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Invariant manifolds, entropy and billiards; smooth maps with singularities, by Anatole Katok and Jean-Marie Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki. Lecture Notes in Mathematics, vol. 1222, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1986, viii + 283 pp., $23.60. ISBN 0-387-17190-8

Many dynamical systems arising in physics, meteorology, chemistry, biology, engineering and other fields exhibit chaotic behavior. There is no precise definition of "chaos"; however, in simple terms, chaotic behavior means that a typical orbit seems to wander aimlessly in the phase space with no identifiable pattern and its future is unpredictable although the system itself is deterministic in nature. The only known cause of chaos is hyperbolicity. Suppose we begin moving along a hyperbolic orbit with the speed prescribed by the system and observing the relative motion of nearby orbits that start on a codimension 1 transversal to our orbit. Then in an appropriate coordinate system the relative motion up to first-order terms will be the same as in a neighborhood of a saddle point $\dot{x} = \Lambda x$, $\dot{y} = My$. The eigenvalues of Λ have strictly negative real parts; the eigenvalues of M have strictly positive real parts. These real parts are called Lyapunov characteristic exponents (LCEs) and give us the exponential rates with which nearby trajectories move to or away from our orbit. If all orbits are hyperbolic, all LCEs are uniformly separated from 0 and all estimates are uniform, then we have an Anosov system; a good example is the geodesic flow on a compact surface of curvature -1. D. Anosov and Ya. Sinaĭ studied such systems about 20 years ago. They constructed invariant families (or foliations) of stable and unstable manifolds and used them to prove ergodicity (i.e., chaos) for Anosov systems preserving an absolutely continuous measure. In the mid-70s Ya. Pesin generalized the whole theory for smooth nonuniformly hyperbolic dynamical systems, i.e., systems for which LCEs are not bounded away from 0 and some may actually equal 0. He followed a similar path by constructing and using the stable and unstable manifolds and proved that the measure-theoretic entropy equals $\sum_j \int \mu_j \, dm$, where the μ_j's are positive exponents and m is an absolutely continuous invariant measure. Later D. Ruelle, R. Mane and others simplified and generalized some of Pesin's results.

In their book A. Katok and J. M. Strelcyn generalize the Pesin theory for the case of a dynamical system with singularities. An example of such a system and one of the main motivations for the book is a billiard system, i.e., the motion of a particle in a region with piecewise smooth boundary when the particle bounces off the walls as light off mirrors. L. Bunimovich and Ya. Sinaĭ proved hyperbolicity and ergodicity for plane regions with concave boundaries, e.g., the unit square with a disk removed. Note that for convex plane regions with smooth enough boundary, V. Lazutkin showed nonergodicity using the KAM theorem. It seems likely that hyperbolicity created in plane billiards by concavities and boundary corners (i.e. singularities) is different in nature from hyperbolicity in traditional Anosov systems like the geodesic flow of a negatively curved surface. Even in smooth flows hyperbolicity may be caused by the singularities of the Poincaré return map. A good example of this situation is the Lorenz system, where hyperbolicity is created in a small neighborhood of the singular set of the Poincaré map.

The book is divided into five chapters. In the first four chapters Pesin's theory is adapted for systems with singularities. One of the merits of the book is that some minor gaps and mistakes in Pesin's arguments are fixed here. Stable and unstable manifolds are constructed in Chapter 1. Chapter 2, which is over one hundred pages long, is devoted entirely to absolute continuity, the property that allows one to use invariant foliations to study ergodic properties of dynamical systems. To understand what this property means, consider a neighborhood U in $\mathbf{R}^n$ foliated by smooth k-dimensional surfaces $S(x)$ and let T be a smooth $(n-k)$-dimensional transversal to all of them. Suppose a set $A \subset U$ is such that for almost every $x \in T$ the intersection $A \cap S(x)$ has measure 0. To conclude now that A has measure 0 one needs an analogue of the Fubini theorem. If the leaf $S(x)$ depends smoothly on x, it is certainly true. However the stable and unstable manifolds usually do not depend smoothly on the point even for real analytic Anosov systems. The absolute continuity allows one to make the above conclusion. Chapter 2 is very technical and certainly the hardest one to read. However, the original argument by Pesin is even more difficult because of several minor gaps and mistakes.

Pesin's entropy formula for dynamical systems with singularities is proved in Chapters 3 and 4. Chapter 3 contains the estimate from below and Chapter 4 the estimate from above.

Plane billiards—the main example of dynamical systems with singularities—are considered in detail in Chapter 5. It turns out that the conditions imposed on singularities at the beginning of the book are satisfied for a very wide class of plane billiards. It takes a long time to prove that, but complicated multi-case arguments are typical for "billiards" proofs. After the conditions are verified one immediately gets the stable and unstable manifolds and all ergodic properties.

The book under review is the only complete detailed treatment in book form of nonuniformly hyperbolic dynamical systems from the point of view of smooth ergodic theory that I know of. I recommend it to motivated and experienced readers.

MICHAEL BRIN