Bernoulli **8**(5), 2002, 627–642

Irregular sets and central limit theorems

GONZALO PERERA

Universidad de la República, Julio Herrera y Reissig 565, Montevideo, Uruguay.
E-mail: gperera@fing.edu.uy

In previous papers we have studied the asymptotic behaviour of $S_N(A; X) = (2N+1)^{-d/2}\sum_{n\in A_N} X_n$, where X is a centred, stationary and weakly dependent random field, and $A_N = A \cap [-N, N]^d$, $A \subset \mathbb{Z}^d$. This leads to the definition of asymptotically measurable sets, which enjoy the property that $S_N(A; X)$ has a (Gaussian) weak limit for any X belonging to a certain class. We present here an application of this technique. Consider a regression model $X_n = \varphi(\xi_n, Y_n)$, $n \in \mathbb{Z}^d$, where X_n is centred, φ satisfies certain regularity conditions, and ξ and Y are independent random fields; for any $m \in \mathbb{N}$, and $(y_1, \ldots, y_m)$ the central limit theorem holds for $(\varphi(\xi, y_1), \ldots, \varphi(\xi, y_m))$, but Y satisfies only the strong law of large numbers as it applies to $(Y_m, Y_{m-n})_{m\in\mathbb{Z}^d}$, for any $n \in \mathbb{Z}^d$. Under these conditions, it is shown that the central limit theorem holds for X.

Keywords: asymptotically measurable collections of sets; central limit theorems; level sets; regression models; weakly dependent random fields

1. Introduction

The notion of an 'asymptotically measurable set' (AMS) was introduced in Perera (1994a; 1994b), motivated by statistical problems concerning random fields.

Let us denote by $\mathbb{Z}^d$ the lattice of points in $\mathbb{R}^d$ with integer coordinates. A subset A of $\mathbb{Z}^d$ is said to be an AMS if, for each $n \in \mathbb{Z}^d$, the limit, as N tends to infinity, of $F_N(n; A) = (2N+1)^{-d}\,\mathrm{card}\{A_N \cap (n + A_N)\}$ exists, where $A_N = A \cap [-N, N]^d$; furthermore, we will denote $F(n; A)$ this limit, and $M(\mathbb{Z}^d)$ the class of asymptotically measurable sets.

The main property of this class of sets is the following: denote by $\mathcal{F}$ the class of centred, stationary random fields with finite second moment which satisfy certain weak-dependence conditions, and let

$$S_N(A; X) = \frac{1}{\sqrt{(2N+1)^d}} \sum_{n\in A_N} X_n.$$

Then $S_N(A; X)$ has a non-trivial weak limit for any $X \in \mathcal{F}$ if and only if $A \in M(\mathbb{Z}^d)$.

More precisely, if $A \in M(\mathbb{Z}^d)$, the central limit theorem (CLT) applies and $S_N(A; X)$ converges to a Gaussian law whose covariance depends on the covariances of X and the function $F(\cdot; A)$; if $A \notin M(\mathbb{Z}^d)$, there exists a Gaussian m-dependent random field such that $S_N(A; X)$ has different weak limits for two different subsequences of Ns (see Perera 1994a; 1997). Sets with regular borders (in the sense that their borders are negligible), periodic sets and certain random sets are examples of elements of $M(\mathbb{Z}^d)$.

1350–7265

Let us look in more detail at two examples of AMSs.

Example 1. If dist is the distance induced in $\mathbb{Z}^d$ by the supremum norm (denoted by $\|\cdot\|$), and if $A \subset \mathbb{Z}^d$, then define $\partial A = \{n \in A : \mathrm{dist}(n, A^c) = 1\}$. It is easy to prove the following inequality (see Perera 1997):

$$\mathrm{card}\{A_N \cap (A_N^c - n)\} \leqslant d(\|n\| + 1)\mathrm{card}\{(\partial A)_N\}, \qquad \forall n \in \mathbb{Z}^d. \tag{1}$$

Assume that A satisfies

$$\lim_N \frac{\mathrm{card}(A_N)}{(2N+1)^d} = \upsilon(A) > 0, \qquad \lim_N \frac{\mathrm{card}(\partial A_N)}{(2N+1)^d} = 0$$

(this is what we meant earlier by the expression 'sets with regular borders'). It follows from (1) that A is an AMS with $F(n; A) = \upsilon(A)$ for all $n \in \mathbb{N}$.

Example 2. The most interesting examples are provided by level sets of random fields. Let $Y = (Y_n)_{n \in \mathbb{Z}^d}$ be a random field and let $B^1, \ldots, B^k$ be real Borel sets; define

$$A^j(\omega) = \{n \in \mathbb{Z}^d : Y_n(\omega) \in B^j\}.$$

Note that

$$\frac{\mathrm{card}\left\{A_N^i \cap (A_N^j - n)\right\}}{(2N+1)^d} = \frac{1}{(2N+1)^d} \sum_{m \in A_N} \mathbf{1}_{\{Y_m \in B^i, Y_{m+n} \in B^j\}}. \tag{2}$$

Observe that if Y is stationary, then by ergodic theorem (see Guyon 1995, p. 108) the limit of the expression in (2) exists for any n with probability one (hence $A^1, \ldots, A^k$ is almost surely an AMS) and its limit equals $R_n(B^i \times B^j)$, where R_n is a (random) probability measure on $\mathbb{R}^2$. Further, since a finite number of coordinates does not affect limits of averages, it is clear that, as a function of ω, R_n is σ_∞^Y-measurable, where

$$\sigma_\infty^Y = \bigcap_{h=1}^{\infty} \sigma(\{Y_m : \|m\| \geqslant h\}).$$

We conclude that, if Y satifies the ergodic property

$$\sigma_\infty^Y \text{ trivial (i.e. } P(A)(1 - P(A)) = 0 \forall A \in \sigma_\infty^Y),$$

then R_n is non-random and $R_n(B \times C) = P(Y_0 \in B, Y_n \in C)$ for any Borel sets B, C in $\mathbb{R}$. We will return to this example later on.

On the other hand, a direct construction of a family of sets (with the power of the continuum) which do not belong to $M(\mathbb{Z}^d)$ has been given (see Perera 1994a; 1997). The basic idea of this construction is presented in the following example.

