RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

CONTRIBUTIONS TO THE K-THEORY OF C^*-ALGEBRAS OF TOEPLITZ AND SINGULAR INTEGRAL OPERATORS

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Let X be a compact Hausdorff space upon which the real line $\mathbf{R}$ acts continuously, giving a transformation group or flow, and for $x \in X$ and $t \in \mathbf{R}$, let $x + t$ denote the translate of x by t. We assume throughout that the action is strictly ergodic in the sense that it is minimal and there exists a unique probability measure on X that is invariant under it. For $\varphi \in C(X)$ and $x \in X$, we write φ_x for the function on $\mathbf{R}$ defined by the formula $\varphi_x(t) = \varphi(x+t)$, $t \in \mathbf{R}$. For $x \in X$ fixed, we let $\mathfrak{S}\mathfrak{I}_x$ denote the C^*-algebra on $L^2(\mathbf{R})$ generated by the Hilbert transform and all the multiplication operators M_{φ_x} obtained by letting φ run over $C(X)$ and we let $\mathfrak{T}_x$ denote the compression of $\mathfrak{S}\mathfrak{I}_x$ to the classical Hardy space $H^2(\mathbf{R})$. These algebras arise in a number of contexts and have been objects of intensive study since the late sixties. (See the references at the end for a sampling of the literature.) In this note, we announce our results which lead to a description of the K-theory of these algebras. The algebras $\mathfrak{S}\mathfrak{I}_x$ and $\mathfrak{T}_x$ are closely related and the derivation of their K-theories involves a kind of play off between the two. In order to keep our presentation simple, we concentrate our attention on $\mathfrak{S}\mathfrak{I}_x$.

As in shown in [CMX], $\mathfrak{S}\mathfrak{I}_x$ does not depend upon x, but only on X. However, as defined, $\mathfrak{S}\mathfrak{I}_x$ is not congenial for analysis and it is helpful to represent it on different Hilbert spaces. When [CMX] was written, this was not an easy task. The following analysis remedies the situation. Let $C^*(X, \mathbf{R})$ denote the transformation group C^*-algebra associated with the flow and let $W^*(X, \mathbf{R})$ denote the double dual of $C^*(X, \mathbf{R})$. This is a huge von Neumann algebra acting on a nonseparable Hilbert space, but it is

Received by the editors January 31, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L80, 47G05, 46M20, 47B35.

The first and third authors were supported in part by grants from the National Science Foundation.

The second author was supported in part by the Natural Sciences and Engineering Research Council of Canada.

0273-0979/89 $1.00 + $.25 per page

generated by a covariant representation $\{\pi, u\}$ of $C^*(X, \mathbf{R})$. We write u in its spectral form, $u_t = \int e^{i\lambda t} dE(\lambda)$, and we let $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ denote the C^*-subalgebra of $W^*(X, \mathbf{R})$ generated by $\pi(C(X))$ and $E([0, \infty))$. This algebra, which is separable if $C(X)$ is, is called the *algebra of singular integral operators on* X and *the algebra of Toeplitz operators on* X, denoted $\mathfrak{T}(X, \mathbf{R})$, is simply the compression of $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ to the range of $E([0, \infty))$. Finally, we write $\mathscr{C}(X, \mathbf{R})$ for the commutator ideal in $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$.

THEOREM 1. *The algebra $\mathscr{C}(X, \mathbf{R})$ is a simple C^*-algebra that contains $C^*(X, \mathbf{R})$ properly. The quotient $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})/\mathscr{C}(X, \mathbf{R})$ is canonically isomorphic to $C(X) \oplus C(X)$ and any representation of $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ that does not annihilate $\mathscr{C}(X, \mathbf{R})$ is faithful. In particular, the map that sends $\pi(\varphi)$ to M_{φ_x} and $E([0, \infty))$ to the projection of $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$ extends to an isomorphism from $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ onto $\mathfrak{S}\mathfrak{I}_x$.*

When one views $C^*(X, \mathbf{R})$ as the C^*-algebra of the foliated space determined by the flow, it is clear that $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ is related to Connes' C^*-algebra of order zero pseudodifferential operators on the foliated space **[C1,2]**. The difference between his algebra and ours is that his operators have singularities that are compactly supported in the leaf direction, while ours don't. Remember, the Hilbert transform has a singularity at zero *and* at infinity. This difference is something of a mystery. It is interesting in its own right, but it also intrudes in annoying ways in applications. In the literature the difficulties with $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ have been circumvented using a somewhat ad hoc truncation process to eliminate the singularity at infinity (cf. **[C1,2, D1,2, DHK1,2, FS, JK, R** and **X]**).

