matrix ideal) from a semifir is considered. There are some nice results here. For example it is shown that the group algebra of a free group is a fir (there are other proofs) and that the algebra of rational power series is also a fir.

The notes at the end of each chapter give a good account of the history of the subject. The exercises are plentiful, and range from fairly difficult to open problems (the reader is warned of which category he is dealing with).

This text is an invaluable tool for the researcher and the diligent reader will find it quite rewarding. The reader interested in more examples and applications (some spectacular) is directed to Cohn's companion volume [**1**], and especially to Schofield's lovely monograph [**2**]. Both of these give accounts of Bergman's indispensable coproduct theorems.

REFERENCES

1. P. M. Cohn, *Skew field constructions*, London Math. Soc. Lecture Notes no. 27, Cambridge Univ. Press, Cambridge, 1977.

2. A. H. Schofield, *Representations of the ring over skew fields*, London Math. Soc. Lecture Notes no. 92, Cambridge Univ. Press, Cambridge, 1985.

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Sphere packings, lattices and groups by J. H. Conway and N. J. A. Sloane. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988, xxviii + 663 pp., $87.00. ISBN 0-387-96617-X, and ISBN 3-540-96617-X

Most of us, mathematicians or not, playing with pennies or a compass at an early age, learnt that six circles fit exactly round an equal one; and most of us, whether we have the mathematical language or not, know that you can't do better: each outer circle subtends just one sixth of a revolution at the centre of the inner one. The **kissing number**, in two dimensions, is six.

In three dimensions the situation is much less clear. A theorem of Archimedes tells us that the solid angle subtended by one sphere at the centre of an equal touching sphere is $(2-\sqrt{3})\pi$. Divide this into 4π and get $8+4\sqrt{3}$, so the kissing number in three dimensions is less than 15. But it's clear, when you try to arrange billiard balls round another one, that you have to leave **holes**: you can always stare through the interstices at the central ball. It's not difficult, by taking this into account, to see that the kissing number is less than 14, but to prove that it is less than 13 is far from trivial. Indeed, as eminent mathematicians as David Gregory and Isaac Newton had an inconclusive discussion about it in 1694. The earliest published proofs seem to occur 180 years later [**2**, **22**, **23**]; "the best... now available," according to the authors, is that of Leech [**24**]. The kissing number in three dimensions is 12, because you *can* arrange 12 equal spheres simultaneously touching a thirteenth.

Part of the difficulty is the lack of uniqueness of such an arrangement. Spheres in three dimensions may be packed in layers, each layer being arranged like the packing of pennies in two dimensions, one sphere touching six in its own layer and three in each of the adjacent layers. When a layer is placed on another, its spheres rest in alternate holes, so that there are two options in placing each layer, and uncountably many ways of packing spheres in 3-space, each of density (fraction of space occupied) $\pi/\sqrt{18} \approx 0.74048$, and "many mathematicians believe, and all physicists know, that the density cannot exceed this" [**30**, p. 610].

If the first layer is in position A, and positions B and C are the options for the next layer, then the special arrangement of the layers, ABCABCA..., is a **lattice packing**, and Gauss [**21**] showed that, among lattice packings, this **spherical close packing**, or **face-centred cubic** lattice packing, is the densest possible. The authors have recently given a particularly simple proof [**20**]. But the maximum density for nonlattice packings remains a notorious unsolved problem.

Start again: place 13 spheres of unit radius with their centres at the 12 vertices and centre of a regular icosahedron of diameter 4, so that the first 12 each touch the central one. However, the edge-length of the icosahedron is $\sqrt{(8 - 8/\sqrt{5})} \approx 2.10292$ so no pair of the 12 touch: there is a certain amount of room to manoeuvre. The authors (and see also [**26**]) show that there is in fact enough room to permute the 12 spheres in any way you like, while still keeping them in contact with the central sphere. So it is possible to crowd the spheres together slightly, and locally to obtain a higher density than the best known global one [**4**]: "there are partial packings that are denser than the face-centred cubic lattice over larger regions of space than one might have supposed."

So it's no surprise to learn that these problems become even more difficult in higher dimensions. The kissing number in four dimensions is 24 or 25. No doubt, if it were a matter on which the defence of nations depended, the possibility 25 would be removed after a few months of diligent and ingenious computation.

