References

[H] I. N. Herstein, *Noncommutative rings*, Carus Mathematical Monographs 15, MAA, 1968.

[J] N. Jacobson, *Structure of rings*, vol. 37, Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, R. I., 1956.

[M-R] J. C. McConnell and J. C. Robson, *Noncommutative noetherian rings*, Wiley, New York, 1987.

[P] D. S. Passman, *The algebraic structure of group rings*, Wiley, New York, 1977.

[Pi] R. S. Pierce, *Associative algebras*, Springer-Verlag, Berlin and New York, 1982.

[R] L. H. Rowen, *Polynomial identities in ring theory*, Academic Press, New York, 1980.

EDWARD FORMANEK
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK
MSRI, BERKELEY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Real variable methods in harmonic analysis, by Alberto Torchinsky. Pure and Applied Mathematics, vol. 123, Academic Press, Inc., Orlando, 1986, xii + 462 pp., $89.00 ($45.00 paper). ISBN 0-12-695460-7

Weighted norm inequalities and related topics, by J. García-Cuerva and J. L. Rubio de Francia. Mathematical Studies, vol. 116, North-Holland, Amsterdam, New York, Oxford, 1985, x + 604 pp., ISBN 0-444-87804-1

In the preface of their book, García-Cuerva and Rubio de Francia say: "After the classical monograph of A. Zygmund **[31]**, the standard references for the important developments that occurred in Fourier Analysis during the second half of this century are E. M. Stein **[25]** and Stein and Weiss **[29]**, both published around 1970." Since that date there have been dramatic advances in several areas of Harmonic Analysis. We shall describe some of these. We begin with the **Theory of Hardy spaces** and shall present a more elaborate description of this development since we can use some of the material presented to help explain the progress made in other areas.

Classical Harmonic Analysis in one dimension is either associated with the Real line $\mathbf{R}$ or the **Torus** $\mathbf{T}= [0, 2\pi)$, often identified with the **Circle Group** $\{z \in \mathbf{C}\colon z = e^{i\theta},\ 0 \le \theta < 2\pi\}$. Let us concentrate on $\mathbf{R}$, which we consider embedded in $\mathbf{R}^2$ (or $\mathbf{C}$) as the boundary of the **Upper Half Plane** $\mathbf{R}^2_+ \equiv \{z = (x, y) \in \mathbf{R}^2\colon y > 0\} = \{x + iy \in \mathbf{C}\colon y > 0\}$. If $0 < p \le \infty$ the **Hardy Space** H^p consists of all holomorphic functions $F(x + iy)$ on $\mathbf{R}^2_+$ such that

$$(1)\qquad \|F\|_{H^p} \equiv \sup_{y>0} \left\{ \int_{-\infty}^{\infty} |F(x + iy)|^p\, dx \right\}^{1/p} < \infty.$$

These spaces are related to the Lebesgue spaces $L^p(\mathbf{R})$ via the following theorems:

EXISTENCE OF BOUNDARY VALUES. *If $F \in H^p$ then* $\lim_{y\to 0} F(x+iy) \equiv F(x)$ *exists for almost every $x \in \mathbf{R}$. Moreover, if $0 < p < \infty$, then* $\lim_{y\to 0} \int_{-\infty}^{\infty} |F(x+iy) - F(x)|^p \, dx = 0$.

When $0 < p$ the real part, $u(z)$, of $F(z)$ converges, as $y \to 0$, to a real valued function $f(x)$ on $\mathbf{R}$ that belongs to L^p. When $1 < p < \infty$ one can obtain a converse result: if f is a real valued function in L^p then the function on $\mathbf{R}^2_+$ defined by

$$F(z) = u(z) + iv(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{z-t} \, dt \tag{2}$$

belongs to H^p, the real part of its boundary value equals $f(x)$ a.e. and the mapping $f \to F$ is a bounded linear transformation.

