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Self-decomposability and Lévy processes in free probability

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In this paper we study the bijection, introduced by Bercovici and Pata, between the classes of infinitely divisible probability measures in classical and in free probability. We prove certain algebraic and topological properties of that bijection (in the present paper denoted Λ), and those properties are then used to show, in particular, that Λ maps the class of classically self-decomposable probability measures onto the natural free counterpart which we define here. Further, we study Lévy processes in free probability and use the properties of Λ to construct stochastic integrals with respect to such processes. In particular, we derive the free analogue of the integral representation of self-decomposable random variables.

Keywords: free additive convolution; free and classical infinite divisibility; free self-decomposability; free Lévy processes; free stochastic integrals; free Ornstein–Uhlenbeck processes

1. Introduction

The concept of self-decomposability of probability measures is due to Paul Lévy. In this paper we study a free analogue of self-decomposability, i.e. a self-decomposability concept formulated in the theory of non-commutative probability and free independence. In that theory, free independence, which was introduced by Voiculescu in 1982 (see Voiculescu 1985), plays a role somewhat similar to that of independence in classical probability.

Voiculescu's pioneering papers have led to an extensive body of work; see the papers cited below and references given therein. For survey material, see Voiculescu *et al.* (1992), Voiculescu (2000), Biane (1998b) and Hiai and Petz (2000). In particular, close analogies as well as intriguing differences between infinite divisibility in the classical and in the non-commutative sense have been uncovered, as we shall indicate.

The origin of the idea of free independence came from Voiculescu's study of the free group von Neumann factors, in which free independence may be naturally encountered. Voiculescu later discovered that free independence also appears in the study of the asymptotic behaviour of *independent* large (Gaussian) random matrices. The starting point of the latter approach to free independence is Wigner's semi-circle law, which occurs as a limiting distribution of eigenvalue distributions of large self-adjoint random matrices with complex entries. This law plays in the theory of free probability the same role as the

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normal or Gaussian law in classical probability. Wigner's approach was through the study of the asymptotic behaviour of the mean values $\mathrm{E}\{\mathrm{tr}_n[(X^{(n)})^p]\}$, where $(X^{(n)})^p$ is the pth power of the $n \times n$ random matrix $X^{(n)}$, and tr_n is the normalized trace on the set $M_n(\mathbb{C})$ of complex $n \times n$ matrices. Voiculescu took the broader view of looking at mean values of the form

$$\mathrm{E}\{\mathrm{tr}_n(X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)})\},$$

where the $X_i^{(n)}$ are independent $n \times n$ random matrices, with i ranging over a finite set $\{1, 2, \ldots, r\}$. Under suitable conditions these moments will, as in the case $r = 1$, converge and determine a limit object, and free independence expresses how the independence of $X_1^{(n)}$, $X_2^{(n)}, \ldots, X_r^{(n)}$ is reflected in properties of that object; see Voiculescu (1991) for the precise formulation. Since in general the matrices do not commute, free independence constitutes a truly 'non-commutative' probabilistic concept. However, the most general and concise way to define free independence is through operator algebra theory, and this links the theory of free independence more closely to quantum mechanics. We refer, in passing, to recent related work on random matrices: see Hiai and Petz (1999), Thorbjørnsen (2000), Geman (1980), Silverstein (1985) and Haagerup and Thorbjørnsen (1998; 1999), and references given therein.

Of key importance to the theory of classical infinite divisibility is the Lévy–Khinchine formula for the logarithm of the characteristic function of an element of the class $\mathcal{ID}(*)$ of infinitely divisible laws. There is a similar formula for free infinite divisibility, and the two Lévy–Khinchine formulae are linked, in a natural way, by a bijection Λ – which we shall refer to as the Bercovici–Pata bijection – between the elements of $\mathcal{ID}(*)$ and the elements of the free counterpart $\mathcal{ID}(\boxplus)$ of $\mathcal{ID}(*)$. In particular, under this bijection the Gaussian law corresponds to the Wigner (or semi-circle) law, and, as was shown by Bercovici and Pata (1999), the class $\mathcal{S}(*)$ of stable laws corresponds to the class $\mathcal{S}(\boxplus)$ of free stable laws.

In this paper we establish some basic properties of Λ. Further, we introduce a concept of free self-decomposability, defined in operator algebraic terms, and show, using those properties, that – with $\mathcal{L}(*)$ denoting the class of self-decomposable laws in the classical sense – the subclass $\Lambda(\mathcal{L}(*))$ of $\mathcal{ID}(\boxplus)$ corresponds exactly to free self-decomposability.

Infinite divisibility is intimately connected to the concept of Lévy processes, i.e. stochastic processes with independent and identically distributed increments. A recent account of the theory of infinite divisibility and Lévy processes is given by Sato (1999); see also Bertoin (1996; 1997; 2000), Le Gall (1999) and Barndorff-Nielsen *et al.* (2001) for more specialized aspects. The properties of Λ, which we derive, also provide the possibility of translating from classical Lévy processes to free counterparts of those processes. We begin an investigation of this. In particular, we establish the existence of stochastic integrals (of functions) with respect to free Lévy processes, and we use this to prove the free analogue of the integral representation of self-decomposable random variables (cf. Wolfe 1982; Jurek and Verwaat 1983). (Stochastic integration (of processes) with respect to free Lévy processes has also, in recent independent work, been introduced by Anshelevich (2001a), using a quite different technique. The latter extends previous work by Biane and Speicher (1998) which established stochastic integration with respect to free Brownian motion.)

The paper is organized as follows. In Section 2 we provide background material from classical probability, free probability and operator theory. Section 2.1 is a short summary of the basic theory of self-decomposability in classical probability. In Section 2.2 we introduce the notion of free independence, and in Section 2.3 we summarize the basic results on free additive convolution and the main tool thereof: the Voiculescu transform. In Section 2.4 we introduce the concept of free infinite divisibility and the free version of the Lévy–Khinchine formula. In the first part of the main body of the paper (Sections 3–4), the exposition is based on the analytical function tools described in Sections 2.3–2.4. In particular, this avoids stating the results in terms of unbounded operators. However, the last two sections of the paper (Sections 5–6) deal with free Lévy processes, which are by definition processes of (generally) unbounded operators. Consequently, we give, in Section 2.5, a short account of the theory of unbounded operators affiliated with a finite von Neumann algebra.

In Section 3 we introduce the Bercovici–Pata bijection Λ, and study its basic properties. We prove that Λ is a homomorphism, in the sense that it preserves the affine structure on the set $\mathcal{ID}(*)$. We also prove that Λ is a homeomorphism with respect to weak convergence of probability measures. These properties of Λ form the key tools for the results derived in the following sections. In Section 4 we define self-decomposability in free probability, and prove that this notion implies free infinite divisibility. Subsequently, we prove that free self-decomposability corresponds exactly to classical self-decomposability via the mapping Λ. In Section 5 we introduce the notion of Lévy processes in free probability, and we show how the mapping Λ gives rise in a natural way to a one-to-one (in law) correspondence between classical and free Lévy processes. Finally, in Section 6, we use the properties of Λ to carry over the construction of stochastic integrals of continuous functions with respect to classical Lévy processes to a corresponding integral with respect to free Lévy processes. We prove then that the integral representation of a classically self-decomposable random variable also holds, verbatim, in the free case. We end by mentioning the connection to Ornstein–Uhlenbeck-type processes.

2. Preliminaries

The present section briefly reviews relevant background material on classical self-decomposability, free independence and operator theory.

2.1. Self-decomposability in classical probability

Denoting, for the classical case, the classes of Gaussian, stable, self-decomposable and infinitely divisible laws by $\mathcal{G}(*)$, $\mathcal{S}(*)$, $\mathcal{L}(*)$ and $\mathcal{ID}(*)$, we have the hierarchy

$$\mathcal{G}(*) \subset \mathcal{S}(*) \subset \mathcal{L}(*) \subset \mathcal{ID}(*). \tag{2.1}$$

Briefly, the stable laws are those that occur as limiting distributions for $n \to \infty$ of affine transformations of sums $X_1 + \ldots + X_n$ of independent and identically distributed random variables (subject to the assumption of uniform asymptotic neglibility). Dropping the assumption of identical distribution, one arrives at the class $\mathcal{L}(*)$. Finally, the class $\mathcal{ID}(*)$ of all infinitely divisible distributions consists of the limiting laws for sums of independent random variables of the form $X_{n1} + \ldots + X_{nk_n}$ (again subject to the assumption of uniform asymptotic neglibility). An alternative characterization of self-decomposability says that (the distribution of) a random variable Y is self-decomposable if and only if, for all c in $]0, 1[$, the characteristic function f of Y (i.e. the Fourier transform of the distribution of Y) can be factorized as

$$f(\zeta) = f(c\zeta)f_c(\zeta), \tag{2.2}$$

for some characteristic function f_c (which then, as can be proved, necessarily corresponds to an infinitely divisible random variable Y_c). In other words, considering Y_c as independent of Y, we have a representation in law

$$Y \stackrel{\mathrm{d}}{=} cY + Y_c.$$

The latter formulation makes the idea of self-decomposability of immediate appeal from the viewpoint of mathematical modelling. Yet another key characterization is given by the following result which was first proved by Wolfe (1982) and later generalized and strengthened by Jurek and Verwaat (1983; cf. also Jurek and Mason, 1993, Theorem 3.3.6): a random variable Y has law in $\mathcal{L}(*)$ if and only if Y has a representation of the form

$$Y \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t, \tag{2.3}$$

where X_t is a Lévy process satisfying $\mathrm{E}\{\log(1 + |X_1|)\} < \infty$. The process $X = (X_t)_{t\geqslant 0}$ is termed the *background driving Lévy process* corresponding to Y.

We mention next how the self-decomposable measures on $\mathbb{R}$ are characterized in terms of their Lévy–Khinchine representation. Recall that a probability measure μ on $\mathbb{R}$ (with the Borel σ-algebra) is infinitely divisible if and only if its characteristic function f_μ has a representation (the Lévy–Khinchine representation) of the form

$$\log f_\mu(u) = \mathrm{i}\gamma u + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma(\mathrm{d}t), \qquad u \in \mathbb{R}, \tag{2.4}$$

where γ is a real constant and σ is a finite measure on $\mathbb{R}$. In that case, the pair (γ, σ) is uniquely determined.

Definition 2.1. *Let μ be an infinitely divisible probability measure on $\mathbb{R}$, and let γ and σ be respectively the (uniquely determined) real constant and finite measure on $\mathbb{R}$ appearing in* (2.4). *We then say that the pair (γ, σ) is the* generating pair *for μ.*

In the literature, there are several alternative ways of writing the above representation. In recent literature, the following version seems to be preferred (see, for example, Sato 1999):

$$\log f_\mu(u) = \mathrm{i}\gamma' u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t))\;\rho(\mathrm{d}t), \qquad u \in \mathbb{R}, \tag{2.5}$$

where γ' is a real constant, a is a non-negative constant and ρ is a measure on $\mathbb{R}$ satisfying the conditions

$$\rho(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1,\ t^2\}\ \rho(\mathrm{d}t) < \infty,$$

i.e. ρ is a *Lévy measure*. The relationship between the two representations (2.4) and (2.5) is as follows:

$$a = \sigma(\{0\}),$$

$$\rho(\mathrm{d}t) = \frac{1+t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t)\ \sigma(\mathrm{d}t),$$

$$\gamma' = \gamma + \int_{\mathbb{R}} t\left(1_{[-1,1]}(t) - \frac{1}{1+t^2}\right)\ \rho(\mathrm{d}t).$$

Now, it follows from Sato (1999, Corollary 15.11) that a probability measure μ on $\mathbb{R}$ is self-decomposable if and only if its Lévy measure is of the form

$$\rho(\mathrm{d}t) = \frac{k(t)}{|t|}\mathrm{d}t,$$

where k: $\mathbb{R} \to \mathbb{R}$ is a non-negative function which is increasing on $]-\infty, 0[$ and decreasing on $]0, \infty[$.

In this paper we shall use mostly the representation (2.4); however, we have also included (2.5) since some of the results we refer to in Section 6 are formulated in terms of that representation.

The class of classically self-decomposable distributions is wide and includes many special cases of theoretical and applied interest. Among the probability laws on the positive half-line, all those which are convolutions of gamma distributions and limit laws of such convolutions are self-decomposable. This group of distributions is referred to as generalized gamma convolutions and it has been extensively studied by Bondesson (1992). (It is noteworthy, in the present context, that Bondesson uses Pick functions, which are essentially Cauchy transforms, as a major tool in his investigations.) An important class of generalized Gamma convolutions are the generalized inverse Gaussian distributions. Assume that $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in [0, \infty[$ satisfy the following conditions: $\lambda < 0 \Rightarrow \delta > 0$; $\lambda = 0 \Rightarrow \gamma, \delta > 0$; and $\lambda > 0 \Rightarrow \gamma > 0$. Then the *generalized inverse Gaussian distribution* $\mathrm{GIG}(\lambda, \delta, \gamma)$ is the distribution on $\mathbb{R}_+$ with density (with respect to Lebesgue measure) given by

$$g(t; \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta\gamma)}\, t^{\lambda-1} \exp\{-\tfrac{1}{2}(\delta^2 t^{-1} + \gamma^2 t)\}, \qquad t \geqslant 0,$$

where K_λ is the modified Bessel function of the third kind and with index λ. For all λ, δ, γ (subject to the above restrictions) $\mathrm{GIG}(\lambda, \delta, \gamma)$ is self-decomposable, and it is not stable unless $\lambda = -\frac{1}{2}$ and $\gamma = 0$. For special choices of the parameters, one obtains the gamma distributions (and hence the exponential and χ^2 distributions), the inverse Gaussian distributions, the reciprocal inverse Gaussian distributions[1] and the reciprocal gamma distributions. As concerns distributions on the whole real line, a particularly important group of examples are the marginal laws of subordinated Brownian motion with drift, when the subordinator process is generated by one of the generalized gamma convolutions. The induced self-decomposability of the marginals follows from a recent result due to Sato (2000).

There is an extensive literature on the theory and applications of stable laws. A standard reference for the theoretical properties is Samorodnitsky and Taqqu (1994), but see also Feller (1971) and Barndorff-Nielsen *et al.* (2001). In comparison, work on self-decomposability has until recently been somewhat limited. However, a comprehensive account of the theoretical aspects of self-decomposability, and indeed of infinte divisibility in general, is now available in Sato (1999). Applications of self-decomposability are discussed, *inter alia*, in Brockwell *et al.* (1982), Barndorff-Nielsen (1998) and Barndorff-Nielsen and Shephard (2001a; 2001b).

2.2. Free independence

Free probability is the term given to the combination of the concept of free independence with non-commutative probability (see Voiculescu 2000). Non-commutative probability is a field of study of probabilistic structures arising out of quantum mechanics. It is not necessary for present purposes to delineate the field further. However, we do need the precise definition of free independence.

Let $\mathcal{H}$ be a (complex) Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the vector space of continuous linear mappings (or operators) $a\colon \mathcal{H} \to \mathcal{H}$. Consider further a state on $\mathcal{B}(\mathcal{H})$, i.e. a positive linear functional $\tau\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ such that $\tau(\mathbf{1}) = 1$, where $\mathbf{1}$ is the identity mapping on $\mathcal{H}$.[2] Given any self-adjoint operator a in $\mathcal{B}(\mathcal{H})$, the spectrum $\mathrm{sp}(a)$ is contained in $\mathbb{R}$, and there exists a unique probability measure μ_a on $\mathbb{R}$, concentrated on $\mathrm{sp}(a)$, satisfying

$$\tau(f(a)) = \int_{\mathbb{R}} f(t)\, \mu_a(\mathrm{d}t), \tag{2.6}$$

for all bounded Borel functions f on $\mathbb{R}$. The measure μ_a is called the (spectral) distribution of a with respect to τ, and we shall also use the notation $\mathcal{L}\{a\}$ (the law of a) for μ_a.

We say that operators $a_1, \ldots, a_r$ in $\mathcal{B}(\mathcal{H})$ are *freely independent* with respect to τ if they satisfy the following condition: for any p in $\mathbb{N}$ and $i_1, \ldots, i_p$ in $\{1, \ldots, r\}$ with $i_1 \neq i_2, \ldots, i_{p-1} \neq i_p$, we have that

$$\tau(Q_1(a_{i_1}) \cdots Q_p(a_{i_p})) = 0,$$

for all polynomials $Q_1, \ldots, Q_p$ in one variable such that

[1] The inverse Gaussian distributions and the reciprocal inverse Gaussian distributions are, respectively, the first and last passage-time distributions to a constant level by Brownian motion with drift.

[2] In quantum physics, τ is of the form $\tau(a) = \mathrm{tr}(\rho a)$, where ρ is a trace class self-adjoint operator on $\mathcal{H}$ with trace 1 which expresses the state of a quantum system, and a would be an observable, i.e. a self-adjoint operator on $\mathcal{H}$, the mean value of the outcome of observing a being $\tau(a) = \mathrm{tr}\{\rho a\}$.

$$\tau(Q_1(a_{i_1})) = \ldots = \tau(Q_p(a_{i_p})) = 0.$$

The relevance of this definition should be evident from the connection to the study of random matrices mentioned in Section 1. In several respects, free independence is conceptually similar to classical independence. For example, if $a_1, a_2, \ldots, a_r$ are freely independent operators and $k \in \{1, 2, \ldots, r-1\}$, then any polynomial in $a_1, \ldots, a_k$ is freely independent of any polynomial in $a_{k+1}, \ldots, a_r$.