But it *is* a surprise to learn that the kissing number is known to be exactly 240 in eight dimensions and exactly 196560 in twenty-four dimensions, but is not known in other numbers of dimensions. As the authors explain in their preface, many of the results in their subject appear to be little short of miraculous: they list particularly:

the occurrence of the Leech lattice as the unique laminated lattice in 24 dimensions (in contrast to the 23 such lattices in 25 dimensions, and hundreds of thousands in 26);

the one-to-one correspondence between the twenty-three inequivalent deep holes in the Leech lattice with the even unimodular 24-dimensional lattices of minimal norm 2 (which also correspond to the above twenty-three);

Gleason's theorem describing the weight-enumerators of doubly-even self-dual codes and Hecke's theorem describing the theta-series of even unimodular lattices;

the construction of the Leech lattice in the even unimodular Lorentzian lattice $\mathrm{II}_{25,1}$ (i.e., in 26-dimensional space with norm $x_1^2+\cdots+x_{25}^2-x_{26}^2$), as $\mathbf{w}^\perp/\mathbf{w}$, where $\mathbf{w}$ is the vector $(0,1,2,\ldots,24|70)$ of zero norm (why should Lucas's problem, square pyramid made square, appear here?);

the occurrence of the Leech lattice as the Coxeter diagram of the reflexions in the automorphism group of $\mathrm{II}_{25,1}$.

These last two examples are the contents of the short Chapters 26 and 27. Two other miracles are:

(a) the appearance of the unlikely looking number

$$1027637932586061520960267$$

from two different calculations, providing a convincing verification of the completeness of Niemeier's list of even unimodular lattices in twenty-four dimensions, the twenty-three with minimal norm 2, mentioned above, and the Leech lattice with minimal norm 4;

(b) the "monstrous moonshine" phenomena. It's now known (see Chapter 29 of this book) that the monster group, M, can be constructed, as Griess did, as the group of automorphisms of an algebra in 196884-dimensional Euclidean space. Prior to that, Conway & Norton **[9]** had written that:

(A) M. J. T. Guy observed a certain symmetry in the character table of the group, $2^{12}M_{24}$, of monomial automorphisms of the Leech lattice **[7]**.

(B) the elements of M_{24} have "balanced" cycle-shapes, so that $a^\alpha b^\beta c^\gamma \cdots$ is the same as $(N/a)^\alpha(N/b)^\beta(N/c)^\gamma\cdots$ for some N. E.g. $(N=8)$ $1^2{\cdot}2{\cdot}4{\cdot}8^2$.

(C) for each prime p with $p-1$ dividing 24, there's a conjugacy class, p^-, of elements of M, with centralizer of form $p^{1+2d}\cdot G_p$, where $p\cdot G_p$ is the centralizer of a corresponding automorphism of the Leech lattice, p^{1+2d} denotes an extraspecial p-group, and $2d = 24/(p-1)$.

(D) for the same p, there's a second class, p^+, and the characters of p^+ and p^- in the minimal faithful representation differ by p^d.

(E) Ogg noticed that the primes p dividing $|M|$ are just those for which the function field determined by the normalizer of $\Gamma_0(p)$ in $\mathrm{PSL}_2(\mathbb{R})$ has genus zero, and Pizer **[29]** showed that these are exactly the primes satisfying a 1936 conjecture of Hecke, relating modular forms of weight 2 to quaternion algebra theta-series.

(F) McKay noticed the coefficient 196884 in the q-series

$$q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

and J. G. Thompson found that the other coefficients are also simple linear combinations of the character degrees of M.

(G) the Lie group E_8 has dimension $248 = 744/3$.

A few decades ago, when the subject of combinatorics was even less respectable than it is today, there are several bands of researchers, working on a collection of diverse looking topics, and speaking a variety of different languages.

Those trying to fill space with spheres, not necessarily associated with geometrical lattices, speaking the language of Euclidean geometry: packing, covering, density, Voronoi regions or Delaunay (Delone) cells.

Those concerned with error-correcting codes, many of whom realized that they were dealing with vector spaces over finite fields, but speaking the language of information theory: channel, codeword, weight, distance, noise.

A few combinatorial set theoreticians, speaking of Steiner systems, often ignorant of the early work of Kirkman.

Those concerned with certain combinatorial properties of rectangular arrays of zeros and ones, speaking the language of experimental designs: varieties, blocks, treatments.

A handful of recreational mathematicians, playing nim-like games, using the language of game theory: Sprague-Grundy theory, nim-addition. This later resulted, by generalizing Lenstra's coin-turning games [**3**] to coins with many sides, to the discovery of lexicodes by the authors of this book [**19**].