THE M. RIESZ INEQUALITY. *If $f \in L^p(\mathbf{R})$, $1 < p < \infty$, and F is given by equality* (2) *then $\|F\|_{H^p} \le A_p \|f\|_p$, where A_p is independent of f.*

Thus, for this range of p, one can assign to each $f \in L^p(\mathbf{R})$ the H^p-norm of the associated function F (via equality (2)) and one obtains a norm equivalent to the L^p-norm. This is sometimes stated simply by the assertion that "real H^p equals real L^p when $1 < p < \infty$." When $p = 1$ there exist real valued functions L^1 that are not the real part of boundary values of an $F \in H^1$. That is, "real H^1 is a proper subspace of real L^1." This is also reflected by the fact that the constant A_p in the M. Riesz inequality behaves like $1/(p-1)$ as $p \to 1$.

When $0 < p < 1$ the situation is more complicated. Even though the boundary values $F(x)$ exist a.e., the real parts $f(x)$ no longer determine the holomorphic function F on $\mathbf{R}^2_+$. The proper interpretation of these boundary values involves distributions and not functions; thus, the space "real H^p" for $p < 1$ is an appropriate space of (tempered) distributions. It is not a Banach space since the functional in (1) is not a norm; however, it is a "quasinorm" (Minkowski's inequality is replaced by a similar one: $\|a + b\| \le K(\|a\| + \|b\|)$ with $K = K(p) > 1$) and it turns H^p into an interesting topological vector space that is not locally convex. For details about the classical theory of H^p spaces see [**31** and **16**].

One can, however, consider many of these notions without using analytic functions. The boundary values of the imaginary part of an F in H^p can be expressed directly in terms of the **Hilbert Transform** of the boundary values of the real part f:

$$\tilde{f}(x) = (Hf)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|>\varepsilon} \frac{f(x-t)}{t} \, dt \tag{3}$$

(formally, this is the integral in (2) with $\operatorname{Im} z = 0$). A reformulation of the M. Riesz inequality is that the limit defining $\tilde{f}$ exists a.e. when $f \in L^p$, $1 < p < \infty$, and is a bounded operator. This limit does exist when $p = 1$; however, it does not define a bounded operator on L^1. Real H^p, then, is the space of all those $f \in L^p$ whose Hilbert transform belongs to L^p.

When $1 < p < \infty$ these two spaces are the same; when $p = 1$, H^1 is a proper subspace of L^1.

Around 1950 there was a great impetuous to develop Fourier Analysis in higher dimensions. This was created in large part by Calderón and Zygmund through their study of singular integrals (which we shall describe below). The push was accompanied by extensions of the theory of Hardy spaces to several dimensions. There are several more or less obvious settings for such generalizations: (i) the unit disk can be replaced by its n-fold Cartesian product, **the Polydisc.** Holomorphic functions of one variable are then replaced by holomorphic functions of several variables and Fourier series are replaced by multiple Fourier series; (ii) the unit disk can be replaced by the unit ball in $\mathbf{C}^n$. Again, Hardy space theory involves holomorphic functions of several complex variables; the boundary of the ball, however, does not enjoy some of the features of the circle group (when $n = 2$ this boundary can be identified with the group SU(2); but it is not a group when $n > 2$); (iii) in the noncompact case there are parallel directions. The n-fold Cartesian product of $\mathbf{R}$ leads to a theory similar to the one associated with the polydisc. There are many other directions and approaches that have been developed in the twenty year period following 1950. Some of these are considered in the articles [**7**, **30** and **12**].

There is one n-dimensional extension of H^p theory, however, that in addition to being natural, fits in particularly well with later developments. It is based on the following elementary facts about analytic functions of a complex variable. Suppose that $F = u + iv$ is such a function defined on a simply connected domain $\Omega \subset \mathbf{C}$. Then the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ tell us, first, that (v, u) is the gradient of a function h on Ω and, second, that $\Delta h = 0$ (that is, that h is harmonic). Thus, one can consider gradients of harmonic functions on a domain $\Omega \subset \mathbf{R}^n$ to be extensions of the notion of a holomorphic function on a domain in $\mathbf{R}^2 (= \mathbf{C})$. In view of this observation and keeping in mind the definition of Hardy spaces induced by condition (1) we can consider the following spaces to be an extension of the spaces H^p to n dimensions. Let $F = (v_1, v_2, \dots, v_n, u) = (V, u)$ be a mapping from the **Upper Half Space** $\mathbf{R}^{n+1}_{+} \equiv \{(x_1, x_2, \dots, x_n, y) = (x, y) \in \mathbf{R}^{n+1} : y > 0\}$ onto $\mathbf{R}^{n+1}$ satisfying the **Generalized Cauchy-Riemann Equations**