2.3. Free additive convolution and the Voiculescu transform

From a probabilistic point of view, free additive convolution may be considered merely as a new type of convolution on the set of probability measures on $\mathbb{R}$. Let a and b be self-adjoint operators in $\mathcal{B}(\mathcal{H})$ and note that $a+b$ is also self-adjoint. Denote the (spectral) distributions of a, b and $a+b$ by μ_a, μ_b and μ_{a+b}. If a and b are freely independent, it is not hard to see that the moments of μ_{a+b} (and hence μ_{a+b} itself) are uniquely determined by μ_a and μ_b. Hence we may write $\mu_a \boxplus \mu_b$ instead of μ_{a+b}, and we say that $\mu_a \boxplus \mu_b$ is the *free additive*[3] *convolution* of μ_a and μ_b.

Since the distribution μ_a of a self-adjoint operator a in $\mathcal{B}(\mathcal{H})$ is a compactly supported probability measure on $\mathbb{R}$, the definition of free additive convolution, stated above, works at most for all compactly supported probability measures on $\mathbb{R}$. On the other hand, given any two compactly supported probability measures μ_1 and μ_2 on $\mathbb{R}$, it follows from a free product construction (see Voiculescu *et al.* 1992) that it is always possible to find a Hilbert space $\mathcal{H}$, a state τ on $\mathcal{B}(\mathcal{H})$ and freely independent, self-adjoint operators a, b in $\mathcal{B}(\mathcal{H})$, such that a and b have distributions μ_1 and μ_2, respectively. Thus, the operation $\boxplus$ introduced above is, in fact, defined on all compactly supported probability measures on $\mathbb{R}$. To extend this operation to all probability measures on $\mathbb{R}$, one needs to consider unbounded self-adjoint operators in a Hilbert space, and then to proceed with a construction similar to that described above. We postpone a detailed discussion of this matter to Section 2.5 (see Remark 2.14), since, for our present purposes, it is possible to study free additive convolution by virtue of the Voiculescu transform, which we introduce next (in fact, one may even define free additive convolution in terms of the Voiculescu transform; see Voiculescu 2000).

By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the set of complex numbers with strictly positive (strictly negative) imaginary part.

Let μ be a probability measure on $\mathbb{R}$, and consider its Cauchy (or Stieltjes) transform $G_\mu\colon \mathbb{C}^+ \to \mathbb{C}^-$ given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu(\mathrm{d}t), \qquad z \in \mathbb{C}^+.$$

Then define the mapping $F_\mu\colon \mathbb{C}^+ \to \mathbb{C}^+$ by

[3] The reason for the term 'additive' is that there exists another convolution operation called *free multiplicative convolution*, which arises naturally out of the non-commutative setting (i.e. the non-commutative multiplication of operators). In the present paper we do not consider free multiplicative convolution.

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \qquad z \in \mathbb{C}^+,$$

and note that F_μ is analytic on $\mathbb{C}^+$. It was proved by Bercovici and Voiculescu (1993, Proposition 5.4 and Corollary 5.5) that there exist positive numbers η and M, such that F_μ has an (analytic) right inverse F_μ^{-1} defined on the region

$$\Gamma_{\eta,M} := \{z \in \mathbb{C} | \, |\mathrm{Re}(z)| \leqslant \eta \, \mathrm{Im}(z), \, \mathrm{Im}(z) > M\}.$$

In other words, there exists an open subset $G_{\eta,M}$ of $\mathbb{C}^+$ such that F_μ is injective on $G_{\eta,M}$ and such that $F_\mu(G_{\eta,M}) = \Gamma_{\eta,M}$.

Now the *Voiculescu transform* ϕ_μ of μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z,$$

on any region of the form $\Gamma_{\eta,M}$, where F_μ^{-1} is defined. It follows from (Bercovici and Voiculescu 1993, Corollary 5.3) that $\mathrm{Im}(F_\mu^{-1}(z)) \leqslant \mathrm{Im}(z)$ and hence $\mathrm{Im}(\phi_\mu(z)) \leqslant 0$ for all z in $\Gamma_{\eta,M}$.

The Voiculescu transform ϕ_μ should be viewed as a modification of Voiculescu's $\mathcal{R}$-transform (see, for example, Voiculescu *et al.* 1992), since we have the correspondence

$$\phi_\mu(z) = \mathcal{R}_\mu\left(\frac{1}{z}\right).$$

A third variant, which seems worth considering (see Remark 4.3 below), is the *free cumulant tranform*, given by

$$\mathcal{C}_\mu(z) = z\mathcal{R}_\mu(z) = z\phi_\mu\left(\frac{1}{z}\right). \tag{2.7}$$

The key property of the Voiculescu transform is the following important result, which shows that the Voiculescu transform can be viewed as a free analogue of the classical cumulant function (the logarithm of the characteristic function); see also Remark 4.3 below. The result was first proved by Voiculescu (1986) for probability measures with compact support, and then by Maassen (1992) in the case where the measures have variance. Finally, Bercovici and Voiculescu (1993) proved the general case.

Theorem 2.2. *Let μ_1 and μ_2 be probability measures on $\mathbb{R}$, and consider their free additive convolution $\mu_1 \boxplus \mu_2$. Then*

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z),$$

for all z in any region $\Gamma_{\eta,M}$ where all three functions are defined.

Remark 2.3. We shall need the fact that a probability measure on $\mathbb{R}$ is uniquely determined by its Voiculescu transform. To see this, suppose μ and μ' are probability measures on $\mathbb{R}$, such that $\phi_\mu = \phi_{\mu'}$ on a region $\Gamma_{\eta,M}$. It follows then that $F_\mu = F_{\mu'}$ on some open subset of $\mathbb{C}^+$, and hence (by analytic continuation), $F_\mu = F_{\mu'}$ on all of $\mathbb{C}^+$. Consequently, μ and μ'

have the same Cauchy (or Stieltjes) transform, and by the Stieltjes inversion formula (see, for example, Chihara 1978, p. 90), this means that $\mu = \mu'$.

Bercovici and Voiculescu (1993, Proposition 5.6) proved the following characterization of Voiculescu transforms:

Theorem 2.4. *Let ϕ be an analytic function defined on a region $\Gamma_{\eta,M}$, for some positive numbers η and M. Then the following assertions are equivalent:*

(i) *There exists a probability measure μ on $\mathbb{R}$, such that $\phi(z) = \phi_\mu(z)$ for all z in a domain $\Gamma_{\eta,M'}$, where $M' \geqslant M$.*

(ii) *There exists a number M' greater than or equal to M, such that:*
   (a) $\mathrm{Im}(\phi(z)) \leqslant 0$ *for all z in $\Gamma_{\eta,M'}$;*
   (b) $\phi(z)/z \to 0$, *as* $|z| \to \infty$, $z \in \Gamma_{\eta,M}$;
   (c) *for any positive integer n and any points $z_1, \ldots, z_n$ in $\Gamma_{\eta,M}$, the $n \times n$ matrix*

$$\left[\frac{z_j - \overline{z_k}}{z_j + \phi(z_j) - \overline{z_k} - \overline{\phi(z_k)}}\right]_{1 \leqslant j,k \leqslant n}$$

is positive definite.

Recall that a sequence (σ_n) of *finite* measures on $\mathbb{R}$ is said to converge weakly to a finite measure σ on $\mathbb{R}$ if

$$\int_{\mathbb{R}} f(t)\, \sigma_n(\mathrm{d}t) \to \int_{\mathbb{R}} f(t)\, \sigma(\mathrm{d}t), \qquad \text{as } n \to \infty, \tag{2.8}$$

for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{C}$. In that case, we write $\sigma_n \xrightarrow{\mathrm{w}} \sigma$, as $n \to \infty$.

Remark 2.5. For later use, we note that since the convergence in (2.8) is with respect to a metric, it follows immediately from the above definition that $\sigma_n \xrightarrow{\mathrm{w}} \sigma$ if and only if any subsequence $(\sigma_{n'})$ has a subsequence $(\sigma_{n''})$ which converges weakly to σ. This follows also from the fact that weak convergence can be viewed as convergence with respect to certain metric on the set of bounded measures on $\mathbb{R}$ (the Lévy metric).

The relationship between weak convergence of probability measures and the Voiculescu transform was settled in (Bercovici and Voiculescu 1993, Proposition 5.7) and (Bercovici and Pata 1996, Proposition 1):

Proposition 2.6. *Let (μ_n) be a sequence of probability measures on $\mathbb{R}$. Then the following assertions are equivalent:*

(i) *The sequence (μ_n) converges weakly to a probability measure μ on $\mathbb{R}$.*

(ii) *There exist positive numbers η and M, and a function ϕ, such that all the functions ϕ, ϕ_{μ_n} are defined on $\Gamma_{\eta,M}$, and such that:*

(a) $\phi_{\mu_n}(z) \to \phi(z)$, *as* $n \to \infty$, *uniformly on compact subsets of* $\Gamma_{\eta,M}$*;*
(b) $\sup_{n\in\mathbb{N}}|\phi_{\mu_n}(z)/z| \to 0$, *as* $|z| \to \infty$, $z \in \Gamma_{\eta,M}$.

(iii) *There exist positive numbers* η *and* M*, such that all the functions* ϕ_{μ_n} *are defined on* $\Gamma_{\eta,M}$*, and such that:*
(a) $\lim_{n\to\infty} \phi_{\mu_n}(\mathrm{i}y)$ *exists for all* y *in* $[M, \infty[$*;*
(b) $\sup_{n\in\mathbb{N}}|\phi_{\mu_n}(\mathrm{i}y)/y| \to 0$, *as* $y \to \infty$.

If conditions (i)–(iii) are satisfied, then $\phi = \phi_\mu$ *on* $\Gamma_{\eta,M}$.

Remark 2.7. *Cumulants I.* Under the assumption of finite moments of all orders, both classical and free convolution can be handled completely by a combinatorial approach based on cumulants. Suppose, for simplicity, that μ is a compactly supported probability measure on $\mathbb{R}$. Then for $n \in \mathbb{N}$, the classical cumulant c_n of μ may be defined as the nth derivative at 0 of the cumulant transform $\log f_\mu$. In other words, we have the Taylor expansion

$$\log f_\mu(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n.$$

Consider further the sequence $(m_n)_{n\in\mathbb{N}_0}$ of moments of μ. Then the sequence (m_n) is uniquely determined by the sequence (c_n) (and vice versa). The formulae determining m_n from (c_n) are generally quite complicated. However, by viewing the sequences (m_n) and (c_n) as multiplicative functions M and C on the lattice of all partitions of $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$ (see, for example, Speicher 1997), the relationship between (m_n) and (c_n) can be elegantly expressed by the formula

$$C = M \star \mathrm{Moeb},$$

where Moeb denotes the Möbius transform and where $\star$ denotes *combinatorial convolution* of multiplicative functions on the lattice of all partitions (see Speicher 1997; Rota 1964; or Barndorff-Nielsen and Cox 1989).

The *free* cumulants (k_n) of μ were introduced by Speicher (1994). They may similarly be defined as the coefficients in the Taylor expansion of the free cumulant transform $\mathcal{C}_\mu$:

$$\mathcal{C}_\mu(z) = \sum_{n=1}^{\infty} k_n z^n$$

(see (2.7)). Viewing (k_n) and (m_n) as multiplicative functions k and m on the lattice of all *non-crossing* partitions of $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, the relationship between (k_n) and (m_n) is expressed by exactly the same formula,

$$k = m \star \mathrm{Moeb}, \tag{2.9}$$

where now $\star$ denotes combinatorial convolution of multiplicative functions on the lattice of all *non-crossing* partitions (see Speicher 1997).

For a family $a_1, a_2, \ldots, a_r$ of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ and a state τ on $\mathcal{B}(\mathcal{H})$, it is also possible to define generalized cumulants which are related to the family of all mixed moments (with respect to τ) of $a_1, a_2, \ldots, a_r$ by a formula similar to (2.9) (see, for example, Speicher 1997). In terms of these multivariate cumulants, free independence of

$a_1, a_2, \ldots, a_r$ has a rather simple formulation, and using this formulation, Speicher (1994) gave a simple and completely combinatorial proof of the fact that the free cumulants (and hence the free cumulant transform) linearize free convolution. A treatment of the theory of classical multivariate cumulants can be found in Barndorff-Nielsen and Cox (1989).

2.4. Infinite divisibility with respect to free additive convolution

In this subsection we recall the definition and some basic facts about infinite divisibility with respect to free additive convolution. By complete analogy with the classical case, a probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if, for any $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}$, such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

It was proved in Pata (1996) that the class $\mathcal{ID}(\boxplus)$ of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$ is closed with respect to weak convergence. For the corresponding classical result, see Gnedenko and Kolmogorov (1968, §17, Theorem 3). As in classical probability, $\boxplus$-infinitely divisible probability measures are characterized as those probability measures that have a (free) Lévy–Khinchine representation:

Theorem 2.8. (Voiculescu 1986; Maassen 1992; Bercovici and Voiculescu 1993). *Let μ be a probability measure on $\mathbb{R}$. Then μ is $\boxplus$-infinitely divisible if and only if there exist a finite measure σ on $\mathbb{R}$ and a real constant γ, such that*

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}t) \tag{2.10}$$

$$= \gamma + \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right)\nu(\mathrm{d}t), \qquad z \in \mathbb{C}^+, \tag{2.11}$$

where $\nu(\mathrm{d}t) = (1+t^2)\sigma(\mathrm{d}t)$. Moreover, for a $\boxplus$-infinitely divisible probability measure μ on $\mathbb{R}$, γ and σ are uniquely determined.

Proof. Note first that (2.11) follows from (2.10) and the elementary formula

$$\frac{1+tz}{(z-t)(1+t^2)} = \frac{1}{z-t} + \frac{t}{1+t^2}.$$

The equivalence between $\boxplus$-infinite divisibility and the existence of a representation in the form (2.10) was proved (in the general case) by Bercovici and Voiculescu (1993, Theorem 5.10). They first proved that μ is $\boxplus$-infinitely divisible if and only if ϕ_μ has an extension to a holomorphic function of the form $\phi\colon \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$, i.e. a Pick function multiplied by -1. Equation (2.10) (and its uniqueness) then follows from the existence (and uniqueness) of the integral representation of Pick functions (see Donoghue 1974, Chapter 2, Theorem I). Compared to the general integral representation for Pick functions just referred to, there is a linear term missing on the right-hand side of (2.10), but this corresponds to the fact that $\phi(\mathrm{i}y)/y \to 0$, as $y \to \infty$, if ϕ is a Voiculescu transform (cf. Theorem 2.4 above). □

Definition 2.9. *Let μ be a $\boxplus$-infinitely divisible probability measure on $\mathbb{R}$, and let γ and σ be respectively the (uniquely determined) real constant and finite measure on $\mathbb{R}$ appearing in (2.10). We then say that the pair (γ, σ) is the* free generating pair *for μ.*

The next result, due to Bercovici and Pata (2000), is the free analogue of Khinchine's characterization of classically infinitely divisible probability measures. It plays an important role in Section 4 of the present paper.

Definition 2.10. *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, and let*

$$A = \{\mu_{nj} | n \in \mathbb{N}, j \in \{1, 2, \ldots, k_n\}\}$$

be an array of probability measures on $\mathbb{R}$. We say that A is a null array *if the following condition is fulfilled:*

$$\forall\epsilon > 0\colon \lim_{n\to\infty} \max_{1\leqslant j\leqslant k_n} \mu_{nj}(\mathbb{R}\setminus[-\epsilon, \epsilon]) = 0.$$

Theorem 2.11. *Let $\{\mu_{nj} | n \in \mathbb{N}, j \in \{1, 2, \ldots, k_n\}\}$ be a null array of probability measures on $\mathbb{R}$, and let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Let δ_{c_n} denote the Dirac measure at c_n. If the probability measures $\mu_n = \delta_{c_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converge weakly, as $n \to \infty$, to a probability measure μ on $\mathbb{R}$, then μ has to be $\boxplus$-infinitely divisible.*

We recall, finally, the definition of $\boxplus$-stable probability measures. For a probability measure μ on $\mathbb{R}$, we denote by $T(\mu)$ the *type* of μ, i.e. the class of probability measures on $\mathbb{R}$ given by

$$T(\mu) = \{\psi(\mu) \,|\, \psi\colon \mathbb{R} \to \mathbb{R} \text{ is an increasing affine transformation}\}.$$

Exactly as in classical probability theory, a probability measure μ on $\mathbb{R}$ is called $\boxplus$-stable if the class $T(\mu)$ is closed under $\boxplus$. We denote by $\mathcal{S}(\boxplus)$ the class of $\boxplus$-stable probability measures on $\mathbb{R}$.

As was noted in Bercovici and Voiculescu (1993, Section 7), $\boxplus$-stability implies $\boxplus$-infinite divisibility, i.e. we have the inclusion $\mathcal{S}(\boxplus) \subseteq \mathcal{ID}(\boxplus)$, just as in the classical case.

2.5. Unbounded operators affiliated with a W^*-probability space

In this section we give, for the reader's convenience, a brief account of the theory of closed, densely defined operators affiliated with a finite von Neumann algebra. We start by introducing von Neumann algebras. For a detailed introdution to von Neumann algebras, we refer to Kadison and Ringrose (1983; 1986), but Nelson (1974) also has a nice short introduction to the subject. For background material on unbounded operators, see Rudin (1991).