Example 3. For the sake of simplicity, take $\mathbb{N}$ instead of $\mathbb{Z}^d$ and replace $(2N+1)^d$ by N in the definition of AMS. Define $I(n) = [100^{2^{n-1}}, 100^{2^n})$, $A(n, 0) = I(n) \cap (5\mathbb{N})$, $A(n, 1) = I(n) \cap [(10\mathbb{N}) \cup (10\mathbb{N} + 1)]$, $n \in \mathbb{N}$, and let $A = \bigcup_{i=1}^{\infty} (A(2i, 0) \cup A(2i+1, 1))$. By a straightforward computation, we can check that, for this set A $F(1; A)$ does not exist, and hence A is not an AMS.

For statistical purposes, a generalization of the notion of an AMS is needed. We will say that a collection $\{A^i: i = 1, \ldots, r\}$ of subsets of $\mathbb{Z}^d$ is an asymptotically measurable collection (AMC) if

$$\lim_N F_N(n; A^i, A^j) = F(n; A^i, A^j) \qquad \forall n \in \mathbb{Z}^d,\ i, j = 1, \ldots, r,$$

where $F_N(n; A^i, A^j) = (2N+1)^{-d}\,\mathrm{card}\{A^i_N \cap (n + A^j_N)\}$.

Let us introduce some notation. If Z is an $\mathbb{R}^d$-valued random vector, P^Z will denote its probability distribution. The weak convergence of probability measures will be denoted by '$\overset{w}{\Longrightarrow}$'. If U is a fixed random variable with values in some measurable space, then '$Z_N \underset{N}{\overset{w/U}{\Longrightarrow}} Z$' refers to a conditional weak convergence: the law of Z_N conditioned upon U converges almost surely to the law of Z conditioned upon U.

The symbol '0' will represent both the real zero and the zero element of $\mathbb{R}^d$; the context will make its meaning clear. $N_r(\mu, \Sigma)$ denotes a Gaussian distribution in $\mathbb{R}^r$ with mean vector μ and covariance matrix Σ; when $r = 1$, the index will be omitted. We also denote by $N_r(\mu, \Sigma)$ a random vector which follows this distribution; for instance, $P(N(0, 1) \in B)$ denotes the probability under the standard Gaussian law of the Borel set B.

The symbol '$:=$' will be used to indicate an entity that is defined implicitly in the middle of a computation; δ_{ij} denotes Kronecker's function. C will denote a generic constant that may change from line to line; when needed, expressions of the type $C(J, X)$ will indicate its dependence on certain parameters.

Now consider $X = (X^1, \ldots, X^r)$, an $\mathbb{R}^r$-valued, centred, stationary and weakly dependent random field, and define

$$M_N(A^1, \ldots, A^r; X^1, \ldots, X^r) = (S_N(A^1; X^1), \ldots, S_N(A^r; X^r)).$$

Then $M_N(A^1, \ldots, A^r; X^1, \ldots, X^r)$ converges weakly for any X in a suitable class if and only if $A^1, \ldots, A^r$ is an AMC.

For instance, take $\|x\| = \max_{1 \leqslant i \leqslant r} |x^i|$, $x = (x^1, \ldots, x^r) \in \mathbb{R}^r$, and denote by S the class of centred, stationary random fields with finite second moments such that the following conditions hold:

(C1) $\sum_{n \in \mathbb{Z}^d} |\mathrm{E}\{X^i_0 X^j_n\}| < \infty$, $\sum_{n \in \mathbb{Z}^d} \mathrm{E}\{X^i_0 X^i_n\} > 0$, $i, j = 1, \ldots, r$.

(C2) There exists a sequence $b(J)$ such that $\lim_J b(J) = 0$ and, for each $A \subset \mathbb{Z}^d$, we have

$$\mathrm{E}\{(S_N(A; X - X^J))^2\} \leqslant b(J) \frac{\mathrm{card}(A_N)}{(2N+1)^d},$$

where X^J is the truncation by J of the random field X; that is $X^J_n = X_n \mathbf{1}_{\{\|X_n\| \leqslant J\}} - \mathrm{E}\{X_n \mathbf{1}_{\{\|X_n\| \leqslant J\}}\}$.

(C3) For each $J > 0$, there is a number $c(X, J)$, depending only on X and J, such that, for all $N \geqslant 1$ and $A \subset [-N, N]^d$, we have

$$\mathrm{E}\{(S_N(A;\, X^i))^4\} \leqslant c(X,\, J)\left(\frac{|A_N|}{(2N+1)^d}\right)^2, \qquad i = 1,\, \ldots,\, r.$$

(C4) There exists a sequence $d(J)$ such that $\lim_J d(J) = 0$, and a bounded real function g such that, for any pair $(A,\, B)$, where $A,\, B \subset \mathbb{Z}^d$ satisfy $\mathrm{dist}(A,\, B) \geqslant J$, we have

$$|\mathrm{cov}\{\exp[\mathrm{i}S_{\mathrm{N}}(A;\, \langle t,\, X\rangle)],\, \exp[iS_N(B;\, \langle t,\, X\rangle)]\}| \leqslant d(J)g(t), \qquad t \in \mathbb{R}^r.$$

Proposition 1. *Suppose conditions* (C1)–(C4) *are fulfilled.:*

(*a*) *If* $A^1,\, \ldots,\, A^r$ *is an AMC, then* $M_N(A^1,\, \ldots,\, A^r;\, X^1,\, \ldots,\, X^r)) \underset{N}{\overset{w}{\Longrightarrow}} N_r(0,\, \sum)$, *where* $\Sigma(i,\, j) = \sum_{n\in\mathbb{Z}^d} F(n;\, A^i,\, A^j)\mathrm{E}\{X_0^i\, X_n^j\}$.

(*b*) *If* $A^1,\, \ldots,\, A^r$ *is not an AMC, then there exists a Gaussian m-dependent random field* X *such that* $M_N(A^1,\, \ldots,\, A^r;\, X^1,\, \ldots,\, X^r)$ *does not have a weak limit.*

The proof of this proposition is obtained by Bernshtein's (1944) 'large and small blocks' method and is similar to the proof of Proposition 2.2 in Perera (1997), where the role of $d(J)$ is played by the strong mixing coefficient

$$\alpha^X(J) = \sup\{|P(A \cap B) - P(A)P(B)|:\, A \in \sigma^X(R),\, B \in \sigma^X(T),$$
$$R \text{ rectangle},\, T \subset \mathbb{Z}^d,\, \mathrm{dist}(R,\, T) \geqslant J\},$$

and where $\sigma^X(T)$ denotes the σ-algebra generated by $\{X_n:\, n \in T\}$. Of course, to check condition (C4) is not a trivial matter, but it can be done under various weak-dependence assumptions. We can also find similar results based on other weak-dependence assumptions which in many examples are easier to check than (C4). We refer to Doukhan and Louhichi (1997) for a survey of weak-dependence models where (C4) holds, and, in what follows, we present a brief summary of different weak-dependence assumptions and more precise references to alternative results.