$$\text{(4)} \qquad \text{(i)}\ \frac{\partial u}{\partial y} + \sum_{1}^{n} \frac{\partial v_j}{\partial x_j} = 0, \quad \text{(ii)}\ \frac{\partial v_j}{\partial x_k} = \frac{\partial v_k}{\partial x_j}, \qquad \frac{\partial v_j}{\partial y} = \frac{\partial u}{\partial x_j}$$

for $j, k = 1, 2, \dots, n$. We say that such an F belongs to the space $H^p(\mathbf{R}^n)$, $p > 0$, if and only if

$$\text{(5)} \qquad \|F\|_{H^p} \equiv \sup_{y>0} \left\{ \int_{\mathbf{R}^n} |F(x, y)|^p \, dx \right\}^{1/p} < \infty.$$

These spaces were introduced in [**28**]; there it was shown that many of the properties of the classical Hardy spaces are valid for these n-dimensional extensions. For example, the boundary values

$$F(x) = (v_1(x), v_2(x), \dots, v_n(x), u(x)) = \lim_{y \to 0} F(x, y)$$

exist for almost every $x \in \mathbf{R}^n$ as well as in the L^p norm, provided $p > (n-1)/n$ (for lower values of the index p, one has to consider more general versions of the Cauchy-Riemann equations [**28**, **6**, **29**]). As is the case when $1 < p < \infty$, the boundary function $u(x)$ is the general element of $L^p(\mathbf{R}^n)$ and it uniquely determines the boundary values of the "conjugates" $v_j(x)$,

$$v_j(x) = (R_j u)(x) = \lim_{\varepsilon \to 0} c_n \int_{|t|>\varepsilon} u(x-t) \frac{t_j}{|t|^{n+1}}\, dt. \tag{6}$$

R_j is a natural extension of the Hilbert transform (compare with (3)) and is called the jth **Riesz Transform** (it was M. Riesz who first considered this n-dimensional version of the Hilbert transform).

Though many other properties of the one dimensional spaces were extended to these higher dimensional analogs, some basic questions were not resolved. In fact, many were unresolved even in the one dimensional case. We have seen how the spaces H^p correspond to the boundary spaces L^p when $1 < p < \infty$. A natural problem is to characterize the subspace of $L^1(\mathbf{R})$ consisting of the real parts of boundary values of functions in $H^1(\mathbf{R}^2_+)$. Another question is: What is the dual of this subspace of L^1? Once these problems are resolved one can then consider their natural extension to $\mathbf{R}^n$, $n > 1$. In the decade 1970–1980 there occurred many major breakthroughs in the theory of Hardy spaces; in particular, the solutions of the above questions were obtained.

Burkholder, Gundy and Silverstein [**1**], using probability theory methods, characterized the harmonic functions $u(x, y)$ on $\mathbf{R}^2_+$ that are the real parts of functions in H^1. Let

$$u^*(x) = \sup_{|w-x|<y} |u(w, y)|.$$

They showed that u is such a function if and only if $u^* \in L^1(\mathbf{R})$. C. Fefferman and Stein [**18**] extended this result to n dimension for the Hardy spaces associated with $\mathbf{R}^{n+1}_+$ we described above. At just about the same time, C. Fefferman identified the dual of H^1 with the space of functions of **Bounded Mean Oscillation** (BMO) that was introduced a few years earlier by John and Nirenberg [**21**]. Fefferman and Stein extended this result to the n-dimensional case. Appropriate versions of all these results extend to the case $p > 0$.