Let $\mathcal{H}$ be a Hilbert space, and consider, as in Section 2.3, the vector space $\mathcal{B}(\mathcal{H})$ of bounded (or continuous) operators $a: \mathcal{H} \to \mathcal{H}$. Recall that composition of operators constitutes a multiplication on $\mathcal{B}(\mathcal{H})$, and that the adjoint operation $a \mapsto a^*$ is an involution on $\mathcal{B}(\mathcal{H})$ (i.e. $(a^*)^* = a$). Altogether $\mathcal{B}(\mathcal{H})$ is a $*$-algebra. (Throughout this section, the $*$ refers to the adjoint operation and not to classical convolution.) For any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{S}'$ the *commutant* of $\mathcal{S}$, i.e.

$$\mathcal{S}' = \{b \in \mathcal{B}(\mathcal{H}) \,|\, by = yb \text{ for all } y \text{ in } \mathcal{S}\}.$$

A *von Neumann algebra* acting on $\mathcal{H}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the multiplicative unit $\mathbf{1}$ of $\mathcal{B}(\mathcal{H})$, and which is closed under the adjoint operation and closed in the weak operator topology (see Kadison and Ringrose 1983, Definition 5.1.1). By von Neumann's fundamental double commutant theorem, a von Neumann algebra may also be characterized as a subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, which is closed under the adjoint operation and equals the commutant of its commutant: $\mathcal{A}'' = \mathcal{A}$.

A trace (or tracial state) on a von Neumann algebra $\mathcal{A}$ is a postive linear functional $\tau: \mathcal{A} \to \mathbb{C}$, satisfying $\tau(\mathbf{1}) = 1$ and $\tau(ab) = \tau(ba)$ for all a, b in $\mathcal{A}$. We say that τ is a normal trace on $\mathcal{A}$ if, in addition, τ is continuous with respect to the weak operator topology on the unit ball of $\mathcal{A}$. We say that τ is faithful if $\tau(a^*a) > 0$ for any non-zero operator a in $\mathcal{A}$.

Throughout this paper, we shall use the terminology W^*-*probability space* for a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $\tau: \mathcal{A} \to \mathbb{C}$ is a faithful normal tracial state on $\mathcal{A}$. In the remaining part of this subsection, $(\mathcal{A}, \tau)$ denotes a W^*-probability space acting on the Hilbert space $\mathcal{H}$.

By a linear operator *in* $\mathcal{H}$ we shall mean a (not necessarily bounded) linear operator $a: \mathcal{D}(a) \to \mathcal{H}$, defined on a subspace $\mathcal{D}(a)$ of $\mathcal{H}$. For an operator a in $\mathcal{H}$, we say that:

- a is *densely defined* if $\mathcal{D}(a)$ is dense in $\mathcal{H}$;
- a is *closed* if the graph $\mathcal{G}(a) = \{(h, ah) \,|\, h \in \mathcal{D}(a)\}$ of a is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$;
- a is *preclosed* if the norm closure $\overline{\mathcal{G}(a)}$ is the graph of a (uniquely determined) operator, denoted $[a]$, in $\mathcal{H}$;
- a is *affiliated with* $\mathcal{A}$ if $au = ua$ for any unitary operator u in the commutant $\mathcal{A}'$.

If a is bounded, a is affiliated with $\mathcal{A}$ if and only if $a \in \mathcal{A}$. In general, a self-adjoint operator a in $\mathcal{H}$ is affiliated with $\mathcal{A}$ if and only if $f(a) \in \mathcal{A}$ for any *bounded Borel* function $f: \mathbb{R} \to \mathbb{C}$ (here $f(a)$ is defined in terms of spectral theory). As in the bounded case, if a is a self-adjoint operator affiliated with $\mathcal{A}$, there exists a unique probability measure μ_a on $\mathbb{R}$, concentrated on the spectrum $\mathrm{sp}(a)$, and satisfying

$$\int_{\mathbb{R}} f(t)\, \mu_a(\mathrm{d}t) = \tau(f(a)),$$

for any bounded Borel function $f: \mathbb{R} \to \mathbb{C}$. We call μ_a the (spectral) distribution of a, and we shall denote it also by $\mathcal{L}\{a\}$. Unless a is bounded, $\mathrm{sp}(a)$ is an unbounded subset of $\mathbb{R}$ and, in general, μ_a is not compactly supported.

By $\overline{\mathcal{A}}$ we denote the set of closed, densely defined operators in $\mathcal{H}$ which are affiliated with $\mathcal{A}$. In general, dealing with unbounded operators is somewhat unpleasant, compared to the bounded case, since one needs constantly to take the domains into account. However, the following two important propositions (cf. Nelson 1974) allow us to deal with operators in $\overline{\mathcal{A}}$ in a quite relaxed manner.

Proposition 2.12. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. If $a, b \in \overline{\mathcal{A}}$, then $a + b$ and ab are densely defined, preclosed operators affiliated with $\mathcal{A}$, and their closures $[a + b]$ and $[ab]$ belong to $\overline{\mathcal{A}}$. Furthermore, $a^* \in \overline{\mathcal{A}}$.*

By virtue of the proposition above, the adjoint operation may be restricted to an involution on $\overline{\mathcal{A}}$, and we may define operations, the *strong sum* and the *strong product*, on $\overline{\mathcal{A}}$, as follows:

$$(a, b) \mapsto [a + b] \quad \text{and} \quad (a, b) \mapsto [ab], \qquad a, b \in \overline{\mathcal{A}}.$$

Proposition 2.13. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Equipped with the adjoint operation and the strong sum and product, $\overline{\mathcal{A}}$ is a $*$-algebra.*

The effect of the above proposition is that, with respect to the adjoint operation and the strong sum and product, we can work with operators in $\overline{\mathcal{A}}$ without worrying about domains etc. So, for example, we have rules like

$$[[a + b]c] = [[ac] + [bc]], \qquad [a + b]^* = [a^* + b^*], \qquad [ab]^* = [b^* a^*],$$

for operators a, b, c in $\overline{\mathcal{A}}$. Note, in particular, that the strong sum of two self-adjoint operators in $\overline{\mathcal{A}}$ is again a self-adjoint operator. In the following, we shall omit the brackets in the notation for the strong sum and product, and it will be understood that all sums and products are formed in the strong sense.

Remark 2.14. If $a_1, a_2, \ldots, a_r$ are self-adjoint operators in $\overline{\mathcal{A}}$, we say that $a_1, a_2 \ldots, a_r$ are *freely independent* if, for any bounded Borel functions $f_1, f_2, \ldots, f_r$: $\mathbb{R} \to \mathbb{R}$, the bounded operators $f_1(a_1), f_2(a_2), \ldots, f_r(a_r)$ in $\mathcal{A}$ are freely independent in the sense defined in Section 2.2. Given any two probability measures μ_1 and μ_2 on $\mathbb{R}$, it follows from a free product construction (see Voiculescu *et al.* 1992), that one can always find a W^*-probability space $(\mathcal{A}, \tau)$ and freely independent self-adjoint operators a and b affiliated with $\mathcal{A}$, such that $\mu_1 = \mathcal{L}\{a\}$ and $\mu_2 = \mathcal{L}\{b\}$. As noted above, for such operators $a + b$ is again a self-adjoint operator in $\overline{\mathcal{A}}$, and, as was proved in Bercovici and Voiculescu 1993, Theorem 4.6), the (spectral) distribution $\mathcal{L}\{a + b\}$ depends only on μ_1 and μ_2. We may thus define the free additive convolution $\mu_1 \boxplus \mu_2$ of μ_1 and μ_2 to be $\mathcal{L}\{a + b\}$.

Next, we shall equip $\overline{\mathcal{A}}$ with a topology, the so-called measure topology, which was introduced by Segal (1953) and later studied by Nelson (1974). For any positive numbers ϵ, δ, we denote by $N(\epsilon, \delta)$ the set of operators $a \in \overline{\mathcal{A}}$ for which there exists an orthogonal projection p in $\mathcal{A}$, satisfying

$$p(\mathcal{H}) \subseteq \mathcal{D}(a), \qquad \|ap\| \leqslant \epsilon, \qquad \tau(p) \geqslant 1 - \delta. \tag{2.12}$$

Definition 2.15. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. The* measure topology *on $\overline{\mathcal{A}}$ is the topology on $\overline{\mathcal{A}}$ for which the sets $N(\epsilon, \delta)$, $\epsilon, \delta > 0$, form a neighbourhood basis for 0.*

It is clear from the definition of the sets $N(\epsilon, \delta)$ that the measure topology satisfies the first axiom of countability. In particular, all convergence statements can be expressed in terms of sequences rather than nets.

Proposition 2.16. (cf. Nelson, 1974). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and consider the $*$-algebra $\overline{\mathcal{A}}$.*

- (i) *Scalar multiplication, the adjoint operation and strong sum and product are all continuous operations with respect to the measure topology. Thus, $\overline{\mathcal{A}}$ is a topological $*$-algebra with respect to the measure topology.*
- (ii) *The measure topology on $\overline{\mathcal{A}}$ is a complete Hausdorff topology.*

We shall note, next, that the measure topology on $\overline{\mathcal{A}}$ is, in fact, the topology for convergence in probability. Recall, first, that for a closed, densely defined operator a in $\mathcal{H}$, we put $|a| = (a^*a)^{1/2}$. In particular, if $a \in \overline{\mathcal{A}}$, then $|a|$ is a self-adjoint operator in $\overline{\mathcal{A}}$ (see Kadison and Ringrose 1986, Theorem 6.1.11), and we may consider the probability measure $\mathcal{L}\{|a|\}$ on $\mathbb{R}$.

Definition 2.17. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. We say that $a_n \to a$* in probability, *as $n \to \infty$, if $|a_n - a| \to 0$ in distribution, i.e. if $\mathcal{L}\{|a_n - a|\} \to \delta_0$ weakly.*

If a and a_n, $n \in \mathbb{N}$, are *self-adjoint* operators in $\overline{\mathcal{A}}$, then, as noted above, $a_n - a$ is self-adjoint for each n, and $\mathcal{L}\{|a_n - a|\}$ is the transformation of $\mathcal{L}\{a_n - a\}$ by the mapping $t \mapsto |t|$, $t \in \mathbb{R}$. In this case, it follows thus that $a_n \to a$ in probability if and only if $a_n - a \to 0$ in distribution, i.e. if and only if $\mathcal{L}\{a_n - a\} \to \delta_0$ weakly.

From the definition of $\mathcal{L}\{|a_n - a|\}$, it follows immediately that we have the following characterization of convergence in probability:

Lemma 2.18. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. Then $a_n \to a$ in probability if and only if*

$$\forall \epsilon > 0\colon \tau[1_{]\epsilon,\infty[}(|a_n - a|)] \to 0, \qquad \text{as } n \to \infty.$$

Proposition 2.19. (cf. Terp 1981). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Then for any positive numbers ϵ, δ, we have*

$$N(\epsilon, \delta) = \{a \in \overline{\mathcal{A}} |\ \tau\ [1_{]\epsilon,\infty[}(|a|)] \leqslant \delta\}, \tag{2.13}$$

where $N(\epsilon, \delta)$ is defined via (2.12). *In particular, a sequence a_n in $\overline{\mathcal{A}}$ converges, in the measure topology, to an operator a in $\overline{\mathcal{A}}$ if and only if $a_n \to a$ in probability.*

Proof. The last statement of the proposition follows immediately from (2.13) and Lemma

2.18. To prove (2.13), note first that by considering the polar decomposition of an operator a in $\overline{\mathcal{A}}$ (cf. Kadison and Ringrose 1986, Theorem 6.1.11), it follows that $N(\epsilon, \delta) = \{a \in \overline{\mathcal{A}} \,|\, |a| \in N(\epsilon, \delta)\}$. From this, the inclusion $\supseteq$ in (2.13) follows easily. Regarding the reverse inclusion, suppose $a \in N(\epsilon, \delta)$, and let p be a projection in $\mathcal{A}$, such that (2.12) is satisfied with a replaced by $|a|$. Then, using spectral theory, it can be shown that the ranges of the projections p and $1_{]\epsilon,\infty[}(|a|)$ only have 0 in common. This implies that $\tau[1_{]\epsilon,\infty[}(|a|)] \leqslant \tau(\mathbf{1} - p) \leqslant \delta$. We refer to Terp (1981) for further details. □

Finally, we shall need the fact that convergence in probability also implies convergence in distribution in the non-commutative setting. The key point in the proof given below is that weak convergence can be expressed in terms of the Cauchy transform (cf. Maassen 1992, Theorem 2.5).

Proposition 2.20. *Let (a_n) be a sequence of self-adjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and assume that a_n converges in probability, as $n \to \infty$, to a self-adjoint operator a affiliated with $(\mathcal{A}, \tau)$. Then $a_n \to a$ in distribution too, i.e. $\mathcal{L}\{a_n\} \xrightarrow{\text{w}} \mathcal{L}\{a\}$, as $n \to \infty$.*

Proof. Let x, y be real numbers such that $y > 0$, and put $z = x + \mathrm{i}y$. Then define the function $f_z \colon \mathbb{R} \to \mathbb{C}$ by

$$f_z(t) = \frac{1}{t - z} = \frac{1}{(t - x) - \mathrm{i}y}, \qquad t \in \mathbb{R},$$

and note that f_z is continuous and bounded with $\sup_{t\in\mathbb{R}} |f_z(t)| = y^{-1}$. Thus, we may consider the bounded operators $f_z(a_n)$, $f_z(a) \in \mathcal{A}$. Note then that (using strong products and sums),

$$\begin{aligned} f_z(a_n) - f_z(a) &= (a_n - z\mathbf{1})^{-1} - (a - z\mathbf{1})^{-1} \\ &= (a_n - z\mathbf{1})^{-1}((a - z\mathbf{1}) - (a_n - z\mathbf{1}))(a - z\mathbf{1})^{-1} \\ &= (a_n - z\mathbf{1})^{-1}(a - a_n)(a - z\mathbf{1})^{-1}. \end{aligned} \tag{2.14}$$

Now, given any positive numbers ϵ, δ, we may choose $N \in \mathbb{N}$ such that $a_n - a \in N(\epsilon, \delta)$, whenever $n \geqslant N$. Moreover, since $\|f_z(a_n)\|$, $\|f_z(a)\| \leqslant y^{-1}$, we have that $f_z(a_n)$, $f_z(a) \in N(y^{-1}, 0)$. Using the rule $N(\epsilon_1, \delta_1)N(\epsilon_2, \delta_2) \subseteq N(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$, which holds for all ϵ_1, ϵ_2 in $]0, \infty[$ and δ_1, δ_2 in $[0, \infty[$ (see Nelson 1974, formula 17′), it follows from (2.14) that $f_z(a_n) - f_z(a) \in N(\epsilon y^{-2}, \delta)$ whenever $n \geqslant N$. We may thus conclude that $f_z(a_n) \to f_z(a)$ in the measure topology, i.e. that $\mathcal{L}\{|f_z(a_n) - f_z(a)|\} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$. Using the Cauchy–Schwarz inequality for τ, it now follows that

$$|\tau(f_z(a_n) - f_z(a))|^2 \leqslant \tau(|f_z(a_n) - f_z(a)|^2) \cdot \tau(\mathbf{1}) = \int_0^\infty t^2 \mathcal{L}\{|f_z(a_n) - f_z(a)|\}(\mathrm{d}t) \to 0,$$

as $n \to \infty$, since $\operatorname{supp}(\mathcal{L}\{|f_z(a_n) - f_z(a)|\}) \subseteq [0, 2y^{-1}]$ for all n, and since $t \mapsto t^2$ is a continuous bounded function on $[0, 2y^{-1}]$.

Finally, let G_n and G denote the Cauchy transforms for $\mathcal{L}\{a_n\}$ and $\mathcal{L}\{a\}$, respectively. From what we have established above, it follows that

$$G_n(z) = -\tau(f_z(a_n)) \to -\tau(f_z(a)) = G(z), \qquad \text{as } n \to \infty,$$

for any complex number $z = x + \mathrm{i}y$ for which $y > 0$. By Maassen (1992, Theorem 2.5), this means that $\mathcal{L}\{a_n\} \stackrel{\mathrm{w}}{\to} \mathcal{L}\{a\}$, as desired. □

3. The Bercovici–Pata bijection

The bijection defined next was introduced by Bercovici and Pata (1999).

Definition 3.1. *By the* Bercovici–Pata bijection Λ*:* $\mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ *we denote the mapping defined as follows. Let* μ *be a measure in* $\mathcal{ID}(*)$*, and consider its generating pair* (γ, σ) (*see Definition 2.1*)*. Then* $\Lambda(\mu)$ *is the measure in* $\mathcal{ID}(\boxplus)$ *that has* (γ, σ) *as free generating pair* (*see Definition 2.9*)*.*

Since the $*$-infinitely divisible ($\boxplus$-infinitely divisible) probability measures on $\mathbb{R}$ are exactly those measures that have a unique Lévy–Khinchine representation (unique free Lévy–Khinchine representation), it follows immediately that Λ is a well-defined bijection between $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$. In this section we shall study some of the algebraic and topological properties of Λ.

Let ν be a measure on $\mathbb{R}$. Then for any constant c in $\mathbb{R}\backslash\{0\}$, we denote by $D_c\nu$ the measure on $\mathbb{R}$ given by

$$D_c\nu(B) = \nu(c^{-1}B),$$

for any Borel set B. Moreover, we put $D_0\nu = \delta_0$, the Dirac measure at 0. Thus, using integration terminology, we have $D_c\nu(\mathrm{d}t) = \nu(c^{-1}\mathrm{d}t)$, whenever $c \neq 0$.

The following lemma is contained (implicitly) in Feller (1971, Section XVII.8). Since the lemma plays an important role in the proof of Theorem 3.5 below, and for the sake of completeness, we include a proof.