Mixing random fields have been extensively studied in the last four decades (see Rosenblatt 1956; Kolmogorov and Rozanov 1960; Bradley 1986; Doukhan 1995). It is well known that α-mixing conditions, such as those employed in Perera (1997), are too strong for many interesting models, such as Gibbs fields (see Dobrushin 1968). Indeed, for random fields, α-mixing conditions on arbitrary large sets are equivalent to ρ-mixing conditions by the Kolmogorov–Rozanov–Bradley inequality (see Bradley 1993). These remarks have stimulated several results obtained by Stein's method, where mixing conditions are assumed only over small sets (see, for instance, Bolthausen 1982). However, rates of mixing are often required in Steins's method. Results similar to Proposition 1 can also be obtained using Stein's method and Rio's (1993) inequalities for covariances (see Perera 1997, Propositions 2.4 and 2.5). For a nice related result, see also Dedecker (1998). In the last three decades, an asymptotic theory of other notions of weak dependence has been developed. This is the case for association (see Esary *et al*. 1967; Birkel 1988; Roussas 1994; Yu 1993), where non-correlation and coordinatewise independence are found to be equivalent (see Newman 1984). However, association is also a restrictive condition for many interesting models: for instance, a Gaussian random field is associated if and only if its covariances are non-negative (see Pitt 1982). Furthermore, it is easy to show that some very simple linear processes are neither mixing nor associated (see Rosenblatt 1980; Andrews

1984). More recently, Doukhan and Louhichi (1997) proposed a definition of 'weak dependence' for time series which includes mixing, association, linear processes and Markov models as particular cases, and which enables one to obtain CLTs and other asymptotic results; this type of a framework could provide a unified approach to weakly dependent random fields.

Here we shall establish a CLT regression model of a certain specific type. We will consider a weakly dependent random field ξ. In this context, we will not assume a particular setting of weak dependence; we will simply assume that certain CLTs hold for ξ, a condition that can be checked for mixing, associated, linear or Markov random fields using a suitable version of Proposition 1. We shall also consider another random field Y, independent of ξ. Y will be assumed to satisfy only a certain law of large numbers. The random field X which we will 'observe' is a smooth (in a sense to be specified later on) function of ξ and Y. We will call this type of regression models *I-decomposable* and *reachable*. The 'I' in 'I-decomposable' emphasizes the key fact that both 'components' are *independent*. The term 'reachable' comes from the fact that we can approximate Y in a suitable way by a process taking values on a finite set. Note that the dependence structure of (ξ, Y) (hence that of X) can fail to be weakly dependent in the sense of mixing, association, linearity, etc.

The proof of our result is simple and is based on the following idea: as already stated, the problem can be reduced to the case of a finite-valued Y. If Y takes values in a finite set, X can be decomposed into a finite number of random fields observed over an AMC, and the CLT will be derived.

We will give all the details of the proof when the regression is instantaneous; that is,

$$X_n = \varphi(\xi_n, Y_n).$$

The extension to the case when X_n is a function of a finite number of coordinates of (ξ, Y) is straightforward. The extension to functions of the whole trajectory (for instance $X_n = \sum_{m\in\mathbb{Z}^d} a_m f(\xi_{n+m}) g(Y_{n+m})$, for suitable f, g, $(a_m)_{m\in\mathbb{Z}^d}$) follows by standard arguments.

The main purpose of this paper is to show that the study of the asymptotic behaviour of additive functionals of dependent fields over 'irregular sets' gives, as very simple consequences, asymptotic results for some interesting non-stationary models. A second part of this work presents analogous results for compound Poisson limit theorems for high-level exceedances (see Bellanger and Perera 1997).

2. Definitions and results

Definition 1. *Let $Y = \{Y_n : n \in \mathbb{Z}^d\}$ be a (real-valued) random field. We will say that Y is consistent if, for any $r \in \mathbb{N}$, the following conditions hold:*

(H1) *There exists a probability measure R_0 on the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}$ such that*

$$\frac{1}{(2N+1)^d} \sum_{m\in[-N,N]^d} P^{Y_m}(B) \underset{N}{\to} R_0(B), \qquad \forall B \in \mathcal{B}.$$

(H2) *For each* $n \in \mathbb{Z}^d - \{0\}$, *there exists a probability measure* R_n *defined on* $\mathcal{B}_2$, *the Borel σ-algebra in* $\mathbb{R}^2$, *such that*

$$\frac{1}{(2N+1)^d} \sum_{m\in[-N,N]^d} P^{(Y_m, Y_{m-n})}(B \times C) \underset{N}{\to} R_n(B \times C), \qquad \forall B, C \in \mathcal{B}.$$

(H3) *For each pair* B, C *of Borel sets, and for any* $n \in \mathbb{Z}^d$

$$\frac{1}{(2N+1)^d} \sum_{m\in[-N,N]^d} [\mathbf{1}_B(Y_m)\mathbf{1}_C(Y_{m-n}) - P(Y_m \in B, Y_{m-n} \in C)] \overset{\text{a.s.}}{\underset{N}{\to}} 0.$$

Remark 1. A straightforward computation shows that $R_n(B \times C) = R_{-n}(C \times B)$ for any $n \in \mathbb{Z}^d - \{0\}$ and B, C Borel sets. Applying the Cauchy–Schwarz inequality twice in (H2), we deduce the inequality $R_n(B \times C) \leqslant \sqrt{R_0(B)}\sqrt{R_0(C)}$.