We calculate the K-theory of $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ using another representation in which the truncations can be monitored. Let $L^2(m)$ be the L^2-space on X built with the unique invariant probability measure m and let $H^2(m)$ be the closure in $L^2(m)$ of all the functions φ in $C(X)$ with the property that φ_x belongs to the classical Hardy space $H^\infty(\mathbf{R})$ for each x in X. Then the map which sends $\pi(\varphi)$ to multiplication by φ on $L^2(m)$, $\varphi \in C(X)$, and $E([0, \infty))$ to the projection from $L^2(m)$ onto $H^2(m)$ extends to a faithful representation of $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ on $L^2(m)$. Careful analysis of the operators in the image of $\mathfrak{S}\mathfrak{I}(X, \mathbf{R})$ on $L^2(m)$ yields the following theorem which was first proved by Ji and Xia **[JX]** under the assumption that X is a quotient of the Bohr group with $\mathbf{R}$ acting in the usual way. Their techniques of proof rely very heavily on the theory of almost periodic functions and are quite different from ours.

THEOREM 2. *The inclusion mapping, $j: C^*(X, \mathbf{R}) \to \mathscr{C}(X, \mathbf{R})$, induces an order isomorphism*

$$j_*: K_0(C^*(X, \mathbf{R})) \to K_0(\mathscr{C}(X, \mathbf{R}))$$

and a short exact sequence

$$0 \to K_1(C^*(\mathbf{R})) \xrightarrow{i_*} K_1(C^*(X, \mathbf{R})) \xrightarrow{j_*} K_1(\mathscr{C}(X, \mathbf{R})) \to 0$$

where i is the canonical embedding of $C^*(\mathbf{R})$ into $C^*(X, \mathbf{R})$.

With this and Connes' analogue of the Thom isomorphism **[C3]**, the first pair of assertions in the next theorem are immediate; the second pair are proved in the process of proving Theorem 2; and the third pair are easy consequences of the second.

THEOREM 3.
(a) $K_0(\mathscr{C}(X,\mathbf{R})) \simeq K^1(X)$.
(b) $K_1(\mathscr{C}(X,\mathbf{R})) \simeq K^0(X)/\mathbf{Z}[1]$, *where* $[1]$ *denotes the class of the trivial line bundle on* X.
(c) $K_0(\mathfrak{T}(X,\mathbf{R})) \simeq \mathbf{Z}$.
(d) $K_1(\mathfrak{T}(X,\mathbf{R}) = 0$.
(e) $K_0(\mathfrak{S}\mathfrak{I}(X,\mathbf{R})) \simeq \mathbf{Z} \oplus K^0(X)$.
(f) $K_1(\mathfrak{S}\mathfrak{I}(X,\mathbf{R})) \simeq K^1(X)$.

Another representation of $\mathfrak{S}\mathfrak{I}(X,\mathbf{R})$ on the L^2-space of the graph of the foliation determined by the flow, i.e., the representation of $\mathfrak{S}\mathfrak{I}(X,\mathbf{R})$ in the group-measure von Neumann algebra determined by the flow and the measure m, is used to prove

THEOREM 4. *Up to scalar multiples, the* C^**-algebra* $\mathscr{C}(X,\mathbf{R})$ *has a unique trace. It is induced by* m.

Theorems 2 and 4 imply the uniqueness of the index theory developed in **[CMX]** which, in turn, generalizes the results in **[CP, CDSS** and **GF]**.

While the hypothesis that our flow is strictly ergodic may seem somewhat technical and perhaps a little excessive—the assumption of minimality only would be aesthetically more pleasing—it seems to intervene in our proofs frequently and in essential ways. It is used to show that a number of mean values that we have to compute actually converge uniformly for functions in $C(X)$.

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