Lemma 3.2. *Let* μ *be a* $*$*-infinitely divisible probability measure on* $\mathbb{R}$ *with Lévy–Khinchine representation given by*

$$\begin{aligned}\log f_\mu(u) &= \mathrm{i}\gamma u + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma(\mathrm{d}t)\\ &= \mathrm{i}\gamma u + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\right)\frac{1}{t^2}\nu(\mathrm{d}t), \qquad u \in \mathbb{R},\end{aligned}$$

where γ *is a real constant,* σ *is a finite measure on* $\mathbb{R}$ *and* $(1+t^2)\sigma(\mathrm{d}t) = \nu(\mathrm{d}t)$*. Then for any* c *in* $\mathbb{R}$ *the Lévy–Khinchine representation for* $D_c\mu$ *is given by*

$$\begin{aligned}\log f_{D_c\mu}(u) &= \mathrm{i}\rho_c u + c^2\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{1}{t^2}\,D_c\nu(\mathrm{d}t)\\ &= \mathrm{i}\rho_c u + \int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{c^2+t^2}{t^2}\,D_c\sigma(\mathrm{d}t), \qquad u\in\mathbb{R}, \qquad (3.1)\end{aligned}$$

where

$$\rho_c = \gamma c + c(1-c^2)\int_{\mathbb{R}}\frac{t}{1+(ct)^2}\,\sigma(\mathrm{d}t).$$

Proof. We note first that the second equality in (3.1) follows from the first by a standard calculation. To prove the first equality in (3.1), note that for any u in $\mathbb{R}$,

$$\begin{aligned}\log f_{D_c\mu}(u) &= \log\Big(\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}ut}\,D_c\mu(\mathrm{d}t)\Big) = \log\Big(\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}cut}\mu(\mathrm{d}t)\Big) = \log f_\mu(cu)\\ &= \mathrm{i}\gamma(cu) + \int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}(cu)t} - 1 - \frac{\mathrm{i}(cu)t}{1+t^2}\Big)\frac{1}{t^2}\nu(\mathrm{d}t),\end{aligned}$$

and that

$$\begin{aligned}c^2\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{1}{t^2}\,D_c\nu(\mathrm{d}t) &= c^2\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}u(ct)} - 1 - \frac{\mathrm{i}u(ct)}{1+(ct)^2}\Big)\frac{1}{(ct)^2}\nu(\mathrm{d}t)\\ &= \int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}cut} - 1 - \frac{\mathrm{i}cut}{1+(ct)^2}\Big)\frac{1}{t^2}\nu(\mathrm{d}t).\end{aligned}$$

Therefore,

$$\begin{aligned}\log f_{D_c\mu}(u) &- c^2\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{1}{t^2}\,D_c\nu(\mathrm{d}t)\\ &= \mathrm{i}\gamma cu + \int_{\mathbb{R}}\Big[\Big(\mathrm{e}^{\mathrm{i}cut} - 1 - \frac{\mathrm{i}cut}{1+t^2}\Big) - \Big(\mathrm{e}^{\mathrm{i}cut} - 1 - \frac{\mathrm{i}cut}{1+(ct)^2}\Big)\Big]\frac{1}{t^2}\nu(\mathrm{d}t)\\ &= \mathrm{i}u\Big(\gamma c + c\int_{\mathbb{R}}\Big(\frac{t}{1+(ct)^2} - \frac{t}{1+t^2}\Big)\frac{1}{t^2}\nu(\mathrm{d}t)\Big)\\ &= \mathrm{i}\rho_c u,\end{aligned}$$

where ρ_c is a constant (not depending on u). Since

$$\frac{t}{1+(ct)^2} - \frac{t}{1+t^2} = \frac{(1-c^2)t^3}{(1+(ct)^2)(1+t^2)},$$

we find that

$$\rho_c = \gamma c + c\int_{\mathbb{R}} \left(\frac{(1-c^2)t^3}{(1+(ct)^2)(1+t^2)} \right) \frac{1}{t^2} \nu(\mathrm{d}t) = \gamma c + c(1-c^2)\int_{\mathbb{R}} \frac{t}{1+(ct)^2} \sigma(\mathrm{d}t),$$

and this completes the proof. □

Our next objective is to prove the free analogue of Lemma 3.2. We start with the following:

Lemma 3.3. *Let μ be a probability measure on $\mathbb{R}$, and let η and M be positive numbers such that the Voiculescu transform ϕ_u is defined on $\Gamma_{\eta,M}$ (see Section 2.3). Then, for any constant c in $\mathbb{R}\backslash\{0\}$, $\phi_{D_c\mu}$ is defined on $|c|\Gamma_{\eta,M} = \Gamma_{\eta,|c|M}$, and:*

(i) *if $c > 0$, then $\phi_{D_c\mu}(z) = c\phi_\mu(c^{-1}z)$ for all z in $c\Gamma_{\eta,M}$;*
(ii) *if $c < 0$, then $\phi_{D_c\mu}(z) = \overline{c\phi_\mu(c^{-1}\bar{z})}$ for all z in $|c|\Gamma_{\eta,M}$.*

In particular, for a constant c in $[-1, 1]$, the domain of $\phi_{D_c\mu}$ contains the domain of ϕ_μ.

Proof. (i) This is a special case of Bercovici and Voiculescu (1993, Lemma 7.1).

(ii) Note first that, by virute of (i), it suffices to prove (ii) in the case $c = -1$. We start by noting that the Cauchy transform G_μ (see Section 2.3) is actually well defined for all z in $\mathbb{C}\backslash\mathbb{R}$ (even for all z outside $\mathrm{supp}(\mu)$), and that $G_\mu(\bar{z}) = \overline{G_\mu(z)}$, for all such z. Similarly, F_μ is defined for all z in $\mathbb{C}\backslash\mathbb{R}$, and $F_\mu(z) = \overline{F_\mu(\bar{z})}$, for such z.

Note next that, for any z in $\mathbb{C}\backslash\mathbb{R}$, $G_{D_{-1}\mu}(z) = -G_\mu(-z)$, and consequently

$$F_{D_{-1}\mu}(z) = -F_\mu(-z) = -\overline{F_\mu(-\bar{z})}.$$

Now, since $-\overline{\Gamma_{\eta,M}} = \Gamma_{\eta,M}$, it follows from the equation above, that $F_{D_{-1}\mu}$ has a right inverse on $\Gamma_{\eta,M}$, given by $F^{-1}_{D_{-1}\mu}(z) = -\overline{F^{-1}_\mu(-\bar{z})}$, for all z in $\Gamma_{\eta,M}$. Consequently, for z in $\Gamma_{\eta,M}$, we have

$$\phi_{D_{-1}\mu}(z) = F^{-1}_{D_{-1}\mu}(z) - z = -\overline{F^{-1}_\mu(-\bar{z})} - z = -\overline{(F^{-1}_\mu(-\bar{z}) - (-\bar{z}))} = -\overline{\phi_\mu(-\bar{z})},$$

as desired. □

Lemma 3.4. *Let μ be a $\boxplus$-infinitely divisible probability measure on $\mathbb{R}$ with free Lévy–Khinchine representation given by*

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} \sigma(\mathrm{d}t) = \gamma + \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) \nu(\mathrm{d}t), \qquad z \in \mathbb{C}^+,$$

where γ is a real constant, σ is a finite measure on $\mathbb{R}$ and $\nu(\mathrm{d}t) = (1+t^2)\sigma(\mathrm{d}t)$. Then, for any c in $\mathbb{R}$, the free Lévy–Khinchine representation for $D_c\mu$ is given by

$$\begin{aligned}\phi_{D_c\mu}(z) &= \rho_c + c^2\int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) D_c\nu(\mathrm{d}t) \\ &= \rho_c + \int_{\mathbb{R}} \left(\frac{1+tz}{z-t} \right) \left(\frac{c^2+t^2}{1+t^2} \right) D_c\sigma(\mathrm{d}t), \qquad (3.2)\end{aligned}$$

where

$$\rho_c = \gamma c + c(1-c^2)\int_{\mathbb{R}} \frac{t}{1+(ct)^2}\,\sigma(\mathrm{d}t).$$

Proof. Note first that the second equality in (3.2) follows easily from the first by a standard calculation.

We start by proving the first equality in (3.2) in the case where $c > 0$. Note for this case that by Lemma 3.3,

$$\begin{aligned}\phi_{D_c\mu}(z) = c\phi_\mu(c^{-1}z) &= c\gamma + c\int_{\mathbb{R}}\Big(\frac{1}{c^{-1}z-t} + \frac{t}{1+t^2}\Big)\nu(\mathrm{d}t)\\ &= c\gamma + \int_{\mathbb{R}}\Big(\frac{c^2}{z-ct} + \frac{ct}{1+t^2}\Big)\nu(\mathrm{d}t).\end{aligned}$$

Note next that

$$\begin{aligned}c^2\int_{\mathbb{R}}\Big(\frac{1}{z-t} + \frac{t}{1+t^2}\Big)D_c\nu(\mathrm{d}t) &= c^2\int_{\mathbb{R}}\Big(\frac{1}{z-ct} + \frac{ct}{1+(ct)^2}\Big)\nu(\mathrm{d}t)\\ &= \int_{\mathbb{R}}\Big(\frac{c^2}{z-ct} + \frac{c^3t}{1+(ct)^2}\Big)\nu(\mathrm{d}t).\end{aligned}$$

From the two calculations above, it follows that

$$\phi_{D_c\mu}(z) - c^2\int_{\mathbb{R}}\Big(\frac{1}{z-t} + \frac{t}{1+t^2}\Big)D_c\nu(\mathrm{d}t) = c\gamma + \int_{\mathbb{R}}\Big(\frac{ct}{1+t^2} - \frac{c^3t}{1+(ct)^2}\Big)\nu(\mathrm{d}t) = \rho_c,$$

where ρ_c is a constant (not depending on z). Using the equality

$$\frac{ct}{1+t^2} - \frac{c^3t}{1+(ct)^2} = \frac{c(1-c^2)t}{(1+t^2)(1+(ct)^2)},$$

it now follows that

$$\rho_c = \gamma c + \int_{\mathbb{R}} \frac{c(1-c^2)t}{(1+t^2)(1+(ct)^2)}\nu(\mathrm{d}t) = \gamma c + c(1-c^2)\int_{\mathbb{R}}\frac{t}{1+(ct)^2}\sigma(\mathrm{d}t). \tag{3.3}$$

This completes the proof in the case $c > 0$.

It remains to consider the case where $c \in\,]-\infty, 0]$. Note here that the case $c = 0$ follows trivially. We proceed to the case $c = -1$. By Lemma 3.3, we obtain that

$$\begin{aligned}
\phi_{D_{-1}\mu}(z) = -\overline{\phi_\mu(-\bar{z})} &= \overline{-\gamma - \int_{\mathbb{R}} \left(\frac{1}{-\bar{z} - t} + \frac{t}{1+t^2} \right) \nu(\mathrm{d}t)} \\
&= -\gamma - \int_{\mathbb{R}} \left(\frac{1}{-z - t} + \frac{t}{1+t^2} \right) \nu(\mathrm{d}t) \\
&= -\gamma + \int_{\mathbb{R}} \left(\frac{1}{z - (-t)} + \frac{-t}{1+(-t)^2} \right) \nu(\mathrm{d}t) \\
&= -\gamma + \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{1+t^2} \right) D_{-1}\nu(\mathrm{d}t),
\end{aligned}$$

where we have used the fact that γ is real. The above calculation shows that the lemma holds for $c = -1$. Finally, for general c in $]-\infty, 0[$, note that $D_c\mu = D_{|c|}D_{-1}\mu$, and therefore, by virtue of the cases $c = -1$ and $c > 0$, it follows that

$$\begin{aligned}
\phi_{D_c\mu}(z) &= \rho_c + |c|^2 \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{1+t^2} \right) D_{|c|}D_{-1}\nu(\mathrm{d}t) \\
&= \rho_c + c^2 \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{1+t^2} \right) D_c\nu(t),
\end{aligned}$$

where (cf. (3.3)),

$$\begin{aligned}
\rho_c &= (-\gamma)|c| + \int_{\mathbb{R}} \frac{|c|(1-|c|^2)t}{(1+t^2)(1+(|c|t)^2)} D_{-1}\nu(\mathrm{d}t) = \gamma c + \int_{\mathbb{R}} \frac{c(1-c^2)t}{(1+t^2)(1+(ct)^2)} \nu(\mathrm{d}t) \\
&= \gamma c + c(1-c^2) \int_{\mathbb{R}} \frac{t}{1+(ct)^2} \sigma(\mathrm{d}t).
\end{aligned}$$

This concludes the proof. □

Theorem 3.5. *The Bercovici–Pata bijection* $\Lambda \colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$, *has the following* (*algebraic*) *properties:*

(i) *If* $\mu_1, \mu_2 \in \mathcal{ID}(*)$, *then* $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.
(ii) *If* $\mu \in \mathcal{ID}(*)$ *and* $c \in \mathbb{R}$, *then* $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.
(iii) *For any constant* c *in* $\mathbb{R}$, *we have* $\Lambda(\delta_c) = \delta_c$.

Proof. (i) For j in $\{1, 2\}$, let (γ_j, σ_j) be the generating pair for μ_j (so that γ_j is a real constant and σ_j is a finite measure on $\mathbb{R}$). Then since

$$\log f_{\mu_1 * \mu_2}(u) = \log f_{\mu_1}(u) + \log f_{\mu_2}(u),$$

it follows readily that the generating pair for $\mu_1 * \mu_2$ is $(\gamma_1 + \gamma_2, \sigma_1 + \sigma_2)$. Similarly, since the free generating pair for $\Lambda(\mu_j)$ is (γ_j, σ_j), and since

$$\phi_{\Lambda(\mu_1) \boxplus \Lambda(\mu_2)}(z) = \phi_{\Lambda(\mu_1)}(z) + \phi_{\Lambda(\mu_2)}(z),$$

it follows that the free generating pair for $\Lambda(\mu_1) \boxplus \Lambda(\mu_2)$ is $(\gamma_1 + \gamma_2, \sigma_1 + \sigma_2)$. By the definition of Λ, it thus follows that $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$, as desired.

(ii) Suppose μ has generating pair (γ, σ). Then (γ, σ) is the free generating pair for $\Lambda(\mu)$. Now, by Lemma 3.2, the generating pair for $D_c\mu$ is $(\rho_c, (c^2 + t^2)/(1 + t^2) \cdot D_c\sigma(\mathrm{d}t))$, where

$$\rho_c = \gamma c + c(1 - c^2) \int_{\mathbb{R}} \frac{t}{1 + (ct)^2} \sigma(\mathrm{d}t).$$

According to Lemma 3.4, that same pair is also the *free* generating pair for $D_c\Lambda(\mu)$. Hence, by definition of Λ, $\Lambda(D_c\mu) = D_c\Lambda(\mu)$, as desired.

(iii) This follows from the fact that $(c, 0)$ is both the generating pair and the free generating pair for δ_c. $\square$

Corollary 3.6. *The bijection* $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ *is invariant under affine transformations, i.e. if* $\mu \in \mathcal{ID}(*)$ *and* $\psi\colon \mathbb{R} \to \mathbb{R}$ *is an affine transformation, then*

$$\Lambda(\psi(\mu)) = \psi(\Lambda(\mu)).$$

Proof. Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be an affine transformation, i.e. $\psi(t) = ct + d$, $t \in \mathbb{R}$, for some constants c, d in $\mathbb{R}$. Then for a probability measure μ on $\mathbb{R}$, $\psi(\mu) = D_c\mu * \delta_d$, and also $\psi(\mu) = D_c\mu \boxplus \delta_d$. Assume now that $\mu \in \mathcal{ID}(*)$. Then by Theorem 3.5,

$$\Lambda(\psi(\mu)) = \Lambda(D_c\mu * \delta_d) = D_c\Lambda(\mu) \boxplus \Lambda(\delta_d) = D_c\Lambda(\mu) \boxplus \delta_d = \psi(\Lambda(\mu)),$$

as desired. $\square$

As a consequence of the above corollary, we obtain a short proof of the following result, which was proved by Bercovici and Pata (1999).

Corollary 3.7. *The bijection* $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ *maps the* $*$*-stable probability measures on* $\mathbb{R}$ *onto the* $\boxplus$*-stable probability measures on* $\mathbb{R}$.

Proof. Assume that μ is a $*$-stable probability measure on $\mathbb{R}$, and let $\psi_1, \psi_2\colon \mathbb{R} \to \mathbb{R}$ be increasing affine transformations on $\mathbb{R}$. Then $\psi_1(\mu) * \psi_2(\mu) = \psi_3(\mu)$, for yet another increasing affine transformation $\psi_3\colon \mathbb{R} \to \mathbb{R}$. Now by Corollary 3.6 and Theorem 3.5(i),

$$\begin{aligned}\psi_1(\Lambda(\mu)) \boxplus \psi_2(\Lambda(\mu)) &= \Lambda(\psi_1(\mu)) \boxplus \Lambda(\psi_2(\mu)) = \Lambda(\psi_1(\mu) * \psi_2(\mu)) \\ &= \Lambda(\psi_3(\mu)) = \psi_3(\Lambda(\mu)),\end{aligned}$$

which shows that $\Lambda(\mu)$ is $\boxplus$-stable.

The same line of argument shows that μ is $*$-stable if $\Lambda(\mu)$ is $\boxplus$-stable. $\square$

We end this section by studying some topological properties of Λ. The key result is the following theorem, which is the free analogue of a result due to B.V. Gnedenko (see Gnedenko and Kolmogorov 1968, §19, Theorem 1).