Example 4. If the coordinates of Y have a common probability distribution, (H1) holds with $R_0 = P^{Y_0}$. In addition, if Y is stationary, (H2) also holds with $R_n = P^{(y_n, y_0)}$. Many weak-dependence assumptions imply (H3). For instance, Rosenthal inequalities and a Borel–Cantelli argument guarantee (H3): Rosenthal inequalities can be obtained under different mixing assumptions (see Bryc and Smolenski 1993; Rio 1993), association (see Birkel 1988), and for linear or Markov random fields. More precisely, we say that a centred random field $Z = \{Z_n : n \in \mathbb{Z}^d\}$ with bounded second moment satisfies a Rosenthal inequality of order $q > 2$ if there exists a constant $C(q)$ such that, for any finite $F \subset \mathbb{Z}^d$, we have

$$\mathrm{E}\left(\left|\sum_{m\in F} Z_m\right|^q\right) \leqslant C(q)\left[\sum_{m\in F} \mathrm{E}(|Z_m|^q) + \left(\sum_{m\in F} \mathrm{E}(Z_m^2)\right)^{q/2}\right].$$

We refer to Doukhan and Louhichi (1997) for a review of different situations where Rosenthal inequalities apply.

A more precise example: if the process Y satisfies the mixing condition $\rho^X(1) < 1$, where

$$\rho^X(J) = \sup\{|\mathrm{corr}(X, Y)| : X \in L^2(\sigma^X(R)), Y \in L^2(\sigma^X(S)), R, S \subset \mathbb{Z}^d, d(R, S) \geqslant J\},$$

then, for any Borel sets B, C and for any $n \in \mathbb{Z}^d$, the strong law of large numbers applies to

$$Z_m = \mathbf{1}_B(Y_m)\mathbf{1}_C(Y_{m-n}) - P(Y_m \in B, Y_{m-n} \in C),$$

and hence (H3) holds (see Bryc and Smolenski 1993, Theorem 1).

Lemma 1. *Let* Y *be a consistent random field, and let* $B^1, \ldots, B^r$ *be real Borel sets; define*

$$A^i(\omega) = \{n \in \mathbb{Z}^d : Y_n(\omega) \in B^i\}.$$

Then $A^1, \ldots, A^r$ *is an AMC almost surely, with, for* $i, j = 1, \ldots, r$,

$$F(n; A^i, A^j) = R_n(B^i \times B^j), \qquad n \in \mathbb{Z}^d - \{0\},$$
$$F(0; A^i, A^j) = R_0(A^i)\delta_{ij}.$$

Proof. Fix $n \in \mathbb{Z}^d - \{0\}$. We have

$$F_N(n;\, A^i,\, A^j) = \frac{1}{(2N+1)^d} \sum_{m \in [-N,N]^d \cap (n+[-N,N]^d)} \mathbf{1}_{B^i}(Y_m)\mathbf{1}_{B^j}(Y_{m-n}).$$

It is easy to check that

$$|\mathrm{card}([-N,\, N]^d) - \mathrm{card}([-N,\, N]^d \cap (n + [-N,\, N]^d))| \leqslant C(d,\, n)N^{d-1};$$

it follows that the asymptotic behaviour of $F_N(n;\, A^i,\, A^j)$ is the same as that of

$$\frac{1}{(2N+1)^d} \sum_{m \in [-N,N]^d} \mathbf{1}_{B^i}(Y_m)\mathbf{1}_{B^j}(Y_{m-n}).$$

By (H3), we conclude that this quantity has almost surely the same limit as

$$\frac{1}{(2N+1)^d} \sum_{m \in [-N,N]^d} P(Y_m \in B^i,\, Y_{m-n} \in B^j);$$

(H2) implies that the limit equals $R_n(B^i \times B^j)$. For $n = 0$, the proof is even simpler. □

Definition 2. *A family of stationary, centred random fields $\{X^{(y)}\colon y \in \mathbb{R}\}$; is said to be totally pre-Gaussian if the following conditions hold:*

(H4) *For each pair $y, z \in \mathbb{R}$, we have*

$$\sum_{n \in \mathbb{Z}^d} |\mathrm{E}\{X_0^{(y)} X_n^{(z)}\}| < \infty.$$

(H5) *If we define: $\Psi\colon \mathbb{R}^2 \to \mathbb{R}$, $\psi\colon \mathbb{R}^2 \to \mathbb{R}$, by*

$$\Psi(y,\, z) = \sum_{n \in \mathbb{Z}^d} |\mathrm{E}\{X_0^{(y)} X_n^{(z)}\}|,\ \psi(y,\, z) = \sum_{n \in \mathbb{Z}^d} \mathrm{E}\{X_0^{(y)} X_n^{(z)}\},$$

then Ψ is bounded and ψ is continuous in $\mathbb{R}^2$.

(H6) *For each $r = 1, 2, \ldots,$ for each AMC $A^1, \ldots, A^r$ and any $(y_1, \ldots, y_r) \in \mathbb{R}^r$, we have*

$$M_N(A^{(1)}, \ldots, A^{(r)}, X^{(y_1)}, \ldots, X^{(y_r)}) \underset{N}{\overset{w}{\Longrightarrow}} N_r(0, \Sigma),$$

where $\Sigma(i,\, j) = \sum_{n \in \mathbb{Z}^d} \mathrm{E}\{X_0^{(y_i)} X_n^{(y_j)}\} F(n,\, A^i,\, A^j)$.

Definition 3. *A random field $X = \{X_n\colon n \in \mathbb{Z}^d\}$ is said to be I-decomposable if the following conditions hold:*

(H7) *There exist two random fields $\xi = \{\xi_n\colon n \in \mathbb{Z}^d\}$ and $Y\{Y_n\colon n \in \mathbb{Z}^d\}$ and a continuous function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ such that ξ is stationary, Y is consistent, $\xi \perp\!\!\!\perp Y$ and X follows the regression model $X_n = \varphi(\xi_n,\, Y_n)$ for all $n \in \mathbb{Z}^d$.*

(H8) *For each $y \in \mathbb{R}$, we have $\mathrm{E}\{\varphi(\xi_0,\, y)\} = 0$.*

(H9) *If we set $X_n^{(y)} = \varphi(\xi_n,\, y)$, $y \in \mathbb{R}$, then the family $\{X^{(y)}\colon y \in \mathbb{R}\}$ is totally pre-Gaussian.*

To simplify the notation, if X is I-decomposable, we will write

$$\Gamma(n;\, y,\, z) = \mathrm{E}\{\varphi(\xi_0,\, y)\varphi(\xi_n,\, z)\}, \qquad n \in \mathbb{Z}^d,\ y,\, z \in \mathbb{R}.$$

