for all ω to W, we also have convergence in law of $(M^n(k_n), W)$ to (U, W), and thus $\mathrm{E}_x(\Phi\zeta M^n(k_n))\Psi(W)) \to \bar{\mathrm{E}}_x(\Phi(U)\Psi(W))$ for all continuous bounded functions Φ, Ψ on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. But any bounded random variable Z on $(\Omega, \mathscr{F}_\infty, P_x)$ is the $\mathbb{L}^1$-limit of a sequence of variables of the form $\Psi_p(W)$ with Ψ_p continuous, uniformly bounded in p: it readily follows that $\mathrm{E}_x(\Phi(M^n(k_n))Z) \to \bar{\mathrm{E}}_x(\Phi(U)Z)$, that is we have (9.3).

Due to (8.15), the third expression in (9.7) is smaller than C/n, so it remains to prove the first two convergences in (9.7).

(c) With the notation of (8.11), we have $C_t^n = \frac{1}{n}\sum_{i=1}^{[nt]}\delta^n(k_n, X_{(i-1)/n})$. Setting $\tilde{\delta}^n(x,u) = \delta_{f_{n,x},f_{n,x}}(\sigma(x), \beta_n, u)$, we can apply (8.18) to get

$$|C_t^n - \frac{1}{n}\sum_{i=1}^{[nt]}\tilde{\delta}^n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\})| \leq Cn^{-3/32}.$$

Next, (7.6) and (7.9) show that the functions $(x,u,y) \to \tilde{\delta}^n(x,u)$ satisfy K_1', except for the convergence of $\tilde{\delta}^n$ to a limit, and $M\tilde{\delta}^n(x) = \Delta(f_n,f_n)(x,\beta_n)$ by (2.2), (2.7) and (2.11). So Remark 9.1 implies that

$$\sup_{t\leq T}\left|\frac{1}{n}\sum_{i=1}^{[nt]}(\tilde{\delta}^n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}) - \Delta(f_n,f_n)(X_{(i-1)/n}, \beta_n))\right| \to 0$$

in $\mathbb{L}^2(P_x)$. Finally, the functions $(x,u,y) \to \Delta(f_n,f_n)(x,\beta_n)$ also satisfy K_1', with the limiting function $(x,u,y) \to \Delta(f,f)(x,\beta)$ by Lemma 7.3 and (2.11). Hence Theorem 2.1 implies that

$$\sup_{t\leq T}\left|\frac{1}{n}\sum_{i=1}^{[nt]}\Delta(f_n,f_n)(X_{(i-1)/n}, \beta_n) - \int_0^t \Delta(f,f)(X_s,\beta)\mathrm{d}s\right| \to 0$$

in $\mathbb{L}^2(P_x)$. Therefore the second convergence in (9.7) takes place.

(d) Let us denote by $\tilde{\mu}_i^n(k)$ the variable defined by (8.6), with the function f_n substituted by $f'(x,u,y) = y/\sigma(x)$ (the stationary sequence (f') also satisfies K_2', with possibly different constants K, p), and set

$$\tilde{B}_t^n = \frac{1}{n}\sum_{i=1}^{[nt]} \mathrm{E}_{X_{(i-1)/n}}(\mu_1^n(k_n)\tilde{\mu}_1^n(k_n)).$$

Denote also by $C^{+,n}$ (or $C^{-,n}$) the processes defined by (9.6), except that f_n is substituted by $f_n^+ = f_n + f'$ (or $f_n^- = f_n - f'$). If $f^+ = f + f'$ and $f^- = f - f'$, (b) above implies that $C_t^{\pm,n} \to \int_0^t \Delta(f^\pm, f^\pm)(X_s,\beta)\mathrm{d}s$ in P_x-probability. Now, $\Delta(f,f') = \frac{1}{4}(\Delta(f^+,f^+) - \Delta(f^-,f^-))$ and $\tilde{B}^n = \frac{1}{4}(C^{+,n} - C^{-,n})$, so we deduce that

$$\tilde{B}_t^n \to \int_0^t \Delta(f,f')(X_s,\beta)\mathrm{d}s \qquad \text{in } P_x\text{-probability.}$$

Since $\Delta(f,f')(x,\beta) = Rf(x)$ by (2.11) and (7.8), if we prove that

$$\tilde{B}_t^n - B_t^n \to 0 \qquad \text{in } P_x\text{-probability,} \tag{9.9}$$

we will have the first convergence in (9.7), and Theorem 2.2 will be proved.

(e) With f' in place of f_n, we get $\eta_i^n = \gamma_i^n - \mathrm{E}_x(\gamma_i^n|\mathscr{F}_{(i-1)/n})$, where $\gamma_i^n = \sqrt{n}(X_{i/n} - X_{(i-1)/n})/\sigma(X_{(i-1)/n})$ (see (8.1) and (8.5)). Therefore $\tilde{\mu}_1^n(k_n) = \gamma_1^n - \mathrm{E}_{X_0}(\gamma_1^n)$. Then (5.9) yields first $|\mathrm{E}_x(\gamma_1^n)| \le C\sqrt{n}$ and then $\mathrm{E}_x(|\bar{\mu}_1^n(k_n) - \sqrt{n}W_{1/n}|^2) \le C/n$. Using (8.15), we deduce that

$$|\mathrm{E}_x(\mu_1^n(k_n)\tilde{\mu}_1^n(k_n)) - \mathrm{E}_x(\mu_1^n(k_n)\sqrt{n}W_{1/n})| \le C/n.$$