Theorem 3.8. *Let μ be a measure in $\mathcal{ID}(\boxplus)$, and let (μ_n) be a sequence of measures in $\mathcal{ID}(\boxplus)$. For each n, let (γ_n, σ_n) be the free generating pair for μ_n, and let (γ, σ) be the free generating pair for μ. Then the following two conditions are equivalent:*

(i) $\mu_n \xrightarrow{\text{w}} \mu$, *as* $n \to \infty$.
(ii) $\gamma_n \to \gamma$ *and* $\sigma_n \xrightarrow{\text{w}} \sigma$, *as* $n \to \infty$.

Proof. First, assume that (ii) holds. By Theorem 2.6 it is sufficient to show that

(a) $\phi_{\mu_n}(\mathrm{i}y) \to \phi(\mathrm{i}y)$, as $n \to \infty$, for all y in $]0, \infty[$;
(b) $\sup_{n\in\mathbb{N}}|\phi_{\mu_n}(\mathrm{i}y)/y| \to 0$, as $y \to \infty$.

Regarding (a), note that for any y in $]0, \infty[$, the function $t \mapsto (1 + t\mathrm{i}y)/(\mathrm{i}y - t)$, $t \in \mathbb{R}$, is continuous and bounded. Therefore, by the assumptions in (ii),

$$\phi_{\mu_n}(\mathrm{i}y) = \gamma_n + \int_{\mathbb{R}} \frac{1 + t\mathrm{i}y}{\mathrm{i}y - t}\sigma_n(\mathrm{d}t) \underset{n\to\infty}{\longrightarrow} \gamma + \int_{\mathbb{R}} \frac{1 + t\mathrm{i}y}{\mathrm{i}y - t}\sigma(\mathrm{d}t) = \phi_\mu(\mathrm{i}y).$$

Turning to (b), note that for $n \in \mathbb{N}$ and $y \in]0, \infty[$,

$$\frac{\phi_{\mu_n}(\mathrm{i}y)}{y} = \frac{\gamma_n}{y} + \int_{\mathbb{R}} \frac{1 + t\mathrm{i}y}{y(\mathrm{i}y - t)}\sigma_n(\mathrm{d}t).$$

Since the sequence (γ_n) is, in particular, bounded, it suffices to show that

$$\sup_{n\in\mathbb{N}}\left|\int_{\mathbb{R}} \frac{1 + t\mathrm{i}y}{y(\mathrm{i}y - t)}\sigma_n(\mathrm{d}t)\right| \to 0, \qquad y \to \infty. \tag{3.4}$$

For this, note first that since $\sigma_n \xrightarrow{\text{w}} \sigma$, as $n \to \infty$, and since $\sigma(\mathbb{R}) < \infty$, it follows by standard techniques that the family $\{\sigma_n | n \in \mathbb{N}\}$ is tight (cf. Breiman 1992, Corollary 8.11).

Note, next, that for any t in $\mathbb{R}$ and any y in $]0, \infty[$,

$$\left|\frac{1 + t\mathrm{i}y}{y(\mathrm{i}y - t)}\right| \leqslant \frac{1}{y(y^2 + t^2)^{1/2}} + \frac{|t|}{(y^2 + t^2)^{1/2}}.$$

From this estimate it follows that

$$\sup_{y\in[1,\infty[, t\in\mathbb{R}}\left|\frac{1 + t\mathrm{i}y}{y(\mathrm{i}y - t)}\right| \leqslant 2,$$

and that for any $N \in \mathbb{N}$ and $y \in [1, \infty[$,

$$\sup_{t\in[-N,N]}\left|\frac{1 + t\mathrm{i}y}{y(\mathrm{i}y - t)}\right| \leqslant \frac{N + 1}{y}.$$

From the two estimates above, it follows that for any $N \in \mathbb{N}$, and any $y \in [1, \infty[$ we have

$$\sup_{n\in\mathbb{N}}\left|\int_{\mathbb{R}}\frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)}\sigma_n(\mathrm{d}t)\right| \leqslant \frac{N+1}{y}\sup_{n\in\mathbb{N}}\sigma_n([-N,N]) + 2\cdot\sup_{n\in\mathbb{N}}\sigma_n([-N,N]^c)$$

$$\leqslant \frac{N+1}{y}\sup_{n\in\mathbb{N}}\sigma_n(\mathbb{R}) + 2\cdot\sup_{n\in\mathbb{N}}\sigma_n([-N,N]^c). \tag{3.5}$$

Now, given ϵ in $]0,\infty[$, we may, since $\{\sigma_n|n\in\mathbb{N}\}$ is tight, choose $N\in\mathbb{N}$, such that $\sup_{n\in\mathbb{N}}\sigma_n([-N,N]^c)\leqslant\epsilon/4$. Moreover, since $\sigma_n\xrightarrow{\mathrm{w}}\sigma$ and $\sigma(\mathbb{R})<\infty$, the sequence $\{\sigma_n(\mathbb{R})|n\in\mathbb{N}\}$ is, in particular, bounded, and hence, for the chosen N, we may subsequently choose $y_0\in[1,\infty[$ such that $((N+1)/y_0)\sup_{n\in\mathbb{N}}\sigma_n(\mathbb{R})\leqslant\epsilon/2$. Using then the estimate in (3.5), it follows that

$$\sup_{n\in\mathbb{N}}\left|\int_{\mathbb{R}}\frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)}\sigma_n(\mathrm{d}t)\right| \leqslant \epsilon,$$

whenever $y\geqslant y_0$. This verifies (3.4).

Now suppose that (i) holds. Then by Theorem 2.6, there exists a number $M\in]0,\infty[$, such that:

(c) $\forall y\in[M,\infty[$: $\phi_{\mu_n}(\mathrm{i}y)\to\phi_\mu(\mathrm{i}y)$, as $n\to\infty$;
(d) $\sup_{n\in\mathbb{N}}|\phi_{\mu_n}(\mathrm{i}y)/y|\to 0$, as $y\to\infty$.

We show first that the family $\{\sigma_n|n\in\mathbb{N}\}$ is conditionally compact with respect to weak convergence, i.e. that any subsequence $(\gamma_{n'})$ has a subsequence $(\sigma_{n''})$ which converges weakly to some finite measure σ^* on $\mathbb{R}$. By Gnedenko and Kolmogorov (1968, §9, Theorem 3 bis), it suffices to show that $\{\sigma_n|n\in\mathbb{N}\}$ is tight, and that $\{\sigma_n(\mathbb{R})|n\in\mathbb{N}\}$ is bounded. They key step in the argument is the observation that, for any $n\in\mathbb{N}$ and any $y\in]0,\infty[$, we have

$$-\mathrm{Im}\phi_{\mu_n}(\mathrm{i}y) = -\mathrm{Im}\left(\gamma_n+\int_{\mathbb{R}}\frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma_n(\mathrm{d}t)\right)$$

$$= -\mathrm{Im}\left(\int_{\mathbb{R}}\frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma_n(\mathrm{d}t)\right) = y\int_{\mathbb{R}}\frac{1+t^2}{y^2+t^2}\sigma_n(\mathrm{d}t). \tag{3.6}$$

We now show that $\{\sigma_n|n\in\mathbb{N}\}$ is tight. For fixed $y\in]0,\infty[$, note that

$$\{t\in\mathbb{R}|\,|t|\geqslant y\}\subseteq\left\{t\in\mathbb{R}|\frac{1+t^2}{y^2+t^2}\geqslant\frac{1}{2}\right\},$$

so that, for any n in $\mathbb{N}$,

$$\sigma_n(\{t\in\mathbb{R}|\,|t|\geqslant y\})\leqslant 2\int_{\mathbb{R}}\frac{1+t^2}{y^2+t^2}\sigma_n(\mathrm{d}t) = -2\mathrm{Im}\left(\frac{\phi_{\mu_n}(\mathrm{i}y)}{y}\right)\leqslant 2\left|\frac{\phi_{\mu_n}(\mathrm{i}y)}{y}\right|.$$

Combining this estimate with (d), it follows immediately that $\{\sigma_n|n\in\mathbb{N}\}$ is tight.

We next show that the sequence $\{\sigma_n(\mathbb{R})|n\in\mathbb{N}\}$ is bounded. Note first that, with M as in (c), there exists a constant $c\in]0,\infty[$, such that

$$c \leqslant \frac{M(1+t^2)}{M^2+t^2}, \qquad \text{for all } t \text{ in } \mathbb{R}.$$

It follows, by (3.6), that, for any n in $\mathbb{N}$,

$$c\sigma_n(\mathbb{R}) \leqslant \int_{\mathbb{R}} \frac{M(1+t^2)}{M^2+t^2}\sigma_n(\mathrm{d}t) = -\mathrm{Im}\phi_{\mu_n}(\mathrm{i}M),$$

and therefore, by (c),

$$\limsup_{n\to\infty} \sigma_n(\mathbb{R}) \leqslant \limsup_{n\to\infty}\{-c^{-1}\cdot \mathrm{Im}\phi_{\mu_n}(\mathrm{i}M)\} = -c^{-1}\cdot \mathrm{Im}\phi_{\mu}(\mathrm{i}M) < \infty,$$

which shows that $\{\sigma_n(\mathbb{R}) | n \in \mathbb{N}\}$ is bounded.

Having established that the family $\{\sigma_n | n \in \mathbb{N}\}$ is conditionally compact, recall next from Remark 2.5 that in order to show that $\sigma_n \xrightarrow{\mathrm{w}} \sigma$, it suffices to show that any subsequence $(\sigma_{n'})$ has a subsequence which converges weakly to σ. A similar argument works, of course, to show that $\gamma_n \to \gamma$. So consider any subsequence $(\gamma_{n'}, \sigma_{n'})$ of the sequence of generating pairs. Since $\{\sigma_n | n \in \mathbb{N}\}$ is conditionally compact, there is a subsequence (n'') of (n'), such that the sequence $(\sigma_{n''})$ is weakly convergent to some finite measure σ^* on $\mathbb{R}$. Since the function $t \mapsto (1+t\mathrm{i}y)/(\mathrm{i}y - t)$ is continuous and bounded for any $y \in]0, \infty[$, we know that

$$\int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma_{n''}(\mathrm{d}t) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma^*(\mathrm{d}t),$$

for any $y \in]0, \infty[$. At the same time, we know from (c) that

$$\gamma_{n''} + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma_{n''}(\mathrm{d}t) = \phi_{\mu_{n''}}(\mathrm{i}y) \underset{n\to\infty}{\longrightarrow} \phi_\mu(\mathrm{i}y) = \gamma + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma(\mathrm{d}t),$$

for any $y \in [M, \infty[$. From these observations, it follows that the sequence $(\gamma_{n''})$ must converge to some real number γ^*, which then has to satisfy the identity

$$\gamma^* + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma^*(\mathrm{d}t) = \phi_\mu(\mathrm{i}y) = \gamma + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\sigma(\mathrm{d}t),$$

for all $y \in [M, \infty[$. By uniqueness of the free Lévy–Khinchine representation (cf. Theorem 2.8) and uniqueness of analytic continuation, it follows that we must have $\sigma^* = \sigma$ and $\gamma^* = \gamma$. We have thus verified the existence of a subsequence $(\gamma_{n''}, \sigma_{n''})$ which converges (coordinatewise) to (γ, σ) which was our objective. □

As an immediate consequence of Theorem 3.8 and the corresponding result in classical probability, we obtain the following:

Corollary 3.9. *The Bercovici–Pata bijection* $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ *is a homeomorphism with respect to weak convergence. In other words, if* μ *is a measure in* $\mathcal{ID}(*)$ *and* (μ_n) *is a sequence of measures in* $\mathcal{ID}(*)$*, then* $\mu_n \xrightarrow{\mathrm{w}} \mu$*, as* $n \to \infty$*, if and only if* $\Lambda(\mu_n) \xrightarrow{\mathrm{w}} \Lambda(\mu)$*, as* $n \to \infty$.

Proof. Let (γ, σ) be the generating pair for μ and, for each n, let (γ_n, σ_n) be the generating pair for μ_n.

Assume first that $\mu_n \xrightarrow{\text{w}} \mu$. Then by Gnedenko and Kolmogorov (1968, §19, Theorem 1), $\gamma_n \to \gamma$ and $\sigma_n \xrightarrow{\text{w}} \sigma$. Since (γ_n, σ_n) $((\gamma, \sigma))$ is the free generating pair for $\Lambda(\mu_n)$ $(\Lambda(\mu))$, it follows from Theorem 3.8 that $\Lambda(\mu_n) \xrightarrow{\text{w}} \Lambda(\mu)$.

The same argument applies to the converse implication. □

Remark 3.10. *Cumulants II.* Let μ be a probability measure in $\mathcal{ID}(*)$ with moments of any order, and consider its sequence (c_n) of classical cumulants (cf. Remark 2.7). Then the Bercovici–Pata bijection Λ may also be defined as the mapping that sends μ to the probability measure on $\mathbb{R}$ with free cumulants (c_n). In other words, the free cumulants for $\Lambda(\mu)$ are the classical cumulants for μ. This fact has recently been noted by Anshelevich (2001b, Lemma 6.5). In view of the theory of free cumulants for several variables (cf. Remark 2.7), this point of view might be used to generalize the Bercovici–Pata bijection to multidimensional probability measures.

4. Self-decomposability in free probability

Recall from Section 2.1 that a probability measure μ on $\mathbb{R}$ is $*$-self-decomposable if and only if any (classical) random variable Y with distribution μ has, for any c in $]0, 1[$, a decomposition in law of the form $Y \overset{\text{d}}{=} cY + Y_c$, where Y_c is a random variable which is independent of Y. In view of this definition of $*$-self-decomposability, the natural definition of the free counterpart must be as follows: μ is $\boxplus$-self-decomposable if any self-adjoint operator y with (spectral) distribution μ admits, for any c in $]0, 1[$, a decomposition in law of the form $y \overset{\text{d}}{=} cy + y_c$, where y_c is a self-adjoint operator which is *freely independent* of y. If μ has unbounded support, the self-adjoint operator y would have to be unbounded. We prefer, at this point, to avoid dealing with unbounded operators, and instead to define $\boxplus$-self-decomposability in terms of the measures themselves, rather than in terms of corresponding operators. However, our definition of $\boxplus$-self-decomposability, to be given next, is equivalent to the algebraic formulation stated above. Note that with the notation used in Section 3, a probability measure μ on $\mathbb{R}$ is $*$-self-decomposable if and only if it has, for any c in $]0, 1[$, a decomposition of the form $\mu = D_c\mu * \mu_c$, for some probability measure μ_c on $\mathbb{R}$.

Definition 4.1. *Let μ be a probability measure on $\mathbb{R}$. We say that μ is self-decomposable with respect to free additive convolution (or just $\boxplus$-self-decomposable) if, for any c in $]0, 1[$, there exists a probability measure μ_c on $\mathbb{R}$ such that*

$$\mu = D_c\mu \boxplus \mu_c. \tag{4.1}$$

We denote by $\mathcal{L}(\boxplus)$ the class of $\boxplus$-self-decomposable probability measures on $\mathbb{R}$.

Note that, for a probability measure μ on $\mathbb{R}$ and a constant c in $]0, 1[$, there can be only one probability measure μ_c such that $\mu = D_c\mu \boxplus \mu_c$. Indeed, choose positive numbers η and M such that all three Voiculescu transforms ϕ_μ, $\phi_{D_c\mu}$ and ϕ_{μ_c} are defined on the region

$\Gamma_{\eta,M}$. Then by Theorem 2.2, ϕ_{μ_c} is uniquely determined on $\Gamma_{\eta,M}$, and hence, by Remark 2.3, μ_c is uniquely determined too.

Remark 4.2. Let μ be a probability measure on $\mathbb{R}$. It follows from Theorem 2.2, Lemma 3.3 and Remark 2.3 that μ is $\boxplus$-self-decomposable if and only if there exists, for each c in $]0, 1[$, a probability measure μ_c on $\mathbb{R}$ such that

$$\phi_\mu(z) = c\phi_\mu(c^{-1}z) + \phi_{\mu_c}(z),$$

for all z in a region $\Gamma_{\eta,M}$.