Another important breakthrough was the discovery by Coifman of the **Atomic** characterization of the spaces [**8**]. An **atom** is, simply, an $L^1(\mathbf{R})$ function $a(x)$ that is supported in a finite interval I, and satisfies the size and cancellation conditions

$$\text{(i) } |a(x)| \le 1/l(I) \quad \text{and} \quad \text{(ii) } \int a(x)\, dx = 0, \tag{8}$$

where $l(I)$ is the length of I. It is an easy exercise to show that the Hilbert transform $\tilde{a}(x)$ satisfies $\|\tilde{a}\|_1 \le 7$. Thus, an atom belongs to real H^1 and $\|a + i\tilde{a}\|_{H^1} \approx \|a\|_1$. It follows that any function of the form

$$f = \sum_1^\infty \lambda_n a_n, \tag{9}$$

where the a_n's are atoms and $\sum_1^\infty |\lambda_n| < \infty$, must also belong to real H^1. Coifman showed that all functions f in this space have this form and that the expression $\inf \sum |\lambda_n|$, where the infimum is taken over all representations of the form (9), defines a norm that is equivalent to the H^1 norm of $f + i\tilde{f}$. This characterization of real H^1 in terms of these simple building blocks, the atoms, extends to the n dimensional case and to the spaces H^p, $0 < p \leq 1$. Moreover, the notion of an atom makes sense on any metric space that is endowed with a measure (spheres play the role of intervals and, in (8) (i), $l(I)$ is replaced by the measure of the sphere); thus, the definition of an **Atomic Hardy Space** can be given by considering functions having the representation in (9). Much of harmonic analysis can be carried out on such metric spaces, where the metric is related to the measure μ by the inequality

$$\mu(\overline{S}) \leq K\mu(S),$$

where S is a sphere, $\overline{S}$ is the sphere (with same center) of radius twice the radius of S and K is independent of S. Such spaces are known as **Spaces of Homogeneous Type** (see **[9, 11** and **12]**); they include a large class of settings in which harmonic analysis has been studied.

Another area that has seen considerable development recently is the study of the **Calderón-Zygmund Operators**. In 1952 **[5]** Calderón and Zygmund introduced certain convolution operators that extended the Hilbert transform to n dimensions. These operators, the **Calderón-Zygmund Singular Integral Operators**, which include the Riesz transforms (6), have the form

$$(Tf)(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon>0} f(x-y)\frac{\Omega(y)}{|y|^n}\,dy, \tag{10}$$

where Ω is homogeneous of degee 0 ($\Omega(\rho x) = \Omega(x)$ when $\rho > 0$ and $0 \neq x \in \mathbf{R}^n$) and, thus, can be considered to be a function on the surface S^{n-1} of the unit sphere in $\mathbf{R}^n$; moreover, Ω satisfies the cancellation property

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0, \tag{11}$$

where σ is the Lebesgue surface measure on S^{n-1} and $x' \in S^{n-1}$. When $n = 1$, condition (11) is equivalent (up to a multiplicative constant) to $\Omega(x) = \operatorname{sgn} x$ and, thus, (10) is the Hilbert transform. Similarly, $\Omega(t) = c_n t_j/|t|$ gives us the jth Riesz transform (6).

Calderón and Zygmund first focused their attention to the study of the behaviour of these operators on the spaces $L^p(\mathbf{R}^n)$, $1 \leq p$. They did this by introducing the **Calderón-Zygmund Decomposition** of an L^1 function into the sum of an L^2 function and a series of oscillating terms supported by a set of "small measure." The latter is a countable union of disjoint cubes that support each of the oscillating terms (these oscillatory terms are the precursors of the atoms we described above). The basic features of this program are the following: (i) The L^2 theory is easily obtained by applying the Plancherel theorem and using the Fourier transform $m(\xi)$ of $\Omega(y)|y|^{-n}$; (ii) appropriate (weak-type) estimates are obtained for Tf, for $f \in L^1$, using "mild" hypotheses on the kernel $K(x, y) = \Omega(x-y)|x-y|^{-n}$;

general interpolation theorems (the Marcinkiewicz theorem) gives us L^p boundedness, $1 < p \leq 2$, from (i) and (ii); finally, duality gives us the results when $2 \leq p < \infty$ (here we use the fact that convolution operators are "selfadjoint").