Students of modular forms, using the languages of number theory and of functions of a complex variable: theta series.

Algebraists, working with matrices, or with polynomials over finite fields.

Finally, the Rosetta stone of group theory enabled everyone to see the isomorphism of the symmetry groups of the various structures in which they were severally interested.

Contrast the present work with [**32**], which is mainly concerned with historiography, and contains a great deal of reminiscence and of failure to remember. Although the author protests against it himself, he spends much time attempting to assign priorities, with indifferent success. There is nothing on designs, though R. A. Fisher gets a passing mention. Simple groups are in the title, but are not defined in the text. Its best feature, perhaps, is the demonstration of heuristics. For the nonspecialist it is a useful introduction, to be skimmed through before facing the encyclopedic detail of the book under review.

Here, although error-correcting codes don't appear in the title, they are an all-pervading concept. In Chapter 1, spherical codes are constructed both from sphere packings and from binary codes: the uniqueness of certain spherical codes is established in Chapter 14 by Eiichi Bannai and Sloane [**1**]. Chapter 3, on codes, designs and groups, gives a rapid but thorough survey of the subject and its connexions with designs. The close relation with sphere packing [**25**] is detailed in Chapter 5, and with lattices in Chapter 7. Bounds for codes and sphere packings [**31**] are the subject of Chapter 9. The close relation between Golay codes and Mathieu groups is explored in Chapter 11, while their position vis à vis the monster group is explained in Chapter 29.

Updates of previously published papers comprise more than half the chapters. Those not already mentioned are:

Chapters	6	10	12	13	16	18	19	21	22	23	24	26	27	28	30
References	[**17**]	[**7**]	[**6**]	[**28**]	[**13&19**]	[**33**]	[**11**]	[**14**]	[**27**]	[**12**]	[**15**]	[**16**]	[**8**]	[**18**]	[**5**]

However, this is much more than a convenient collection of items already enshrined in the literature. One of many features which make it an invaluable work of reference are the numerous tables:

Densest known sphere packings in dimensions up to one million,
Best known coverings and quantizers up to 24 dimensions,
Best coding gain of lattices in up to 128 dimensions,
The numbers of points in the first 50 shells of the Leech lattice,
The 284 types of shallow hole in the Leech lattice.
Definite binary quadratic forms with $|\det| \leq 50$, indefinite forms with $|\det| \leq 100$,
Indecomposable ternary forms for the same intervals (see corrections below,
Bounds for kissing numbers in dimensions up to 24,
Mass constants for unimodular lattices in dimensions up to 32,
The $1+1+1+1+1+3+1+4+3+12+12+28+49+180$ unimodular lattices of dimension ≤ 24 that contain no norm 1 vectors,
Laminated lattices in dimensions up to 48,
Groups associated with the Leech lattice,
Simple groups arising from centralizers in the Monster,
The deep holes in the Leech lattice,
and much, much more.

There are misprints and mistakes, but most of them can easily be corrected. Two exceptions are supplied by the authors. There is a proof without a theorem on p. 475. The theorem missing from the foot of p. 474 is:

THEOREM 10. *A nonzero vector $v \in \Lambda$ is relevant if and only if $\pm v$ are the only shortest vectors in the coset $v + 2\Lambda$.*

Three entries are missing from Table 15.6 on p. 398: for determinants $d = 20$, 31 and 42 insert the respective ternary forms $2_1 6_1 2$, $2_1 6_1 3$ and $2_1 6_1 4$.

REFERENCES

1. Eiichi Bannai and N. J. A. Sloane, *Uniqueness of certain spherical codes*, Canad. J. Math. **33** (1981), 437–449.
2. C. Bender, *Bestimmung der grössten Anzahl gleich Kugeln, welche sich auf ein Kugel von demselben Radius, wie die übrigen, auflegen lassen*, Arch Math. Phys. (Grunert) **56** (1874), 302–306.
3. Elwyn R. Berlekamp, John H. Conway and Richard K. Guy, *Winning Ways for your Mathematical Plays*, Chapter 14, Academic Press, New York, 1982.
4. A. H. Boerdijk, *Some remarks concerning close-packing of equal spheres*, Philips Res. Rep. **7** (1952), 303–313.
5. R. E. Borcherds, J. H. Conway, L. Queen and N. J. A. Sloane, *A monster Lie algebra*? Adv. in Math. **53** (1984), 75–79.
6. J. H. Conway, *A characterization of Leech's lattice*, Invent. Math. **7** (1969), 137–142.
7. ——, *Three lectures on exceptional groups*, Finite Simple Groups (M. B. Powell and G. Higman, eds.), Academic Press, New York, 1971, pp. 215–247.
8. ——, *The automorphism group of the 26-dimensional even unimodular Lorentzian lattice*, J. Algebra **80** (1983), 159–163.