Remark 4.3. *Free cumulant transform*. As mentioned previously, besides the $\mathcal{R}$-transform and the Voiculescu transform, there is a third variant, $\mathcal{C}_\mu$, which seems worth taking into account. So far it has been studied, in particular, by Nica (1996) and Speicher, but it is also used in Hiai and Petz (2000). For a probability measure μ on $\mathbb{R}$, $\mathcal{C}_\mu$ is given by the equation

$$\mathcal{C}_\mu(z) = z\mathcal{R}(z) = z\phi_\mu\left(\frac{1}{z}\right),$$

and is thus defined on a region of the form $\Gamma_{\eta,M}^{-1}$, for suitable positive numbers η and M. Of course the transformation $\mu \mapsto \mathcal{C}_\mu$ has a property similar to that of the Voiculescu transform stated in Theorem 2.2. In fact, $\mathcal{C}_\mu$ resembles the classical cumulant function more closely than the Voiculescu transform and the $\mathcal{R}$-transform do. In particular, with respect to dilation it behaves exactly as the classical cumulant function, i.e.

$$\mathcal{C}_{D_c\mu}(z) = \mathcal{C}_\mu(cz), \tag{4.2}$$

for any probability measure μ on $\mathbb{R}$ and any positive constant c. This follows easily from Lemma 3.3. As a consequence of (4.2), it follows, as in Remark 4.2, that a probability measure μ on $\mathbb{R}$ is $\boxplus$-self-decomposable if and only if there exists, for any c in $]0, 1[$, a probability measure μ_c on $\mathbb{R}$ such that

$$\mathcal{C}_\mu(z) = \mathcal{C}_\mu(cz) + \mathcal{C}_{\mu_c}(z).$$

In terms of the function $\mathcal{C}_\mu$, the condition for $\boxplus$-self-decomposability is thus exactly the same as the condition for $*$-self-decomposibility expressed in terms of the (classical) cumulant function (cf. (2.2)). We note, finally, that the free Lévy–Khinchine represenation of $\mathcal{C}_\mu$ takes the form

$$\mathcal{C}_\mu(z) = \gamma z + \int_{\mathbb{R}} \frac{z^2 + tz}{1 - tz}\sigma(\mathrm{d}t) = \gamma z + \int_{\mathbb{R}} \left(\frac{tz}{1 + t^2} + \frac{z^2}{1 - tz}\right)\nu(\mathrm{d}t), \tag{4.3}$$

where γ, σ and ν are the same as in Theorem 2.8. Thus, by analogy with the classical case, the free Lévy–Khinchine representation of $\mathcal{C}_\mu$ includes a linear term, rather than a constant one. Furthermore, if we put $a = \sigma(\{0\})$, then we may rewrite (4.3) as

$$\mathcal{C}_\mu(z) = \gamma z + az^2 + \int_{\mathbb{R}} \frac{z^2 + tz}{1 - tz}\sigma'(\mathrm{d}t) = \gamma z + az^2 + \int_{\mathbb{R}} \left(\frac{tz}{1 + t^2} + \frac{z^2}{1 - tz}\right)\nu'(\mathrm{d}t), \tag{4.4}$$

where σ' and ν' have no mass at zero. The representation (4.4) is analogous to the classical representation (2.5); in particular the quadratic term az^2 corresponds to the Gaussian (i.e. semi-circular) part of μ.

Lemma 4.4. *Let μ be a $\boxplus$-self-decomposable probability measure on $\mathbb{R}$, let c be a number in $]0,1[$, and let μ_c be the probability measure on $\mathbb{R}$ determined by the equation*

$$\mu = D_c\mu \boxplus \mu_c.$$

Let η and M be positive numbers such that ϕ_μ is defined on $\Gamma_{\eta,M}$. Then ϕ_{μ_c} is defined on $\Gamma_{\eta,M}$ as well.

Proof. Choose positive numbers η' and M' such that $\Gamma_{\eta',M'} \subseteq \Gamma_{\eta,M}$ and such that ϕ_μ and ϕ_{μ_c} are both defined on $\Gamma_{\eta',M'}$. For z in $\Gamma_{\eta',M'}$, we then have (cf. Lemma 3.3)

$$\phi_\mu(z) = c\phi_\mu(c^{-1}z) + \phi_{\mu_c}(z).$$

Recalling the definition of the Voiculescu transform, the above equation means that

$$F_\mu^{-1}(z) - z = c\phi_\mu(c^{-1}z) + F_{\mu_c}^{-1}(z) - z, \qquad z \in \Gamma_{\eta',M'},$$

so that

$$F_{\mu_c}^{-1}(z) = F_\mu^{-1}(z) - c\phi_\mu(c^{-1}z), \qquad z \in \Gamma_{\eta',M'}.$$

Now put $\psi(z) = F_\mu^{-1}(z) - c\phi_\mu(c^{-1}z)$ and note that ψ is defined and holomorphic on all of $\Gamma_{\eta,M}$ (cf. Lemma 3.3), and that

$$F_{\mu_c}(\psi(z)) = z, \qquad z \in \Gamma_{\eta',M'}. \tag{4.5}$$

We note next that ψ takes values in $\mathbb{C}^+$. Indeed, since F_μ is defined on $\mathbb{C}^+$, we have that $\mathrm{Im}(F_\mu^{-1}(z)) > 0$, for any z in $\Gamma_{\eta,M}$, and furthermore, for all such z, $\mathrm{Im}(\phi_\mu(c^{-1}z)) \leqslant 0$, as noted in Section 2.3.

Now, since F_{μ_c} is defined and holomorphic on all of $\mathbb{C}^+$, both sides of (4.5) are holomorphic on $\Gamma_{\eta,M}$. Since $\Gamma_{\eta',M'}$ has an accumulation point in $\Gamma_{\eta,M}$, it follows, by uniqueness of analytic continuation, that the equality in (4.5) actually holds for all z in $\Gamma_{\eta,M}$. Thus, F_{μ_c} has a right inverse on $\Gamma_{\eta,M}$, which means that ϕ_{μ_c} is defined on $\Gamma_{\eta,M}$, as desired. □

Lemma 4.5. *Let μ be a $\boxplus$-self-decomposable probability measure on $\mathbb{R}$, and let (c_n) be a sequence of numbers in $]0,1[$. For each n, let μ_{c_n} be the probability measure on $\mathbb{R}$ satisfying*

$$\mu = D_{c_n}\mu \boxplus \mu_{c_n}.$$

Then, if $c_n \to 1$, as $n \to \infty$, we have $\mu_{c_n} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$.

Proof. Choose positive numbers η and M such that ϕ_μ is defined on $\Gamma_{\eta,M}$. Note then that, by Lemma 4.4, $\phi_{\mu_{c_n}}$ is also defined on $\Gamma_{\eta,M}$ for each $n \in \mathbb{N}$ and, moreover,

$$\phi_{\mu_{c_n}}(z) = \phi_\mu(z) - c_n\phi_\mu(c_n^{-1}z), \qquad z \in \Gamma_{\eta,M},\ n \in \mathbb{N}. \tag{4.6}$$

Assume now that $c_n \to 1$ as $n \to \infty$. From (4.6) and continuity of ϕ_μ it is then straightforward that $\phi_{\mu_{c_n}}(z) \to 0 = \phi_{\delta_0}(z)$, as $n \to \infty$, uniformly on compact subsets of $\Gamma_{\eta,M}$. Note, furthermore, that

$$\sup_{n\in\mathbb{N}}\left|\frac{\phi_{\mu_{c_n}}(z)}{z}\right| = \sup_{n\in\mathbb{N}}\left|\frac{\phi_\mu(z)}{z} - \frac{\phi_\mu(c_n^{-1}z)}{c_n^{-1}z}\right| \to 0, \qquad \text{as } |z| \to \infty,\ z \in \Gamma_{\eta,M},$$

since $\phi_\mu(z)/z \to 0$ as $|z| \to \infty$, $z \in \Gamma_{\eta,M}$, and since $c_n^{-1} \geqslant 1$ for all n. It follows from Proposition 2.6 that $\mu_c \xrightarrow{\text{w}} \delta_0$, for $n \to \infty$, as desired. □

Theorem 4.6. *Let μ be a probability measure on $\mathbb{R}$. If μ is $\boxplus$-self-decomposable, then μ is $\boxplus$-infinitely divisible.*

Proof. Assume that μ is $\boxplus$-self-decomposable. Then by successive applications on (4.1), we obtain for any $c \in]0, 1[$ and any $n \in \mathbb{N}$ that

$$\mu = D_{c^n}\mu \boxplus D_{c^{n-1}}\mu_c \boxplus D_{c^{n-2}}\mu_c \boxplus \cdots \boxplus D_c\mu_c \boxplus \mu_c. \tag{4.7}$$

The idea now is to show that for a suitable choice of $c = c_n$, the probability measures

$$D_{c_n^n}\mu,\ D_{c_n^{n-1}}\mu_{c_n},\ D_{c_n^{n-2}}\mu_{c_n},\ \ldots,\ D_{c_n}\mu_{c_n},\ \mu_{c_n}, \qquad n \in \mathbb{N}, \tag{4.8}$$

form a null array (cf. Theorem 2.11). Note that, for any choice of $c_n \in]0, 1[$, we have that

$$D_{c_n^j}\mu_{c_n}(\mathbb{R}\backslash[-\epsilon, \epsilon]) \leqslant \mu_{c_n}(\mathbb{R}\backslash[-\epsilon, \epsilon]),$$

for any $j \in \mathbb{N}$ and any $\epsilon \in]0, \infty[$. Therefore, in order for the probability measures in (4.8) to form a null array, it suffices to choose c_n in such a way that

$$D_{c_n^n}\mu \xrightarrow{\text{w}} \delta_0 \quad \text{and} \quad \mu_{c_n} \xrightarrow{\text{w}} \delta_0, \qquad \text{as } n \to \infty.$$

We claim that this will be the case if we put (for example)

$$c_n = \mathrm{e}^{-1/\sqrt{n}}, \qquad n \in \mathbb{N}. \tag{4.9}$$

To see this, note that with the above choice of c_n, we have

$$c_n \to 1 \quad \text{and} \quad c_n^n \to 0, \qquad \text{as } n \to \infty.$$

Thus, it follows immediately from Lemma 4.5 that $\mu_{c_n} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$. Moreover, if we choose a (classical) real-valued random variable X with distribution μ, then, for each n, $D_{c_n^n}\mu$ is the distribution of $c_n^n X$. Now, $c_n^n X \to 0$ almost surely, as $n \to \infty$, and this implies that $c_n^n X \to 0$, in distribution, as $n \to \infty$.

We have verified, that if we choose c_n according to (4.9), then the probability measures in (4.8) form a null array. Hence, by (4.7) (with $c = c_n$) and Theorem 2.11, μ is $\boxplus$-infinitely divisible. □

Proposition 4.7. *Let μ be a $\boxplus$-self-decomposable probability measure on $\mathbb{R}$, let c be a number in $]0, 1[$ and let μ_c be the probability measure on $\mathbb{R}$ satisfying the condition*

$$\mu = D_c\mu \boxplus \mu_c.$$

Then μ_c is $\boxplus$-infinitely divisible.

Proof. As noted in the proof of Theorem 4.6, for any $d \in\,]0, 1[$ and any $n \in \mathbb{N}$ we have

$$\mu = D_{d^n}\mu \boxplus D_{d^{n-1}}\mu_d \boxplus D_{d^{n-2}}\mu_d \boxplus \cdots \boxplus D_d\mu_d \boxplus \mu_d,$$

where μ_d is defined by the case $n = 1$. Using the above equation with $d = c^{1/n}$, we obtain for each $n \in \mathbb{N}$ that

$$D_c\mu \boxplus \mu_c = \mu = D_c\mu \boxplus D_{c^{(n-1)/n}}\mu_{c^{1/n}} \boxplus D_{c^{(n-2)/n}}\mu_{c^{1/n}} \boxplus \cdots \boxplus D_{c^{1/n}}\mu_{c^{1/n}} \boxplus \mu_{c^{1/n}}. \tag{4.10}$$

From this it follows that

$$\mu_c = D_{c^{(n-1)/n}}\mu_{c^{1/n}} \boxplus D_{c^{(n-2)/n}}\mu_{c^{1/n}} \boxplus \cdots \boxplus D_{c^{1/n}}\mu_{c^{1/n}} \boxplus \mu_{c^{1/n}}, \qquad n \in \mathbb{N}. \tag{4.11}$$

Indeed, by taking Voiculescu transforms in (4.10) and using Theorem 2.2, it follows that the Voiculescu transforms of the right- and left-hand sides of (4.11) coincide on some region $\Gamma_{\eta,M}$. By Remark 2.3, this implies the validity of (4.11).

By (4.11) and Theorem 2.11, it remains now to show that the probability measures

$$D_{c^{(n-1)/n}}\mu_{c^{1/n}},\ D_{c^{(n-2)/n}}\mu_{c^{1/n}},\ \ldots,\ D_{c^{1/n}}\mu_{c^{1/n}},\ \mu_{c^{1/n}}$$

form a null array. Since $c^{j/n} \in\,]0, 1[$ for any $j \in \{1, 2, \ldots, n-1\}$, this is the case if and only if $\mu_{c^{1/n}} \xrightarrow{\text{w}} \delta_0$ as $n \to \infty$. But since $c^{1/n} \to 1$ as $n \to \infty$, Lemma 4.5 guarantees the validity of the latter assertion. □

Theorem 4.8. *Let μ be a $*$-self-decomposable probability measure on $\mathbb{R}$ and let $(\mu_c)_{c\in]0,1[}$ be the family of probability measures on $\mathbb{R}$ defined by the equation*

$$\mu = D_c\mu * \mu_c.$$

Then, for any $c \in\,]0, 1[$, we have the decomposition

$$\Lambda(\mu) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c). \tag{4.12}$$

Consequently, a probability measure μ on $\mathbb{R}$ is $*$-self-decomposable if and only if $\Lambda(\mu)$ is $\boxplus$-self-decomposable, and thus the bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ maps the class $\mathcal{L}(*)$ of $*$-self-decomposable probability measures onto the class $\mathcal{L}(\boxplus)$ of $\boxplus$-self-decomposable probability measures.

Proof. For any c in $]0, 1[$, the measures $D_c\mu$ and μ_c are both $*$-infinitely divisible (see Section 2.1), and hence, by (i) and (ii) of Theorem 3.5,

$$\Lambda(\mu) = \Lambda(D_c\mu * \mu_c) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c).$$

Since this holds for all $c \in\,]0, 1[$, it follows that $\Lambda(\mu)$ is $\boxplus$-self-decomposable.

Assume, conversely, that μ' is a $\boxplus$-self-decomposable probability measure on $\mathbb{R}$, and let $(\mu'_c)_{c\in]0,1[}$ be the family of probability measures on $\mathbb{R}$ defined by

$$\mu' = D_c\mu' \boxplus \mu'_c.$$

By Theorem 4.6 and Proposition 4.7, $\mu', \mu'_c \in \mathcal{ID}(\boxplus)$, so we may consider the $*$-infinitely divisible probability measures $\mu := \Lambda^{-1}(\mu')$ and $\mu_c := \Lambda^{-1}(\mu'_c)$. Then by (i) and (ii) of Theorem 3.5,

$$\mu = \Lambda^{-1}(\mu') = \Lambda^{-1}(D_c(\mu') \boxplus \mu'_c) = \Lambda^{-1}(D_c\Lambda(\mu) \boxplus \Lambda(\mu_c))$$
$$= \Lambda^{-1}(\Lambda(D_c\mu * \mu_c)) = D_c\mu * \mu_c.$$

Since this holds for any $c \in]0, 1[$, μ is $*$-self-decomposable. □

The corollary below can be proved directly by using, for example, Bercovici and Voiculescu (1993, Corollary 7.2). However, by using the corresponding classical result as well as Theorem 4.8 and Corollary 3.7, we can argue without doing any computations.

Corollary 4.9. *Let μ be a $\boxplus$-stable probability measure on $\mathbb{R}$. Then μ is necessarily $\boxplus$-self-decomposable.*

Proof. Since μ is $\boxplus$-stable, μ is also $\boxplus$-infinitely divisible, so we may consider the $*$-infinitely divisible probability measure $\mu' = \Lambda^{-1}(\mu)$. By Corollary 3.7, μ' is $*$-stable and since $*$-stability implies $*$-self-decomposability (cf. Sato 1999, Example 15.2), μ' is also $*$-self-decomposable. Hence, by Theorem 4.8, $\mu = \Lambda(\mu')$ is $\boxplus$-self-decomposable. □

To summarize, we note that it follows from Theorem 4.6 and Corollary 4.9 that we have the following free counterpart to the hierarchy (2.1):

$$\mathcal{G}(\boxplus) \subset \mathcal{S}(\boxplus) \subset \mathcal{L}(\boxplus) \subset \mathcal{ID}(\boxplus), \tag{4.13}$$

where $\mathcal{G}(\boxplus)$ denotes the class of semi-circle distributions. Furthermore, the Bercovici–Pata bijection Λ maps each of the classes of probability measures in (2.1) onto the corresponding free class in (4.13).

5. Free Lévy processes

In this section we introduce and study some basic properties of Lévy processes in free probability. We start by recalling the definition of classical Lévy processes.

Definition 5.1. *A real-valued stochastic process $(X_t)_{t\geqslant 0}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, is called a* Lévy process *if it satisfies the following conditions:*

(i) *Whenever $n \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \ldots < t_n$, the increments*

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent random variables.

(ii) *$X_0 = 0$ almost surely.*

(iii) *For any $s, t \in]0, \infty[$, the distribution of $X_{s+t} - X_s$ does not depend on s.*

(iv) (X_t) *is stochastically continuous, i.e. for any* s *in* $[0, \infty[$ *and any positive* ϵ, *we have* $\lim_{t\to 0} P(|X_{s+t} - X_s| > \epsilon) = 0$.

(v) *For almost all* ω *in* Ω, *the sample path* $t \mapsto X_t(\omega)$ *is right-continuous* (*in* $t \geqslant 0$) *and has left limits* (*in* $t > 0$).

If a stochastic process $(X_t)_{t\geqslant 0}$ satisfies conditions (i)–(iv) in Definition 5.1, we say that (X_t) is a *Lévy process in law.* If (X_t) satisfies conditions (i), (ii), (iv) and (v) ((i), (ii), and (iv)) it is called an *additive process* (an *additive process in law*). Any Lévy process in law (X_t) has a modification which is a Lévy process, i.e. there exists a Lévy process (Y_t), defined on the same probability space as (X_t), and such that $X_t = Y_t$ with probability one, for all t. Similarly, any additive process in law has a modification which is a genuine additive process. These assertions can be found in Sato (1999, Theorem 11.5).

Note that condition (iv) is equivalent to the condition that $X_{s+t} - X_s \to 0$ in distribution as $t \to 0$. Note also that under the assumption of (ii) and (iii), this condition is equivalent to saying that $X_t \to 0$ in distribution as $t \searrow 0$.