Proposition 2. *Assume that X is I-decomposable and that the coordinates of Y take values in a finite set $\{y_1, \ldots, , y_r\}$. If $S_N(X) := S_N(\mathbb{Z}^d; X)$ then*

$$S_N(X) \underset{N}{\overset{w}{\Longrightarrow}} N(0, \sigma^2),$$

where

$$\sigma^2 = \sum_{i,j=1}^{r} \sum_{n \in \mathbb{Z}^d - \{0\}} \Gamma(n;\, y_i,\, y_j) R_n(\{y_i\} \times \{y_j\}) + \sum_{i=1}^{r} \Gamma(0;\, y_i,\, y_i) R_0(\{y_i\}).$$

Proof. For $i = 1, \ldots, r$, let us define

$$S_N^{(i)} = \frac{1}{\sqrt{(2N+1)^d}} \sum_{n \in [-N,N]^d} \varphi(\xi_n,\, i) \mathbf{1}_{\{Y_n = y_i\}}.$$

By Lemma 1, and the independence of ξ and Y, we have that, for each ω in a set of probability one, the vector $(S_N^{(1)}, \ldots, S_N^{(r)})$, conditioned upon Y, has the same distribution as $(S_N(A^{(1)};\, X^{(1)}), \ldots, S_N(A^{(r)};\, X^{(r)}))$, where $A^{(1)}, \ldots, A^{(r)}$ is an AMC with

$$F(n;\, A^i,\, A^j) = R_n(\{y_i\} \times \{y_j\}) \qquad \text{for } n \neq 0,$$

$$F(0,\, A^i,\, A^j) = \delta_{ij} R_0(\{y_i\}).$$

By (H6) it follows that $(S_N^{(1)}, \ldots, S_N^{(r)}) \underset{N}{\overset{w/Y}{\Longrightarrow}} N_r(0, \Sigma)$, where

$$\Sigma(i,\, j) = \sum_{n \in \mathbb{Z}^d - \{0\}} \Gamma(n;\, y_i,\, y_j) R_n(\{y_i\} \times \{y_j\}) \qquad \text{if } i \neq j,$$

$$\Sigma(i,\, i) = \sum_{n \in \mathbb{Z}^d - \{0\}} \Gamma(n;\, y_i,\, y_i) R_n(\{y_i\} \times \{y_i\}) + \Gamma(0;\, y_i,\, y_i) R_0(\{y_i\}).$$

Therefore, since $S_N(X) = \sum_{i=1}^{i=r} S_N^{(i)}$, $P(S_N(X) \leqslant t/Y) \underset{N}{\to} P(N(0, \sigma^2) \leqslant t)$ for any real t; since the limit does not depend on Y, the result follows by integration and the dominated convergence theorem. □

Definition 4. *Let X be a random field such that* (H7) *holds. We will say that X is reachable if the following conditions hold:*

(H10) *For each $(x, y) \in \mathbb{R}^2$, $\partial\varphi(x, y)/\partial y$ exists and is a continuous function of the second argument.*

(H11) *For each pair $w, z \in \mathbb{R}$, we have*

$$\eta(w, z) = \sum_{n \in \mathbb{Z}^d} \left| \mathrm{E}\left\{ \frac{\partial \varphi}{\partial y}(\xi_0, w) \frac{\partial \varphi}{\partial y}(\xi_n, z) \right\} \right| < \infty,$$

and η is bounded over compact subsets of $\mathbb{R}^2$.

Remark 2. Many weak-dependence settings ensure the conditions of Definitions 3 and 4 (mixing, association, linear fields); we refer again to Doukhan and Louhichi (1997) for a detailed list of examples.

Theorem 1. *Let X be I-decomposable and reachable. Then*

$$S_N(X) \underset{N}{\overset{w}{\Longrightarrow}} N(0, \sigma^2),$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} \mathrm{E}\{\varphi(\xi_0, x)^2\} \, \mathrm{d}R_0(x, x) + \sum_{n \in \mathbb{Z}^d - \{0\}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{E}\{\varphi(\xi_0, x)\varphi(\xi_n, y)\} \, \mathrm{d}R_n(x, y).$$

Proof. Given a pair of positive integers J, L, we define the random field $Y(J, L)$ as follows:

$$\begin{aligned} Y(J, L)_n &= \frac{i}{2^L} && \text{if } Y_n \in \left[\frac{i}{s^L}, \frac{i+1}{2^L}\right), \ -J2^L \leqslant i \leqslant J2^L - 1, \\ Y(J, L)_n &= -J && \text{if } Y_n < -J, \\ Y(J, L)_n &= J && \text{if } Y_n \geqslant J. \end{aligned}$$

Fix J, L. Since Y is consistent, it is clear that so is $Y(J, L)$, and Proposition 1 applies to the I-decomposable random field $X(J, L)$ defined by

$$X(J, L)_n = \varphi(\xi_n, Y_n(J, L)).$$

It follows that $S_N(X(J, L)) \underset{N}{\overset{w}{\Longrightarrow}} N(0, \sigma(J, L)^2)$, where

$$\sigma(J, L)^2 = A(J, L) + B(J, L) + C(J, L) + D(J, L) + E(J),$$

with

$$A(J,\,L)=\sum_{i,j=-J2^L}^{i,j=J2^L-1}\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma\left(n;\,\frac{i}{2^L},\,\frac{j}{2^L}\right)R_n\left(\left[\frac{i}{2^L},\,\frac{i+1}{2^L}\right)\times\left[\frac{j}{2^L},\,\frac{j+1}{2^L}\right)\right),$$

$$B(J,\,L)=2\sum_{i=-J2^L}^{i=J2^L-1}\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma\left(n;\,\frac{i}{2^L},\,-J)R_n\left(\left[\frac{i}{2^L},\,\frac{i+1}{2^L}\right)\times(-\infty,\,-J)\right)$$

$$+\,2\sum_{i,j=-J2^L}^{i,j=J2^L-1}\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma\left(n;\,\frac{i}{2^L},\,J\right)R_n\left(\left[\frac{i}{2^L},\,\frac{i+1}{2^L}\right)\times[J,\,\infty)\right),$$

$$C(J)=\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma(n;\,-J,\,-J)R_n((-\infty,\,-J))\times(-\infty,\,-J))$$

$$+\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma(n;\,J,\,J)R_n([J,\,\infty))\times[J,\,\infty))$$

$$+\,2\sum_{n\in\mathbb{Z}^d-\{0\}}\Gamma(n;\,-J,\,J)R_n((-\infty,\,-J))\times[J,\,\infty)),$$

$$D(J,\,L)=\sum_{i=-J2^L}^{i=J2^L-1}\Gamma\left(0;\,\frac{i}{2^L}\right)^2R_0\left(\left[\frac{i}{2^L},\,\frac{i+1}{2^L}\right)\right),$$

$$E(J)=\Gamma(0;\,J,\,J)R_0([J,\,\infty))+\Gamma(0;\,-J,\,-J)R_0((-\infty,\,J)).$$