These operators can be used to study partial differential equations. It is not hard to give meaning to and to obtain the formula for the Riesz transforms

$$R_j = -i\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \tag{12}$$

where Δ is the Laplace operator. Calderón **[3]** used such formulae to study the uniqueness of the Cauchy problem. Such considerations involve the composition of two such operators with kernels $\Omega_j(y)|y|^{-n}$, which corresponds to multiplication of their Fourier transforms $m_j(\xi)$, $j = 1, 2$. This may destroy the mean zero property (11). By throwing in the identity operator one obtains an algebra that can be used for obtaining estimates for partial differential equations of elliptic type with constant coefficients.

In order to study partial differential equations with variable coefficients Calderón and Zygmund introduced extensions of their singular integral operators that are no longer of convolution type:

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon>0} K(x, y)f(y)\,dy,$$

where the kernel has the form $K(x, y) = L(x, x-y)$ and satisfies a smoothness condition as well as the homogeneity and cancellation properties:

(a) $L(x, \lambda z) = \lambda^{-n}L(x, z)$ for $\lambda > 0$, $x \in \mathbf{R}^n$ and $0 \neq z \in \mathbf{R}^n$;

(b) $\int_{S^{n-1}} L(x, z')\,d\sigma(z') = 0$ for each x.

Their technique was based on representing these kernels in the form

$$L(x, z) = \sum m_j(x)H_j(z),$$

where the H_j are spherical harmonics, so that these operators can be studied in terms of convolution operators followed by multiplications by (smooth) functions. At about the same time, Kohn and Nirenberg **[23]** arrived at a class that includes such operators, which they called **Pseudo Differential Operators.**

Other extensions of the Calderón-Zygmund operators were introduced in the sixties. For example, Calderón studied the Cauchy integral of an L^2 function on a Lipschitz curve in $\mathbf{R}^2$. An example is

$$Tf(x) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(y)}{z(y)-z(x)-i\varepsilon}\,dy, \tag{13}$$

where $z(x) = x + iA(x)$ with $A' \in L^\infty(\mathbf{R})$ (with sufficiently small norm).

In all these cases, because we are no longer dealing with convolution operators, L^2 theory, via the Plancherel theorem, is no longer available. In fact, it is a nontrivial task to obtain the L^2 boundedness for these operators. We now know that we can rely on the properties of the limiting spaces obtained as $p \to \infty$ and $p \to 1$ (BMO and H^1). It is interesting to note that, before this was understood, Calderón **[2]** did, indeed, obtain the

L^2 boundedness of the operator T in (13) by very clever use of the tools known at the time (this was a real "tour de force.")

Coifman and Meyer in the early seventies began a systematic study of these more general operators, which they called **Calderón Zygmund Operators** (CZO) **[10]**. These belong to the class of linear transformations mapping test functions (C^∞ functions with compact support in $\mathbf{R}^n$) into distributions. Any such operator T has a (distribution) kernel $K(x, y)$ such that

$$\langle T\varphi, \psi\rangle = \langle K(x, y)\varphi(x), \psi(y)\rangle$$

for all test functions φ and ψ. T is a CZO if this kernel is "represented" by a continuous function on $\mathbf{R}^n \times \mathbf{R}^n - \{(x, y): x - y\}$ so that

$$T\varphi(x) = \int_{\mathbf{R}^n} \varphi(y)K(x, y)\,dy$$

if $x \notin \operatorname{Supp}\varphi$. Moreover, K satisfies certain "standard estimates" an example of which are

(a) (size condition) $|K(x, y)| \leq C|x - y|^{-n}$.

(b) (smoothness condition) there exist $\varepsilon > 0$ such that if $2|x' - x| \leq |x - y|$, then

$$|K(x, y) - K(x', y)| + |K(y, x') - K(y, x)| \leq |x' - x|^{\varepsilon}|x - y|^{-(n+\varepsilon)},$$

as well as a general functional analytic condition, called "the weak boundedness property," that is weaker than L^2 boundedness. They showed that these operators are bounded on L^2 (as well as many other spaces) and in so doing provided a unified approach to their study. Their work culminated in a celebrated theorem of G. David and J.-L. Journé **[14]**.