9. J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.

10. J. H. Conway, A. M. Odlyzko and N. J. A. Sloane, *Extremal self-dual lattices exist only in dimensions* 1 *to* 8, 12, 14, 15, 23 *and* 24, Mathematika **25** (1978), 36–43.

11. J. H. Conway, R. A. Parker and N. J. A. Sloane, *The covering radius of the Leech lattice*, Proc. Roy Soc. London **A380** (1982), 261–290.

12. J. H. Conway and N. J. A. Sloane, *On the enumeration of lattices of determinant one*, J. Number Theory **14** (1982), 83–94.

13. ____, *Voronoi regions of lattices, second moments of polytopes, and quantization*, I.E.E.E. Trans. Info. Theory **28** (1982), 211–226.

14. ____, *Twenty-three constructions for the Leech lattice*, Proc. Roy. Soc. London **A381** (1982), 275–283.

15. ____, *Lorentzian forms for the Leech lattice*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 215–217.

16. ____, *Laminated lattices*, Ann. of Math. (2) **116** (1982), 593–620.

17. ____, *Leech roots and Vinberg groups*, Proc. Roy. Soc. London **A384** (1982), 233–258.

18. ____, *The unimodular lattices of dimension up to* 23 *and the Minkowski-Siegel mass constants*, Europ. J. Combin. **3** (1982), 219–231.

19. ____, *Lexicographic codes: error-correcting codes from game theory*, I.E.E.E. Trans. Info. Theory **32** (1986), 337–348.

20. ____, *Low-dimensional lattices.* III *Perfect forms*, Proc. Roy. Soc. London Ser. A **418** (1988), no. 1854, 43–80.

21. C. F. Gauss, *Besprechung des Buchs von L. A. Seeber: Untersuchungen über die Eigenschaften der positiven ternaren quadratischen Formen usw.*, Göttingsche gelehrte Anzeigen (1831-07-09)=Werke, **II** (1876), 188–196.

22. S. Günther, *Ein stereometrisches Problem*, Arch. Math. Phys. (Grunert) **57** (1875), 209–215.

23. R. Hoppe, *Bemerkung der Redaktion*, Arch. Math. Phys. (Grunert) **56** (1874), 307–312.

24. J. Leech, *The problem of the thirteen spheres*, Math. Gaz. **40** (1956), 22–23.

25. J. Leech and N. J. A. Sloane, *Sphere packing and error-correcting codes*, Canad. J. Math. **23** (1971), 718–745.

26. James G. Propp, *Kepler's spheres and Rubik's cube*, Math. Mag. **61** (1988), 231–239.

27. S. P. Norton, *A bound for the covering radius of the Leech lattice*, Proc. Roy. Soc. London **A380** (1982), 259–260.

28. A. M. Odlyzko and N. J. A. Sloane, *New bounds on the number of unit spheres that can touch a unit sphere in n dimensions*, J. Combin. Theory Ser. A **26** (1979), 210–214.

29. Arnold Pizer, *A note on a conjecture of Hecke*, Pacific J. Math. **79** (1978), 541–548; MR 80g:10028.

30. C. A. Rogers, *The packing of equal spheres*, Proc. London Math. Soc. **8** (1958), 609–620.

31. N. J. A. Sloane, *Recent bounds for codes, sphere packings and related problems obtained by linear programming and other methods*, Contemporary Math. vol. 9, Amer. Math. Soc., Providence, R. I., 1982, pp. 153–185.

32. Thomas M. Thompson, *From error-correcting codes through sphere packings to simple groups*, Carus Math. Monograph **21**, Math. Assoc. Amer., Washington DC, 1983.

33. B. B. Venkov, *The classification of integral even unimodular* 24*-dimensional quadratic forms*, Trudy Mat. Inst. Steklov **148** (1978), 65–76=Proc. Steklov Inst. Math. no. 4 (1980), 63–74.

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