Now fix J. By (H5) and the dominated convergence theorem, it follows that

$$A(J,\,L)\underset{L}{\rightarrow}\sum_{n\in\mathbb{Z}^d-\{0\}}\int_{-J}^{J}\int_{-J}^{J}\Gamma(n;\,x,\,y)\,\mathrm{d}R_n(x,\,y):=A(J),$$

$$B(J,\,L)\underset{L}{\rightarrow}\sum_{n\in\mathbb{Z}^d-\{0\}}\int_{-J}^{J}\int_{-J}^{J}\Gamma(n;\,x,\,-J)\,\mathrm{d}R_n(x,\,y)$$

$$+\,2\sum_{n\in\mathbb{Z}^d-\{0\}}\int_{-J}^{J}\int_{J}^{\infty}\Gamma(n;\,x,\,J)\,\mathrm{d}R_n(x,\,y):=B(J),\tag{3}$$

$$D(J,\,L)\underset{L}{\rightarrow}\int_{-J}^{J}\Gamma(0;\,x,\,x)\,\mathrm{d}R_0(x):=D(J).$$

Therefore,

$$\lim_L\sigma^2(J,\,L)=A(J)+B(J)+C(J)+D(J)+E(J).$$

Applying (H5), we easily arrive at

$$A(J) \underset{J}{\to} \sum_{n \in \mathbb{Z}^d - \{0\}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(n;\, x,\, y)\, \mathrm{d}R_n(x,\, y),$$

$$D(J) \underset{J}{\to} \int_{-\infty}^{\infty} \Gamma(0;\, x,\, x)\, \mathrm{d}R_0(x,\, x), \tag{4}$$

$$B(J) \underset{J}{\to} 0, \qquad C(J) \underset{J}{\to} 0, \qquad E(J) \underset{J}{\to} 0.$$

Therefore, from (3) and (4), we conclude that

$$\lim_J \lim_L \sigma^2(J,\, L) = \sigma^2. \tag{5}$$

On the other hand, we have $S_N(X) - S_N(X(J,\, L)) = S_N(J,\, L) + s_N(J)$, where

$$S_N(J,\, L) = \frac{1}{(2N+1)^2} \sum_{n \in [-N,N]^d} [X_n - X_n(J,\, L)] \mathbf{1}_{\{|Y_n| \geqslant J\}},$$

$$s_N(J) = \frac{1}{(2N+1)^d} \sum_{n \in [-N,N]^d} [X_n - X_n(J,\, L)] \mathbf{1}_{\{|Y_n| < J\}}.$$

It follows that

$$\mathrm{E}\{[S_N(X) - S_N(X(J,\, L))]^2\} \leqslant 2(\mathrm{E}\{S_N(J,\, L)^2\} + \mathrm{E}\{s_N(J)^2\}). \tag{6}$$

We have

$$\mathrm{E}\{S_N(J,\, L)^2\} =$$

$$\frac{1}{(2N+1)^d} \sum_{n,m \in [-N,N]^d} \mathrm{E}\big\{[\varphi(\xi_n,\, Y_n) - \varphi(\xi_n,\, Y_n(J,\, L))] \mathbf{1}_{\{|Y_n| < J\}} [\varphi(\xi_m,\, Y_m)$$

$$-\varphi(\xi_m,\, Y_m(J,\, L)) \mathbf{1}_{\{|Y_m| < J\}}]\big\} := \frac{1}{(2N+1)^d} \sum_{n,m \in [-N,N]^d} \mathrm{E}\{\Delta(n,\, m;\, J,\, L)\}.$$

Define

$$I_i(J,\, L) = \left[\frac{i}{2^L}, \frac{i+1}{2^L}\right), \qquad p_i(J,\, L) = \frac{i}{2^L}, \qquad \text{for } -J2^L \leqslant i \leqslant J2^L - 1.$$

Then, using the fact that $\xi \perp\!\!\!\perp Y$ and that ξ is stationary, we obtain

$$\mathrm{E}\{\Delta(n,\, m;\, J,\, L)\} = \mathrm{E}\{\mathrm{E}\{\Delta(n,\, m;\, J,\, L)/(Y_n,\, Y_m)\}\} \tag{7}$$

$$= \sum_{i,j=-J2^L}^{i,j=J2^L-1} \int_{I_i(J,L)} \int_{I_j(J,L)} \mathrm{E}\{[\varphi(\xi_n, x) - \varphi(\xi_n, p_i(J, L))][\varphi(\xi_m, y)$$

$$- \varphi(\xi_m, p_j(J, L))]\} \, \mathrm{d}P^{(Y_n, Y_m)}(x, y)$$

$$= \sum_{i,j=-J2^L}^{i,j=J2^L-1} \int_{I_i(J,L)} \int_{I_j(J,L)} \mathrm{E}\{[\varphi(\xi_0, x) - \varphi(\xi_0, p_i(J, L))][\varphi(\xi_{m-n}, y)$$

$$- \varphi(\xi_{m-n}, p_j(J, L))]\} \, \mathrm{d}P^{(Y_n, Y_m)}(x, y).$$

For $i, J \subset \mathbb{R}$, define

$$\delta(I, J; k) = \sup_{x \in I, y \in J} |\mathrm{E}\{[\varphi(\xi_0, x) - \varphi(\xi_0, p_i(J, L))][\varphi(\xi_k, y) - \varphi(\xi_k, p_j(J, L))]\}|, \qquad k \in \mathbb{Z}^d.$$

From (7), it follows that

$$\mathrm{E}\{\Delta(n, m; J, L)\} \leqslant \sum_{i,j=-J2^L}^{i,j=J2^L-1} \delta(I_i(J, L), I_j(J, L); m-n) P^{(Y_n, Y_m)}(I_i(J, L) \times I_j(J, L)). \quad (8)$$

But, by (H10),

$$\delta(I_i(J, L), I_j(J, L); k) \leqslant \left| \mathrm{E}\left\{ \frac{\partial \varphi}{\partial y}(\theta_0, c_i(J, L; x)) \frac{\partial \varphi}{\partial y}(\xi_k, c_j(J, L; x)) \right\} \right| \left(\frac{1}{2^L} \right)^2$$

$$\text{if } -J2^L \leqslant i \leqslant J2^L - 1, \quad (9)$$

where $c_i(J, L; x)$ is some point between x and $p_i(J, L)$.