This readily gives (9.9), and we are done. □

Proof of Corollary 2.3. Since $M\tilde{f}_n \to M\tilde{f}$ and $|M\tilde{f}_n| \le C$ (see the previous proofs), both processes $\int_0^t M\tilde{f}_n(X_s)\mathrm{d}s$ and $\frac{1}{n}\sum_{i=1}^{[nt]} M\tilde{f}_n(X_{(i-1)/n})$ converge locally uniformly in time, in P_x-probability, to the process $\int_0^t M\tilde{f}(X_s)\mathrm{d}s$, and the result immediately follows from Theorem 2.2. □

Proof of Theorem 3.4. (a) As in Section 5, we can and will assume that in (3.1) the constants $C_q = C$, $r_q = r$ do not depend on q. Set $v_n(x) = \Gamma\varphi_n(x,\beta_n)$ and $w_n(x) = \tilde{\Gamma}\varphi_n'(x,\beta_n)$. Due to Theorem 3.2, we only have to show the following convergences in P_x-probability, locally uniform in t:

$$n^{-1/2}\sum_{i=1}^{[nt]}(v_n(X_{(i-1)/n}) - v_n(X_{(i-1)/n}^{(\alpha_n)} + \alpha_n/2)) \to 0, \tag{9.10}$$

$$\frac{1}{n}\sum_{i=1}^{[nt]}(w_n(X_{(i-1)/n}) - w_n(X_{(i-1)/n}^{(\alpha_n)})) \to 0. \tag{9.11}$$

By the change of variable $z = y\sigma(x)$ in (3.5), we see that w_n is C^1 with $|w_n'(x)| \le C$, hence $|w_n(x) - w_n(x^{(\alpha_n)})| \le C/\sqrt{n}$ and (9.11) is obvious. Similarly, (3.4) yields that v_n is C^2 with $|v_n^{(i)}(x)| \le C$ for $i = 0, 1, 2$, hence by Taylor's formula

$$|v_n(x) - v_n(x^{(\alpha_n)} + \alpha_n/2) - \alpha_n(\{x/\alpha_n\} - 1/2)v_n'(x)| \le C/n.$$

If $A_t^n = \frac{1}{n}\sum_{i=1}^{[nt]}(\{X_{(i-1)/n}/\alpha_n\} - 1/2)v_n'(X_{(i-1)/n})$, to obtain (9.10) it is enough to show that $A_t^n \to 0$ locally uniformly in P_x-measure. Observe that $A_t^n = V(n, \bar{f}_n)_t$, where $\bar{f}_n(x, u, y) = (u - 1/2)v_n'(x)$ satisfies K_1' except for the convergence of $\bar{f}_n$ to a limit. In view of Remark 9.1, we have, by (9.2):

$$\sup_{t\le T}\left|A_t^n - \frac{1}{n}\sum_{i=1}^{[nt]} M\bar{f}_n(X_{(i-1)/n})\right| \to 0 \qquad \text{in } \mathbb{L}^2(P_x).$$

It remains to observe that $M\bar{f}_n = 0$ (see (2.2)), and we have the result.

(b) Suppose now that $\varphi(x, y) = \varphi(x, -y)$. In view of Corollary 3.3, the limiting process for (3.9) is as described after (3.10). The sequence $\bar{\varphi}_n(x, y) = \varphi_n(x + \alpha_n/2, y)$ also satisfies L_2 with the same limit function φ, so we only have to show that the difference between (3.10) for φ_n and (3.9) for $\bar{\varphi}_n$ goes to 0 in P_x-probability, uniformly in time.

First, L_2 implies that φ is C^1 in the first variable, and we have $\varphi'(x, y) = \varphi'(x, -y)$, so the same change of variable as in the proof of Corollary 3.3 readily shows that $\tilde{\Gamma}\varphi'(x, \rho) = \frac{1}{2}\Gamma\varphi'(x, \rho)$. We also have $\bar{\varphi}_n' \to \varphi'$ pointwise, so L_2 again yields that $\tilde{\Gamma}\bar{\varphi}_n'(x, \beta_n) - \frac{1}{2}\Gamma\bar{\varphi}_n'(x - \alpha_n/2, \beta_n)$ converges locally uniformly in x to $\tilde{\Gamma}\varphi'(x, \beta) - \frac{1}{2}\Gamma(x, \beta) = 0$. Then

$$\frac{1}{n}\sum_{i=1}^{[nt]}\left(\tilde{\Gamma}\bar{\varphi}_n'(X_{(i-1)/n}^{(\alpha_n)}, \beta_n) - \frac{1}{2}\Gamma\bar{\varphi}_n'\left(X_{(i-1)/n}^{(\alpha_n)} + \frac{\alpha_n}{2}, \beta_n\right)\right) \to 0$$

locally uniformly in t. So we can replace the process (3.9) by

$$\sqrt{n}\left(U(n, \bar{\varphi}_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\left(\bar{\varphi}_n - \frac{\alpha_n}{2}\bar{\varphi}_n'\right)\left(X_{(i-1)/n}^{(\alpha_n)} + \frac{\alpha_n}{2}, \beta_n\right)\right). \tag{9.12}$$

Now, Taylor's formula, (3.4) and L_2 yield

$$\left|\Gamma\left(\bar{\varphi}_n - \frac{\alpha_n}{2}\bar{\varphi}_n'\right)(x, \rho) - \Gamma\varphi_n(x, \rho)\right| \le g(x, \rho)\alpha_n^2$$

for some locally bounded function g. So we can replace the process (9.12) by

$$\sqrt{n}\left(U(n, \bar{\varphi}_n)_t - \frac{1}{n}\sum_{i=1}^{[nt]}\Gamma\varphi_n\left(X_{(i-1)/n}^{(\alpha_n)} + \frac{\alpha_n}{2}, \beta_n\right)\right). \tag{9.13}$$

It remains to observe that the processes (9.13) and (3.10) are the same. □

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