We turn now to the non-cummutative setting. Let $(\mathcal{A}, \tau)$ be a W^*-probability space acting on a Hilbert space $\mathcal{H}$ (cf. Section 2.5). By a (stochastic) process affiliated with $\mathcal{A}$, we shall simply mean a family $(Z_t)_{t\in[0,\infty[}$ of *self-adjoint* operators in $\overline{\mathcal{A}}$, which is indexed by the non-negative reals. For such a process (Z_t), we let μ_t denotes the (spectral) distribution of Z_t, i.e. $\mu_t = \mathcal{L}\{Z_t\}$. We refer to the family (μ_t) of probability measures on $\mathbb{R}$ as the family of *marginal distributions* of (Z_t). Moreover, if $s, t \in [0, \infty[$, such that $s < t$, then, as was noted in Section 2.5, $Z_t - Z_s$ is again a self-adjoint operator in $\overline{\mathcal{A}}$, and we may consider its distribution $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$. We refer to the family $(\mu_{s,t})_{0\leqslant s<t}$ as the family of *increment distributions* of (Z_t).

Definition 5.2. *A* free Lévy process (in law)*, affiliated with a* W^**-probability space* $(\mathcal{A}, \tau)$*, is a process* $(Z_t)_{t\geqslant 0}$ *of self-adjoint operators in* $\overline{\mathcal{A}}$*, which satisfies the following conditions:*

(i) *Whenever* $n \in \mathbb{N}$ *and* $0 \leqslant t_0 < t_1 < \ldots < t_n$, *the increments*

$$Z_{t_0},\ Z_{t_1} - Z_{t_0},\ Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}$$

are freely independent random variables.

(ii) $Z_0 = 0$.

(iii) *For any* $s, t \in [0, \infty[$*, the (spectral) distribution of* $Z_{s+t} - Z_s$ *does not depend on* s.

(iv) *For any* $s \in [0, \infty[$*,* $Z_{s+t} - Z_s \to 0$ *in distribution as* $t \to 0$*, i.e. the spectral distributions* $\mathcal{L}\{Z_{s+t} - Z_s\}$ *converge weakly to* δ_0 *as* $t \to 0$.

Note that under the assumption of (ii) and (iii) in Definition 5.2, condition (iv) is equivalent to saying that $Z_t \to 0$ in distribution as $t \searrow 0$.

Remark 5.3. *Free additive processes I.* A process (Z_t) of self-adjoint operators in $\overline{\mathcal{A}}$ which satisfies conditions (i), (ii) and (iv) of Definition 5.2, is called a *free additive process* (*in law*). Given such a process (Z_t), let, as above, $\mu_s = \mathcal{L}\{Z_s\}$ and $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$, whenever $0 \leqslant s < t$. It follows that whenever $0 \leqslant r < s < t$, we have

$$\mu_s = \mu_r \boxplus \mu_{r,s} \quad \text{and} \quad \mu_{r,t} = \mu_{r,s} \boxplus \mu_{s,t}, \tag{5.1}$$

and furthermore

$$\mu_{s+t,s} \xrightarrow{\mathrm{w}} \delta_0, \qquad \text{as } t \to 0, \tag{5.2}$$

for any $s \in [0, \infty[$.

Conversely, given any family $\{\mu_t | t \geq 0\} \cup \{\mu_{s,t} | 0 \leq s < t\}$ of probability measures on $\mathbb{R}$, such that (5.1) and (5.2) are satisfied, there exists a free additive process (in law) (Z_t) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, such that $\mu_s = \mathcal{L}\{Z_s\}$ and $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$, whenever $0 \leq s < t$. In fact, for any families (μ_t) and $(\mu_{s,t})$ satisfying (5.1), there exists a process (Z_t) affiliated with some W^*-probability space $(\mathcal{A}, \tau)$, such that conditions (i) and (ii) in Definition 5.2 are satisfied, and such that $\mu_s = \mathcal{L}\{Z_s\}$ and $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$.

This was noted in Biane (1998a) and Voiculescu (2000). Note that with the notation introduced above, the free Lévy processes (in law) are exactly those free additive processes (in law) for which $\mu_{s,t} = \mu_{t-s}$ for all s, t such that $0 \leq s < t$. In this case (5.1) simplifies to

$$\mu_t = \mu_s \boxplus \mu_{t-s}, \qquad 0 \leq s < t. \tag{5.3}$$

In particular, for any family (μ_t) of probability measures on $\mathbb{R}$ such that (5.3) is satisfied, and such that $\mu_t \xrightarrow{\mathrm{w}} \delta_0$ as $t \searrow 0$, there exists a free Lévy process (in law) (Z_t) such that $\mu_t = \mathcal{L}\{Z_t\}$ for all t.

Consider now a free Lévy process $(Z_t)_{t\geq 0}$, with marginal distributions (μ_t). As For (classical) Lévy processes, it follows that each μ_t is necessarily $\boxplus$-infinitely divisible. Indeed, for any $n \in \mathbb{N}$ we have $Z_t = \sum_{j=1}^{n}(Z_{jt/n} - Z_{(j-1)t/n})$, and thus, in view of conditions (i) and (iii) in Definition 5.2, $\mu_t = \mu_{t/n} \boxplus \cdots \boxplus \mu_{t/n}$ (n terms). From the observation just made, it follows that the Bercovici–Pata bijection Λ: $\mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ gives rise to a correspondence between classical and free Lévy processes:

Proposition 5.4. *Let $(Z_t)_{t\geq 0}$ be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and with marginal distributions (μ_t). Then there exists a (classical) Lévy process $(X_t)_{t\geq 0}$ with marginal distributions $(\Lambda^{-1}(\mu_t))$.*

Conversely, for any (classical) Lévy process (X_t) with marginal distributions (μ_t), there exists a free Lévy process (in law) (Z_t) with marginal distributions $(\Lambda(\mu_t))$.

Proof. Consider a free Lévy process (in law) (Z_t) with marginal distributions (μ_t). Then, as noted above, $\mu_t \in \mathcal{ID}(\boxplus)$ for all t, and hence we may define $\mu_t' = \Lambda^{-1}(\mu_t)$, $t \geq 0$. Then, whenever $0 \leq s < t$,

$$\mu_t' = \Lambda^{-1}(\mu_s \boxplus \mu_{t-s}) = \Lambda^{-1}(\mu_s) * \Lambda^{-1}(\mu_{t-s}) = \mu_s' * \mu_{t-s}'.$$

Hence, by the Kolmogorov extension theorem, there exists a (classical) stochastic process (X_t) (defined on some probability space $(\Omega, \mathcal{F}, P)$), with marginal distributions (μ_t'), which satisfies conditions (i)–(iii) of Definition 5.1. Regarding condition (iv), note that since (Z_t) is a free Lévy process, $\mu_t \xrightarrow{w} \delta_0$ as $t \searrow 0$, and hence, by continuity of Λ^{-1} (cf. Corollary 3.9),

$$\mu_t' = \Lambda^{-1}(\mu_t) \xrightarrow{w} \Lambda^{-1}(\delta_0) = \delta_0, \qquad \text{as } t \searrow 0.$$

Thus (X_t) is a (classical) Lévy process in law, and hence we can find a modification of (X_t) which is a genuine Lévy process.

The second statement of the proposition follows by a similar argument, using Λ rather than Λ^{-1}, and the fact that the marginal distributions of a classical Lévy process are necessarily $*$-infinitely divisible. Furthermore, we have to call upon the existence statement for free Lévy processes (in law) in Remark 5.3. □

Remark 5.5. *Free additive processes II*. Though our main objectives in this section are free Lévy processes, we mention, for completeness, that the Bercovici–Pata bijection Λ also gives rise to a correspondence between classical and free additive processes (in law). Thus, to any classical additive process (in law), with corresponding marginal distributions (μ_t) and increment distributions $(\mu_{s,t})_{0\leqslant s<t}$, there corresponds a free additive process (in law), with marginal distributions $(\Lambda(\mu_t))$ and increment distributions $(\Lambda(\mu_{s,t}))_{0\leqslant s<t}$ – and vice versa.

This follows by the same method as used in the proof of Proposition 5.4 above, once it has been established that for a free additive process (in law) (Z_t), the distributions $\mu_t = \mathcal{L}\{Z_t\}$ and $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$, $0 \leqslant s < t$, are necessarily $\boxplus$-infinitely divisible (for the corresponding classical result, see Sato (1999, Theorem 9.1). The key to this result is Theorem 2.11, together with the fact that (Z_t) is actually uniformly stochastically continuous on compact intervals, in the following sence: for any compact interval $[0, b]$ in $[0, \infty[$, and for any positive numbers ϵ, ρ, there exists a positive number δ such that $\mu_{s,t}(\mathbb{R}\backslash[-\epsilon, \epsilon]) < \rho$, for any s, t in $[0, b]$ for which $s < t < s + \delta$. As in the classical case, this follows from condition (iv) in Definition 5.2, by a standard compactness argument (see Sato 1999, Lemma 9.6). Now for any $t \in [0, \infty[$ and any $n \in \mathbb{N}$, we have (cf. (5.1))

$$\mu_t = \mu_{0,t/n} \boxplus \mu_{t/n,2t/n} \boxplus \mu_{2t/n,3t/n} \boxplus \cdots \boxplus \mu_{(n-1)t/n,t}. \tag{5.4}$$

Since (Z_t) is uniformly stochastically continous on $[0, t]$, it follows that the family $\{\mu_{(j-1)t/n,jt/n} | n \in \mathbb{N}, 1 \leqslant j \leqslant n\}$ is a null array, and hence, by Theorem 2.11, (5.4) implies that μ_t is $\boxplus$-infinitely divisible. Applying this fact to the free additive process (in law) $(Z_t - Z_s)_{t\geqslant s}$, it also follows that $\mu_{s,t}$ is $\boxplus$-infinitely divisible whenever $0 \leqslant s < t$.

Remark 5.6. *An alternative concept of free Lévy processes*. For a classical Lévy process (X_t), condition (iii) of Definition 5.1 is equivalent to the condition that whenever $0 \leqslant s < t$, the conditional distribution $P(X_t|X_s)$ depends only on $t - s$. Conditional probabilities in free probability were studied by Biane (1998a), who noted, in particular, that in the free case the condition just stated *is not* equivalent to condition (iii) of Definition 5.2. Consequently, in free probability there are two classes of stochastic processes that may naturally be called Lévy processes: the ones we defined in Definition 5.2 and the ones for which condition (iii) of Definition 5.2 is replaced by the condition on the conditional distributions, mentioned above. In Biane (1998a) these two types of processes were denoted FAL1 and FAL2, respectively. We should mention, here, that in Biane (1998a) the assumption of stochastic continuity (condition (iv) of Definition 5.2) was not included in the definitions of either FAL1 or FAL2. We have included that condition primarily because it is crucial for the definition of the stochastic integral to be constructed in the next section.

6. Free stochastic integrals and $\boxplus$-self-decomposable variates

As mentioned in Section 2.1, a (classical) random variable Y has distribution in $\mathcal{L}(*)$ if and only if it has a representation in law of the form

$$Y \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t} \mathrm{d}X_t, \tag{6.1}$$

where $(X_t)_{t\geq 0}$ is a (classical) Lévy process satisfying the condition $\mathrm{E}[\log(1 + |X_1|)] < \infty$. The main aim of this section is to establish a similar correspondence between self-adjoint operators with (spectral) distribution in $\mathcal{L}(\boxplus)$ and free Lévy processes (in law).

The stochastic integral in (6.1) is the limit, in probability, as $R \to \infty$, of the stochastic integrals $\int_0^R \mathrm{e}^{-t}\, \mathrm{d}X_t$, i.e. we have

$$\int_0^R \mathrm{e}^{-t}\, \mathrm{d}X_t \xrightarrow{\mathrm{p}} \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}X_t, \qquad \text{as } R \to \infty$$

(the convergence actually holds almost surely; see Proposition 6.3 below). The stochastic integral $\int_0^R \mathrm{e}^{-t}\, \mathrm{d}X_t$ is, in turn, defined as the limit of approximating Riemann sums. More precisely, consider a compact interval $[A, B]$ in $[0, \infty[$, and, for each $n \in \mathbb{N}$, let $\mathcal{D}_n = \{t_{n,0}, t_{n,1}, \ldots, t_{n,n}\}$ be a subdivision of $[A, B]$, i.e.

$$A = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = B.$$

Assume that

$$\lim_{n\to\infty} \max_{j=1,2,\ldots,n} (t_{n,j} - t_{n,j-1}) = 0. \tag{6.2}$$

Moreover, for each n, choose intermediate points

$$t^{\#}_{n,j} \in [t_{n,j-1}, t_{n,j}], \qquad j = 1, 2, \ldots, n. \tag{6.3}$$

Then, for any *continuous* function $f : [A, B] \to \mathbb{R}$, the Riemann sums

$$S_n = \sum_{j=1}^n f(t^{\#}_{n,j}) \cdot (X_{t_{n,j}} - X_{t_{n,j-1}})$$

converge *in probability*, as $n \to \infty$, to a random variable S. Moreover, this random variable S does not depend on the choice of subdivisions $\mathcal{D}_n$ (satisfying (6.2)) or on the choice of intermediate points $t^{\#}_{n,j}$. Hence, it makes sense to call S the stochastic integral of f over $[A, B]$ with respect to (X_t), and we denote S by $\int_A^B f(t)\, \mathrm{d}X_t$.

The construction just sketched depends, of course, heavily on the stochastic continuity of the Lévy process in law (X_t) (condition (iv) of Definition 5.1). A proof of the assertions made above can be found in Lukacs (1975, Theorem 6.2.3). We show next how the above construction carries over, via the Bercovici–Pata bijection, to a corresponding stochastic integral with respect to free Lévy processes (in law).

Theorem 6.1. *Let* (Z_t) *be a free Lévy process* (*in law*)*, affiliated with a* W^**-probability space* $(\mathcal{A}, \tau)$. *Then, for any compact interval* $[A, B]$ *in* $[0, \infty[$ *and any continuous function* $f : [A, B] \to \mathbb{R}$, *the stochastic integral* $\int_A^B f(t)\,\mathrm{d}Z_t$ *exists as the limit in probability* (*see Definition 2.17*) *of approximating Riemann sums. More precisely, there exists a* (*unique*) *self-adjoint operator T, affiliated with* $(\mathcal{A}, \tau)$, *such that, for any sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of sub-divisions of* $[A, B]$ *satisfying* (6.2)*, and for any choice of intermediate points* $t^{\#}_{n,j}$ *as in* (6.3)*, the corresponding Riemann sums*

$$T_n = \sum_{j=1}^{n} f\left(t^{\#}_{n,j}\right) \cdot \left(Z_{t_{n,j}} - Z_{t_{n,j-1}}\right)$$

converge in probability to T as $n \to \infty$. *We call T the stochastic integral of f over* $[A, B]$ *with respect to* (Z_t), *and denote it by* $\int_A^B f(t)\,\mathrm{d}Z_t$.

In the proof below, we shall use the notation

$$\mathop{\ast}_{j=1}^{r} \mu_j := \mu_1 \ast \cdots \ast \mu_r \quad \text{and} \quad \mathop{\boxplus}_{j=1}^{r} \mu_j := \mu_1 \boxplus \cdots \boxplus \mu_r,$$

for probability measures $\mu_1, \ldots, \mu_r$ on $\mathbb{R}$.

Proof. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[A, B]$ satisfying (6.2), let $t^{\#}_{n,j}$ be a family of intermediate points as in (6.3), and consider, for each n, the corresponding Riemann sum:

$$T_n = \sum_{j=1}^{n} f\left(t^{\#}_{n,j}\right) \cdot \left(Z_{t_{n,j}} - Z_{t_{n,j-1}}\right) \in \overline{\mathcal{A}}.$$

We show that (T_n) is a Cauchy sequence with respect to convergence in probability or, equivalently, with respect to the measure topology (see Section 2.5). Given any n, m in $\mathbb{N}$, we form the subdivision

$$A = s_0 < s_1 < \ldots < s_{p(n,m)} = B,$$

which consists of the points in $\mathcal{D}_n \cup \mathcal{D}_m$ (so that $p(n, m) \leqslant n + m$). Then, for each j in $\{1, 2, \ldots, p(n, m)\}$, we choose (in the obvious way) $s^{\#}_{n,j}$ in $\{t^{\#}_{n,k} | k = 1, 2, \ldots, n\}$ and $s^{\#}_{m,j}$ in $\{t^{\#}_{m,k} | k = 1, 2, \ldots, m\}$ such that

$$T_n = \sum_{j=1}^{p(n,m)} f\left(s^{\#}_{n,j}\right) \cdot \left(Z_{s_j} - Z_{s_{j-1}}\right) \quad \text{and} \quad T_m = \sum_{j=1}^{p(n,m)} f\left(s^{\#}_{m,j}\right) \cdot \left(Z_{s_j} - Z_{s_{j-1}}\right).$$

It follows that

$$T_n - T_m = \sum_{j=1}^{p(n,m)} \left(f\left(s_{n,j}^{\#}\right) - f\left(s_{m,j}^{\#}\right) \right) \cdot \left(Z_{s_j} - Z_{s_{j-1}} \right).$$

Let (μ_t) denote the family of marginal distributions of (Z_t), and then consider a classical Lévy process (X_t) with marginal distributions $(\Lambda^{-1}(\mu_t))$ (cf. Proposition 5.4). For each n, form the Riemann sum

$$S_n = \sum_{j=1}^{n} f\left(f t_{n,j}^{\#} \right) \cdot \left(X_{t_{n,j}} - X_{t_{n,j-1}} \right)$$

corresponding to the same $\mathcal{D}_n$ and $t_{n,j}^{\#}$ as above. Then for any n, m in $\mathbb{N}$, we also have that

$$S_n - S_m = \sum_{j=1}^{p(n,m)} \left(f\left(s_{n,j}^{\#}\right) - f\left(s_{m,j}^{\#}\right) \right) \cdot \left(X_{s_j} - X_{s_{j-1}} \right).$$

From this expression, it follows that

$$\begin{aligned}
\mathcal{L}\{S_n - S_m\} &= \mathop{*}_{j=1}^{p(n,\,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} \mathcal{L}\{X_{s_j} - X_{s_{j-1}}\} \\
&= \mathop{*}_{j=1}^{p(n,\,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} \Lambda^{-1}(\mu_{s_j - s_{j-1}}),
\end{aligned}$$

so that (by Theorem 3.5)

$$\begin{aligned}
\Lambda(\mathcal{L}\{S_n - S_m\}) &= \mathop{\boxplus}_{j=1}^{p(n,\,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} \mu_{s_j - s_{j-1}} \\
&= \mathcal{L}\left\{ \sum_{j=1}^{p(n,m)} (f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})) \cdot (Z_{s_j} - Z_{s_{j-1}}) \right\} \\
&= \mathcal{L}\{T_n - T_m\}.
\end{aligned}$$

We know from the classical theory (cf. Lukacs 1975, Theorem 6.2.3), that (S_n) is a Cauchy sequence with respect to convergence in probability, i.e. that $\mathcal{L}\{S_n - S_m\} \xrightarrow{\text{w}} \delta_0$, as n, $m \to \infty$. By continuity of Λ, it follows that

$$\mathcal{L}\{T_n - T_m\} = \Lambda(\mathcal{L}\{S_n - S_m\}) \xrightarrow{\text{w}} \Lambda(\delta_0) = \delta_0, \qquad \text{as } n,\ m \to \infty.$$

By Proposition 2.19, this means that (T_n) is a Cauchy sequence with respect to the measure topology, and since $\overline{\mathcal{A}}$ is complete in the measure topology (Proposition 2.16), there exists an operator T in $\overline{\mathcal{A}}$ such that $T_n \to T$ in the measure topology, i.e. in probability. Since T_n is self-adjoint for each n (see Section 2.5) and since the adjoint operation is continuous with respect to the measure topology (Proposition 2.16), T is necessarily a self-adjoint operator.