Therefore, by (8) and (9), we obtain

$$\mathrm{E}\{S_N(J, L)^2\} =$$

$$\left(\frac{1}{2^L} \right)^2 \frac{1}{(2N+1)^d} \sum_{i,,j=-J2^L}^{J2^L-1} \sum_{n,m \in [-N,N]^d} \left| \mathrm{E}\left\{ \frac{\partial \varphi}{\partial y}(\xi_0, c_i(J, L; x)) \frac{\partial \varphi}{\partial y}(\xi_{m-n}, c_j(J, L; x)) \right\} \right|$$

$$\times P^{(Y_n, Y_m)}(I_i(J, L) \times I_j(J, L)).$$

Setting $u = m - n$, we easily obtain

$$\mathrm{E}\{S_N(J, L)^2\} \leqslant$$

$$\frac{1}{4^L} \sum_{i,j=-J2^L}^{i,j=J2^L-1} \sum_{\|u\| \leqslant 2N} \left| \mathrm{E}\left\{ \frac{\partial \varphi}{\partial y}(\xi_0, c_i(J, L; x)) \frac{\partial \varphi}{\partial y}(\xi_u, c_j(J, L; x)) \right\} \right|$$

$$\times \frac{1}{(2N+1)^d} \sum_{n \in [-N,N]^d} P^{(Y_n, Y_{n+u})}(I_i(J, L) \times I_j(J, L)).$$

Using (H11), this leads to

$$\mathrm{E}\{S_N(J,\ L)^2\} \leqslant \frac{C(J)}{4^L}$$

which implies

$$\lim_J \lim_L \limsup_N \mathrm{E}\{S_N(J,\ L)^2\} = 0. \tag{10}$$

In a similar way, setting $b_J(x) = \mathrm{sgn}(x)J$, for all $x \in \mathbb{R}$, we obtain

$$\mathrm{E}\{s_N(J)^2\} =$$

$$\frac{1}{(2N+1)^d} \sum_{n,m\in[-N,N]^d} \int_{[-J,J]^c}\int_{[-J,J]^c} \mathrm{E}\{[\varphi(\xi_0,\ x) - \varphi(\xi_0,\ b_J(x))][\varphi(\xi_{m-n},\ y)$$

$$- \varphi(\xi_{m-n},\ b_J(y))]\}\, \mathrm{d}P^{(Y_n,Y_m)}(x,\ y)$$

$$\leqslant \frac{1}{(2N+1)^d} \sum_{n,m\in[-N,N]^d} \kappa(m-n;\ J) P^{(Y_n,Y_m)}([-J,\ J]^c \times [-J,\ J]^c),$$

where

$$\kappa(n;\ J) = \sup_{|x|\geqslant J, |y|\geqslant J} |\mathrm{E}\{[\varphi(\xi_0,\ x) - \varphi(\xi_0,\ b_J(x))][\varphi(\xi_n,\ y) - \varphi(\xi_n,\ b_J(y))]\}|.$$

By (H5) we have that

$$\limsup_{J\to\infty} \sum_{n\in\mathbb{Z}^d} \kappa(n;\ J) < \infty. \tag{11}$$

Therefore,

$$\mathrm{E}\{s_N(J)^2\} \leqslant \sum_{\|u\|\leqslant 2N} \kappa(u,\ J) \frac{1}{(2N+1)^d} \sum_{n\in[-N,N]^d} P^{(Y_n,Y_{n+u})}([-J,\ J]^c \times [-J,\ J]^c). \tag{12}$$

Using (H1), (H2) and (11), we deduce from (12) that

$$\limsup_N \mathrm{E}\{s_N(J)^2\} \leqslant \sum_{u\in\mathbb{Z}^d-\{0\}} \kappa(u,\ J) R_u([-J,\ J]^c \times [-J,\ J]^c) + \kappa(0,\ J) R_0([-J,\ J]^c. \tag{13}$$

Applying Remark 1 and (11), we finally obtain

$$\lim_J \limsup_N E\{s_N(J)^2\} = 0. \tag{14}$$

The results follows from (5), (6), (10) and (14). □

Example 5. Assume that Y is a stationary random field such that, for each $n \in \mathbb{Z}^d$, $(Y_{\cdot},\ Y_{\cdot-n})$ satisfies the Marcinkiewicz–Zygmund inequality; that is, there exist a $q > 2$ and a constant $C(n,\ q)$ such that, for any function $f : \mathbb{R}^2 \to \mathbb{R}$ bounded by 1,

$$\mathrm{E}\left\{\left(\sum_{m\in[-N,N]^d}[f(Y_m,\, Y_{m-n})-\mathrm{E}\{f(Y_m,\, Y_{m-n})\}]\right)^q\right\}\leqslant C(n,\, q)(2N+1)^{qd/2}.$$

Then Y is consistent. Let φ be a C^1-function $\varphi\colon \mathbb{R}^2\to\mathbb{R}$ which is odd with respect to the first variable and assume

$$H_K(x)=\sup\left\{\left|\frac{\partial\varphi}{\partial y}(x,\, z)\right|:\, |z|\leqslant K\right\}<\infty,$$

$$h(x)=\sup\{|\varphi(x,\, y)|:\, y\in\mathbb{R}\}<\infty,\, K\in\mathbb{R},\, x\in\mathbb{R}.$$

Consider further a stationary random field ξ, independent of Y, such that the law of ξ_0 is symmetric, $\mathrm{E}\{\xi_0^4\}<\infty$, $\mathrm{E}\{H_K(\xi_0)^2\}<\infty$ for all K, $\mathrm{E}\{h(\xi_0)^2\}<\infty$, $\sum_{m=1}^{\infty}m^{d-1}\alpha^{\xi}(m)<\infty$ and $m^2\alpha^{\xi}(m)$ is bounded, where

$$\alpha^{\xi}(m)=\sup\{|P(C\cap D)-P(C)P(D)|:\, A\in\sigma^{\xi}(A),\, B\in\sigma^{\xi}(B),\, d(A,\, B)\geqslant m\},$$

$$m=1,\, 2,\, \ldots$$

and $\sigma^{\xi}(A)$ denotes the σ-algebra generated by $\{\xi_n:\, n\in A\}$ (see Bradley 1986; Doukhan 1995).