THE T1 THEOREM. *A Calderón-Zygmund operator T is bounded on $L^2(\mathbf{R}^n)$ if and only if $T1$ and T^*1 belong to* BMO.

(T^* is the adjoint of T and it is not hard to show that the domains of T and T^* can be extended in a natural way to include the function 1.) Even more recently, David and Journé, together with S. Semmes, extended this result to spaces of homogeneous type **[15]**. This represents, therefore, a unification of many of the ideas and results that are involved in this development of Harmonic Analysis that has taken place during the seventies and eighties.

Many other areas of harmonic analysis have experienced similar developments during the past two decades. Let us list and describe them briefly.

WEIGHTED NORM INEQUALITIES. Such inequalities have the form

$$\int |Tf(x)|^p w(x)\,dx \leq C \int |f(x)|^p w(x)\,dx, \tag{14}$$

where, for example, T is one of the singular integrals we discussed above, or a maximal function, and C depends only on the weight w and p. Such inequalities arise naturally. For example, H^p spaces can be associated with general domains in $\mathbf{C}$ and their study leads to the study of operators analogous to the Hilbert transform, acting on functions defined on the boundaries. Estimates involving such operators can often be reduced, via a change of variables, to estimates for classical operators on the line (or the torus) that involve a measure of the form $w(x)\,dx$ (w, in such a case, is the Jacobian associated with the change of variables). Calderón's original proof of L^2 boundedness of the operator T in (13) involved precisely such estimates. The classical Helson-Szegö theorem characterizes those weights w for which the Hilbert transform is bounded on L^2. It turns out that weights for which inequalities such as (14) are valid have close connections with BMO functions: roughly speaking, the logarithms of such "good weights" belong to BMO. The study of such inequalities and their applications has been one of the very active areas of research during the last twenty years.

Vector valued inequalities. The definition of many operators that arise in Harmonic Analysis can be extended to Banach space valued functions. One can then ask if, by replacing absolute values by norms, one obtains inequalities that are analogous to the classical ones. It turns out that there is a strong connection between such inequalities and weighted norm inequalities. Many estimates for nonlinear operators that are important in Fourier analysis can, very often, be reformulated as estimates for linear operators whose range consists of vector-valued functions. For example, this has long been known to be the case when the operator in question is the "Littlewood-Paley g-function." Furthermore, vector valued inequalities are very important in that area of Harmonic Analysis that is associated with probability theory.

There are other topics that have seen considerable activity recently. We have mentioned a connection between partial differential equations and harmonic analysis. There are many other such connections that involve not only harmonic analysis but geometry and several variables as well. For example, those working on the $\bar{\partial}$ problem draw from these last two areas and are interested in estimates that have much in common with some that we have mentioned.

Finally we would like to point out that, most recently, there has been a considerable unification in the study of the various topological spaces that occur in analysis such as the Lebesgue spaces, Hardy spaces, Lipschitz spaces, and Sobolev spaces. This is being done by the construction of "spanning systems" of elementary functions, much in the same spirit as atoms span the Hardy spaces. This area has been developed, on the one hand, by Coifman, Meyer and their school and, independently, by the collaboration of Frazier and Jawerth. Some of these systems, called **wavelets**, also enjoy many of the properties of orthonormal bases. A (yet unpublished) book on wavelets by Y. Meyer **[24]** presents this material and its application to the various topics we have discussed. The approach of Frazier and Jawerth is described in **[19, 20]**.

Before commenting on the two books being reviewed, it may be helpful to point out that, in addition to the references already cited, there are several excellent expository articles that deal with the topics we have mentioned. We cite the articles in various Proceedings of the International

Congress of Mathematics that were written by Calderón, Stein, C. Fefferman and G. David [**4**, **26**, **17**, **13**]. Other excellent sources are the book by Journé [**22**] and the book edited by Stein [**27**].

The first half of the book by Garcia-Cuerva and Rubio de Francia is an exceptionally well written presentation of much of the theory of Hardy spaces we have touched upon. Their approach is elegant and very well motivated. The last half is devoted to weighted norm inequalities and vector valued inequalities. This is an area in which Rubio de Francia has made a most important contribution and is also exceptionally well written. Moreover, the presentation gives a beautiful unification of all these topics. This book represents considerable effort by two researchers having complete control of their material. It should serve as an example to be emulated for those writing an advanced text in any subject of mathematics.