It remains to show that the operator T, found above, does not depend on the choice of subdivisions $(\mathcal{D}_n)$ or intermediate points $t_{n,j}^{\#}$. Thus, suppose that (T_n) and (T'_n) are two sequences of Riemann sums of the kind considered above. Then by the argument given above, there exist operators T and T' in $\overline{\mathcal{A}}$ such that $T_n \to T$ and $T'_n \to T'$ in probability. Furthermore, if we consider the 'mixed sequence' $T_1, T'_2, T_3, T'_4, \dots,$ then the corresponding sequence of subdivisions also satisfies (6.2), and hence this mixed sequence also converges in probability to an operator T'' in $\overline{\mathcal{A}}$. Since the mixed sequence has subsequences converging in probability to T and T' respectively, and since the measure topology is a Hausdorff topology (cf. Proposition 2.16), we may thus conclude that $T = T'' = T'$, as desired. □

The stochastic integral $\int_A^B f(t)\,\mathrm{d}Z_t$, introduced above, extends to continuous functions $f : [A, B] \to \mathbb{C}$ in the usual way (the result being non-self-adjoint in general). From the construction of $\int_A^B f(t)\,\mathrm{d}Z_t$ as the limit of approximating Riemann sums, it follows immediately that whenever $0 \leqslant A < B < C$, we have

$$\int_A^C f(t)\,\mathrm{d}Z_t = \int_A^B f(t)\,\mathrm{d}Z_t + \int_B^C f(t)\,\mathrm{d}Z_t,$$

for any continuous function $f : [A, C] \to \mathbb{C}$. Another consequence of the construction, given in the proof above, is the following correspondence between stochastic integrals with respect to classical and free Lévy processes (in law).

Corollary 6.2. *Let (X_t) be a classical Lévy process with marginal distributions (μ_t), and let (Z_t) be a corresponding free Lévy process (in law) with marginal distributions $(\Lambda(\mu_t))$ (cf. Proposition 5.4). Then, for any compact interval $[A, B]$ in $[0, \infty[$ and any continuous function $f : [A, B] \to \mathbb{R}$, the distributions $\mathcal{L}\{\int_A^B f(t)\,\mathrm{d}X_t\}$ and $\mathcal{L}\{\int_A^B f(t)\,\mathrm{d}Z_t\}$ are $*$-infinitely divisible and $\boxplus$-infinitely divisible respectively; moreover,*

$$\mathcal{L}\left\{\int_A^B f(t)\,\mathrm{d}Z_t\right\} = \Lambda\left[\mathcal{L}\left\{\int_A^B f(t)\,\mathrm{d}X_t\right\}\right].$$

Proof. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[A, B]$ satisfying (6.2), let $t^{\#}_{n,j}$ be a family of intermediate points as in (6.3), and consider, for each n, the corresponding Riemann sums:

$$S_n = \sum_{j=1}^{n} f\left(t^{\#}_{n,j}\right) \cdot \left(X_{t_{n,j}} - X_{t_{n,j-1}}\right) \quad \text{and} \quad T_n = \sum_{j=1}^{n} f\left(t^{\#}_{n,j}\right) \cdot \left(Z_{t_{n,j}} - Z_{t_{n,j-1}}\right).$$

Since convergence in probability implies convergence in distribution (Proposition 2.20), it follows from Lukacs (1975, Theorem 6.2.3) and from Theorem 6.1 above that $\mathcal{L}\{S_n\} \xrightarrow{\mathrm{w}} \{\int_A^B f(t)\,\mathrm{d}X_t\}$ and $\mathcal{L}\{T_n\} \xrightarrow{\mathrm{w}} \mathcal{L}\{\int_A^B f(t)\,\mathrm{d}Z_t\}$. Since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are closed with respect to weak convergence (as noted in Section 2.4), it follows that $\mathcal{L}\{\int_A^B f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$ and $\mathcal{L}\{\int_A^B f(t)\,\mathrm{d}Z_t\} \in \mathcal{ID}(\boxplus)$. Moreover, by Theorem 3.5, $\mathcal{L}\{T_n\} = \Lambda(\mathcal{L}\{S_n\})$, for each n in $\mathbb{N}$, and hence the last assertion follows by continuity of Λ. □

We next determine the conditions under which the stochastic integral $\int_0^\infty \mathrm{e}^{-t}\mathrm{d}Z_t$ makes sense as the limit of $\int_0^R e^{-t} dZ_t$, for $R \to \infty$. Again, the result we obtain is derived by virtue of the mapping Λ and the following corresponding classical result:

Proposition 6.3. *Let (X_t) be a classical Lévy process defined on some probability space $(\Omega, \mathcal{F}, P)$, and let (γ, σ) be the generating pair for the $*$-infinitely divisible probability measure $\mathcal{L}\{X_1\}$. Then the following conditions are equivalent:*

(i) $\int_{\mathbb{R}\setminus]-1,1[} \log(1 + |t|)\, \sigma(dt) < \infty$.

(ii) $\int_0^R e^{-t} dX_t$ *converges almost surely, as* $R \to \infty$.

(iii) $\int_0^R e^{-t} dX_t$ *converges in distribution, as* $R \to \infty$.

(iv) $E[\log(1 + |X_1|)] < \infty$.

Proof. This was proved in Jurek and Verwaat (1983, Theorem 3.6.6). We note, though, that in that paper the measure σ in condition (i) is replaced by the Lévy measure ρ appearing in the alternative Lévy–Khinchine representation (2.5) for $\mathcal{L}(X_1\}$. However, since

$$\rho(dt) = \frac{1 + t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t)\, \sigma(dt),$$

it is clear that the integrals $\int_{\mathbb{R}\setminus]-1,1[} \log(1 + |t|)\, \rho(dt)$ and $\int_{\mathbb{R}\setminus]-1,1[} \log(1 + |t|)\, \sigma(dt)$ are finite simultaneously. □

Proposition 6.4. *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and let (γ, σ) be the free generating pair for the $\boxplus$-infinitely divisible probability measure $\mathcal{L}\{Z_1\}$. Then the following statements are equivalent:*

(i) $\int_{\mathbb{R}\setminus]-1,1[} \log(1 + |t|)\, \sigma(dt) < \infty$.

(ii) $\int_0^R e^{-t} dZ_t$ *converges in probability, as* $R \to \infty$.

(iii) $\int_0^R e^{-t} dZ_t$ *converges in distribution, as* $R \to \infty$.

Proof. Let (μ_t) be the family of marginal distributions of (Z_t), and consider a classical Lévy process (X_t) with marginal distributions $(\Lambda^{-1}(\mu_t))$ (cf. Proposition 5.4). By the definition of Λ, it follows that (γ, σ) is the generating pair for the $*$-infinitely divisible probability measure $\mathcal{L}\{X_1\}$.

(i) $\Rightarrow$ (ii). Assume that (i) holds. Then condition (i) of Proposition 6.3 is satisfied for the classical Lévy process (X_t). Hence by (ii) of that proposition, $\int_0^R e^{-t} dX_t$ converges almost surely, and hence in probability, as $R \to \infty$. Consider now any increasing sequence (R_n) of positive numbers, such that $R_n \nearrow \infty$ as $n \to \infty$. Then for any m, n in $\mathbb{N}$ such that $m > n$, we have by Corollary 6.2,

$$\begin{aligned}\mathcal{L}\left\{\int_0^{R_m} e^{-t} dZ_t - \int_0^{R_n} e^{-t} dZ_t\right\} &= \mathcal{L}\left\{\int_{R_n}^{R_m} e^{-t} dZ_t\right\} = \Lambda\left[\mathcal{L}\left\{\int_{R_n}^{R_m} e^{-t} dX_t\right\}\right] \\ &= \Lambda\left[\mathcal{L}\left\{\int_0^{R_m} e^{-t} dX_t - \int_0^{R_n} e^{-t} dX_t\right\}\right]. \end{aligned} \tag{6.4}$$

Since the sequence $(\int_0^{R_n} e^{-t} dX_t)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to convergence in probability, it follows, by continuity of Λ, that so is the sequence $(\int_0^{R_n} e^{-t} dZ_t)_{n\in\mathbb{N}}$. Hence, by Proposition 2.16, there exists a self-adjoint operator W affiliated with $(\mathcal{A}, \tau)$, such that $\int_0^{R_n} e^{-t} dZ_t \to W$ in probability. It remains to argue that W does not depend on the sequence (R_n). This follows, for example as in the proof of Theorem 6.1, by considering, for two given sequences (R_n) and (R_n'), a third increasing sequence (R_n'') containing infinitely many elements from both of the original sequences.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. It follows by (6.4) and continuity of Λ^{-1} that for any increasing sequence (R_n), as above, $(\int_0^{R_n} e^{-t} dX_t)$ is a Cauchy sequence with respect to convergence in probability. We deduce that (iii) of Proposition 6.3 is satisfied for (X_t), and hence so is (i) of that proposition. By definition of (X_t), this means precisely that (i) of Proposition 6.4 is satisfied for (Z_t).

(ii) $\Rightarrow$ (iii). This follows from Proposition 2.20.

(iii) $\Rightarrow$ (i). Suppose (iii) holds, and note that the limit distribution is necessarily $\boxplus$-infinitely divisible. Now by Corollary 6.2 and continuity of Λ^{-1}, condition (iii) of Proposition 6.3 is satisifed for (X_t), and hence so is (i) of that proposition. This means again that (i) in Proposition 6.4 is satisfied for (Z_t). □

If (Z_t) is a free Lévy process (in law) affiliated with $(\mathcal{A}, \tau)$, such that (i) of Proposition 6.4 is satisfied, then we denote by $\int_0^\infty e^{-t} dZ_t$ the self-adjoint operator affiliated with $(\mathcal{A}, \tau)$, to which $\int_0^R e^{-t} dZ_t$ converges in probability as $R \to \infty$. We note that $\mathcal{L}\{\int_0^\infty e^{-t} dZ_t\}$ is $\boxplus$-infinitely divisible, and that Corollary 6.2 and Proposition 2.20 yield the relation

$$\mathcal{L}\left\{\int_0^\infty e^{-t} dZ_t\right\} = \Lambda\left[\mathcal{L}\left\{\int_0^\infty e^{-t} dX_t\right\}\right], \tag{6.5}$$

where (X_t) is a classical Lévy process corresponding to (Z_t) as in Proposition 5.4.

Theorem 6.5. *Let y be a self-adjoint operator affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then the distribution of y is $\boxplus$-self-decomposable if and only if y has a representation in law in the form*

$$y \stackrel{d}{=} \int_0^\infty e^{-t} dZ_t, \tag{6.6}$$

for some free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, and satisfying condition (i) of Proposition 6.4.

Proof. Put $\mu = \mathcal{L}\{y\}$. Suppose first that μ is $\boxplus$-self-decomposable and put $\mu' = \Lambda^{-1}(\mu)$.

Then, by Theorem 4.8, μ' is $*$-self-decomposable, and hence by the classical version of this theorem (cf. Jurek and Verwaat 1983, Theorem 3.2), there exists a classical Lévy proces (X_t), defined on some probability space $(\Omega, \mathcal{F}, P)$, such that condition (i) of Proposition 6.3 is satisfied, and such that $\Lambda^{-1}(\mu) = \mathcal{L}\{\int_0^\infty \mathrm{e}^{-t}\mathrm{d}X_t\}$. Let (Z_t) be a free Lévy process (in law) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, and corresponding to (X_t) as in Proposition 5.4. Then, by definition of Λ, condition (i) of Proposition 6.4 is satisfied for (Z_t) and, by formula (6.5), $\mathcal{L}\{\int_0^\infty \mathrm{e}^{-t}\mathrm{d}Z_t\} = \mu$.

Assume, conversely, that there exists a free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, such that condition (i) of Proposition 6.4 is satisfied, and such that $\mu = \mathcal{L}\{\int_0^\infty \mathrm{e}^{-t}\mathrm{d}Z_t\}$. Then consider a classical Lévy process (X_t) defined on some probability space $(\Omega, \mathcal{F}, P)$, and corresponding to (Z_t) as in Proposition 5.4. Condition (i) in Proposition 6.3 is then satisfied for (X_t) and, by (6.5), $\Lambda^{-1}(\mu) = \mathcal{L}\{\int_0^\infty \mathrm{e}^{-t}\mathrm{d}X_t\}$. Thus, by the classical version of this theorem, $\Lambda^{-1}(\mu)$ is $*$-self-decomposable, and hence μ is $\boxplus$-self-decomposable. $\square$

Remark 6.6. *Free Ornstein–Uhlenbeck processes*. Let y be a self-adjoint operator affiliated with some W^*-probability space $(\mathcal{A}, \tau)$, and assume that there exists a free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, such that condition (i) of Proposition 6.4 is satisfied, and such that $y \overset{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t}\mathrm{d}Z_t$. Note that, for any positive numbers s, λ, we have

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-t}\mathrm{d}Z_t = \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda t} &= \int_s^\infty \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda t} + \int_0^s \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda t}\\ &= \mathrm{e}^{-\lambda s}\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda(s+t)} + \int_0^{\lambda s} \mathrm{e}^{-t}\mathrm{d}Z_t,\end{aligned} \tag{6.7}$$

where we have introduced integration with respect to the processes $V_t = Z_{\lambda t}$ and $W_t = Z_{\lambda(s+t)}$, $t \geqslant 0$. The rules of transformation for stochastic integrals, used above, are easily verified by considering the integrals as limits of Riemann sums. That same point of view, together with the fact that (Z_t) has freely independent stationary increments (conditions (i) and (iii) of Definition 5.2), implies, furthermore, that $\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda(s+t)} \overset{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}Z_{\lambda t} \overset{\mathrm{d}}{=} y$. Note also that the two terms in the last expression of (6.7) are freely independent. Thus (6.7) shows that, for any positive numbers s, λ, we have a decomposition of the form $y \overset{\mathrm{d}}{=} \mathrm{e}^{-\lambda s}y(\lambda, s) + u(\lambda, s)$, where $y(\lambda, s)$ and $u(\lambda, s)$ are freely independent, and where $y(\lambda, s) \overset{\mathrm{d}}{=} y$. In particular, we have verified directly that $\mathcal{L}\{y\}$ is $\boxplus$-self-decomposable. Moreover, if we choose a self-adjoint operator Y_0 affiliated with $(\mathcal{B}, \psi)$, which is freely independent of (Z_t), and such that $\mathcal{L}\{Y_0\} = \mathcal{L}\{y\}$ (extend $(\mathcal{B}, \psi)$ if necessary), then the expression

$$Y_s = \mathrm{e}^{-\lambda s}Y_0 + \int_0^{\lambda s} \mathrm{e}^{-t}\mathrm{d}Z_t, \qquad s \geqslant 0,$$

defines an operator-valued stochastic process (Y_s) affiliated with $(\mathcal{B}, \psi)$, satisfying $Y_s \overset{\mathrm{d}}{=} y$ for all s. If we replace (Z_t) above by a classical Lévy process (X_t) satisfying condition (i) of Proposition 6.3, and let Y_0 be a (classical) random variable which is independent of (X_t), then the corresponding process (Y_s) is a solution to the stochastic differential equation

$$\mathrm{d}Y_s = -\lambda Y_s \,\mathrm{d}s + \mathrm{d}X_{\lambda s},$$

and (Y_s) is said to be a process of *Ornstein–Uhlenbeck type*; cf. Barndorff-Nielsen and Shephard (2001a; 2001b) and references therein.

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