Set $X_n=\varphi(\xi_n,\, Y_n)$. Using covariance inequalities for mixing random fields (see Bradley 1986; Doukhan 1995), it is easy to see that X is I-decomposable and reachable, and hence, it satisfies the CLT.

Remark 3. (a) It is easy to verify that a similar result holds for the multidimensional case (i.e. φ $\mathbb{R}^d$-valued, ξ $\mathbb{R}^{\alpha}$-valued, Y $\mathbb{R}^b$-valued). The proof is obtained from the preceding one without substantial changes.

(b) Assume now that φ is defined over $\mathbb{R}^{\mathbb{Z}^d}\times\mathbb{R}^{\mathbb{Z}^d}$; denote by θ the shift on $\mathbb{R}^{\mathbb{Z}^d}$, $\theta(x)_n=x_{n-1}$, $n\in\mathbb{Z}^d$, and consider the regression model $X_n=\varphi(\theta^n(\xi,\, Y))$. Then, if we assume that φ can be suitably approximated by a sequence of functions $(\varphi_k:\, k\in\mathbb{N})$, such that for each k, φ_k is defined over $\mathbb{R}^{(2k+1)^d}$ and the multidimensional CLT applies to $X_n^k=\varphi_k(\xi_n^k,\ Y_n^k)$, $\xi_n^k=(\xi_{n+m}:\, \|m\|\leqslant k)$, $Y_n^k=(Y_{n+m}:\, \|m\|\leqslant k)$, then we deduce by standard approximation arguments (see Billingsley 1968, p. 183) that the CLT holds for X.

(c) Assume that Y is a stationary random field but that σ_{∞}^{Y} is not trivial (see Example 1). If we allow R_n, R_0 to be random measures, the proof of conditional normality given in Proposition 2 can be reproduced here, but the asymptotic variance will depend on Y. Therefore, the asymptotic distribution of $S_N(X)$ is a mixture of normal laws.

Acknowledgements

The author wishes to express his gratitude to Jean Bretagnolle, Didier Dacunha-Castelle, Paul Doukhan and all the Probability and Statistics team at Orsay for many insightful remarks and for their friendly support, and to an anonymous referee for careful revision and valuable comments.

References

Andrews, D. (1984) Non-strong mixing autoregressive processes. *J. Appl. Probab.*, **21**, 930–934.

Bellanger, L. and Perera, G. (1997) Compound Poisson limit theorems for high-level exceedances of some non-stationary processes. Preprint 46, Université de Paris-Sud.

Bernshtein, S.N. (1944) Extension of the central limit theorem of probability theory to sums of dependent random variables. *Uspekhi Mat. Nauk*, **10**, 65–114 (in Russian).

Billingsley, P. (1968) *Convergence of Probability Measures*. New York: Wiley.

Birkel, T. (1988) Moment bounds for associated sequences. *Ann. Probab.*, **16**, 1184–1193.

Bolthausen, E. (1982) On the central limit theorem for stationary mixing random random fields. *Ann. Probab.*, **10**, 1047–1050.

Bradley, R. (1986) Basic properties of strong mixing conditions. In E. Eberlein and M.S. Taqqu (eds), *Dependence in Probability and Statistics: A Survey of Recent Results*, pp. 165–192. Boston: Birkhäuser.

Bradley, R. (1993) Equivalent mixing conditions for random fields. *Ann. Probab.*, **21**, 1921–1926.

Bryc, W. and Smolenski, W. (1993) Moment conditions for almost sure convergence of weakly correlated random variables. *Proc. Amer. Math. Soc.*, **119**, 355–373.

Dedecker, J. (1998) A central limit theorem for stationary random fields. *Probab. Theory Related Fields*, **110**, 397–426.

Dobrushin, R.L. (1968) The description of a random field by its conditional distribution. *Theory Probab. Appl.*, **13**, 201–229.

Doukhan, P. (1995) *Mixing: Properties and Examples*, Lectures Notes in Statist. 85. Berlin: Springer-Verlag.

Doukhan, P. and Louhichi, S. (1997) Weak dependence and moment inequalities. Preprint 8, Université de Paris-Sud.

Esary, J., Proschan, F. and Walkup, D. (1967) Association of random variables, with applications. *Ann. Math. Statist.*, **38**, 1466–1476.

Guyon, X. (1995) *Random Fields on a Network. Modeling, Statistics, and Applications.* New York: Springer-Verlag.

Kolmogorov, A.N. and Rozanov, Y. (1960) On the strong mixing conditions for stationary Gaussian sequences. *Theory Probab. Appl.*, **5**, 204–207.

Newman, C. (1984) Asymptotic independence and limit theorems for postively and negatively dependent random variables. In Y.L. Tong (ed.), *Inequalities in Statistics and Probability*, IMS Lecture Notes Monogr. Ser. 5, pp. 127–140. Hayward, CA: Institute of Mathematical Statistics.

Perera, G. (1994a) Estadística espacial y teoremas centrales del límite. Doctoral thesis, Centro de Matemática, Universidad de la República, Uruguay.

Perera, G. (1994b) Spatial statistics, central limit theorems for mixing random fields and the geometry of $\mathbb{Z}^d$. *C. R. Acad. Sci. Paris Sér. I Math.*, **319**, 1083–1088.

Perera, G. (1997) Geometry of $\mathbb{Z}^d$ and the central limit theorem for weakly dependent random fields. *J. Theoret. Probab.*, **10**, 581–603.

Pitt, L. (1982) Positively correlated normal variables associated. *Ann. Probab.*, **10**, 496–499.

Rio, E. (1993) Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.*, **29**, 587–597.

Rosenblatt, M. (1956) A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA*, **42**, 43–47.

Rosenblatt, M. (1980) Linear processes and bispectra. *J. Appl. Probab.* **17**, 265–270.

Roussas, G. (1994) Asymptotic normality of random fields of postively or negatively associated processes. *J. Multivariate Anal.*, **50**, 152–173.

Yu, H. (1993) A Glivenko–Cantelli lemma and weak convergence for empirical processes of associated sequences. *Probab. Theory Related Fields*, **95**, 357–370.

Received October 1997 and revised February 2001