The book by Torchinsky covers more ground than the book by García-Cuerva and Rubio de Francia. Though it begins on an elementary level, it covers many topics that have appeared only recently in the literature (such as some aspects of the theory of Calderón-Zygmund Operators and boundary value problems on C^1 domains). It is a useful source for many of the topics that we have touched on in this review.

References

1. D. Burkholder, R. Gundy and M. Silverstein, *A maximal function characterization of the class H^p*, Trans. Amer. Math. Soc. **157** (1971), 137–153.

2. A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1091–1099.

3. ——, *Uniqueness in the Cauchy problem for partial differential equations*, Amer. J. Math. **80** (1958), 16–36.

4. ——, *Commutators, singular integrals on Lipschitz curves and applications*, Internat. Congress Math., Helsinki, 1978, pp. 85–96.

5. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

6. ——, *On higher gradients of harmonic functions*, Studia, Math. **24** (1964), 211–226.

7. S.-Y. A. Chang and R. Fefferman, *Some recent developments in Fourier analysis on product domains*, Bull. Amer. Math. Soc. (NS) **12** (1985), 1–44.

8. R. R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274.

9. R. R. Coifman and M. de Guzmán, *Singular integrals and multipliers on homogeneous spaces*, Rev. Un. Mat. Argentina **25** (1970/1971), 137–143.

10. R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudodifférentiels*, Astérisque **57**, Soc. Math De France, 1978.

11. R. R. Coifman and Guido Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.

12. ——, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

13. G. David, *Opérateurs de Calderón-Zygmund*, Internat. Congress Math., Berkeley, 1986, pp. 890–899.

14. G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2) **120** (1984), 371–397.

15. G. David, J.-L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Ibero-Americana **1** (1985), 1–56.

16. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

17. C. Fefferman, *Recent progress in classical Fourier analysis*, Internat. Congress Math., Vancouver, 1974, pp. 95–118.

18. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

19. M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799.

20. ——, *A discrete transform and decomposition of distribution spaces* (submitted).

21. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

22. J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lecture Notes in Math., vol. 994, Springer-Verlag, Berlin and New York, 1983.

23. J. J. Kohn and L. Nirenberg, *On the algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

24. Y. Meyer, *Ondelettes et opérateurs* (to appear).

25. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

26. ——, *Some problems in harmonic analysis suggested by symmetric spaces and semi-simple Lie groups*, Proc. Internat. Congr. Math., Nice **1** (1970), 173–189.

27. ——, *Beijing lectures in harmonic analysis*, edited by Stein, Ann. of Math. Studies, no. 112, Princeton Univ. Press, Princeton, N. J., 1986.

28. E. M. Stein and Guido Weiss, *On the theory of harmonic functions of several variables*, Acta Math. **103** (1960), 26–62.

29. ——, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

30. E. M. Stein, Guido Weiss and Mary Weiss, *H^p classes of holomorphic functions in tube domains*, Proc. Nat. Acad. Sci. U.S.A. (1964), 1035–1039.

31. A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

GUIDO WEISS
WASHINGTON UNIVERSITY IN ST. LOUIS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Correlation theory of stationary and random functions. vol. I; *Basic results*, vol. II, *Supplementary notes and references*, by A. M. Yaglom. Springer Series in Statistics, Springer-Verlag, New York, Berlin and Heidelberg, 1987, vol. I, xiv + 526 pp., $58.00. ISBN 0-387-96268-9, vol. II, vii + 258 pp., $56.50. ISBN 0-387-96331-6

This work has been set forth in two volumes. The first volume is described as *Basic results* and the second *Supplementary notes and references*. The title *Correlation theory of stationary and related random functions* indicates that the exposition does not attempt to discuss general aspects of the study of stationary processes but rather confines itself to the important but more limited aspect dealing with first and second order moment properties.

The object apparently is to give a direct development of results on a heuristic basis supplemented by illustrations in terms of applications and graphical representations in the first volume. The second volume consists