BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

COHOMOLOGY OF THE INFINITE-DIMENSIONAL LIE ALGEBRA L_1 WITH NONTRIVIAL COEFFICIENTS

B. L. FEIGIN AND A. FIALOWSKI

1. Let $\mathcal{L}$ be the Lie algebra of vector fields on the circle of the form $f(\phi)d/d\phi$ where f is a function having finite Fourier expansion (ϕ is the angular parameter on the circle). In $\mathcal{L}$ we can choose the basis

$$e_n = x^{n+1}\frac{d}{dx}, \qquad n \in \mathbf{Z},$$

with the bracket $[e_i, e_j] = (j-i)e_{i+j}$. The Lie algebra $\mathcal{L}$ is naturally graded, the degree of e_i being i. The most natural modules over $\mathcal{L}$ are the so-called tensor field modules. A tensor field on the circle is of the form $g(\phi)(d/d\phi)^\lambda$. A vector field acts on this by infinitesimally changing the coordinate ϕ, where $g(\phi)$ is a section of some line bundle on the circle S^1 with a flat connection. In the space of tensor fields we choose a basis f_i, $i \in \mathbf{Z}$, such that $e_i(f_j) = (-\lambda(i+1)+\mu+j)\cdot f_{i+j}$. Here $\lambda, \mu \in \mathbf{C}$ are the invariants characterizing the module, i.e., the power of $d/d\phi$ and the logarithm of the monodromy of the flat connection. We denote such a module by $\mathcal{F}_{\lambda,\mu}$ (see [**3**]).

Denote by L_1 the subalgebra of $\mathcal{L}$ with basis $(e_1, e_2, e_3, \ldots)$. It is easy to see that L_1 is isomorphic to the Lie algebra of vector fields on the line, with polynomial coefficients, having a two-fold zero at the origin. The strategy of the cohomology computation for L_1 with coefficients in the adjoint module is the following: we first compute the cohomology of L_1 with coefficients in $\mathcal{F}_{\lambda,\mu}$, and then remark that the adjoint representation of L_1 is a submodule of such an $\mathcal{F}_{\lambda,\mu}$. After this the spaces $H^i(L_1, L_1)$ can easily be determined. The computations of $H^1(L_1, L_1)$ and $H^2(L_1, L_1)$ are contained in [**3**]. Deformations of L_1 are studied in [**5**]. In this paper we shall describe a more general method for the computation of $H^*(L_1, \mathcal{F}_{\lambda,\mu})$.

It will be more convenient for us to deal with homology instead of cohomology. It is easy to see that $H^*(L_1, \mathcal{F}_{\lambda,\mu})$ is dual to $H_*(L_1, \mathcal{F}_{-1-\lambda,-\mu})$. Then, using the fact that L_1^* is the factor of some $\mathcal{F}_{\lambda,\mu}$, we can compute $H_i(L_1, L_1^*)$. Notice that for almost every (λ, μ) the module $\mathcal{F}_{\lambda,\mu}$ is an irreducible representation of $\mathcal{L}$, and L_1 is the maximal nilpotent subalgebra in $\mathcal{L}$. That means that the problem of determining $H_*(L_1, \mathcal{F}_{\lambda,\mu})$ is analogous to that of determining the cohomology of the maximal nilpotent subalgebra of a complex semisimple Lie algebra with coefficients in an irreducible representation. We call theorems of this type Bott-Kostant theorems[**1**, **8**]. (Notice that the representations $\mathcal{F}_{\lambda,\mu}$ are reminiscent of the Harish-Chandra modules rather than

Received by the editors April 28, 1986 and, in revised form, June 8, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17B56; Secondary 17B65, 58H10.

Research supported in part by NSF Grant DMS-8508953.

of representations in the category $\mathcal{O}$; see [**1**].) A method analogous to that used in this paper was applied to the current algebras in [**2**].

2. We first recall some pertinent facts. Introduce another Lie algebra $L(0,1)$ which consists of polynomial vector fields on the line with zeros at the points $0,1$. In $L(0,1)$ we can choose the basis

$$\bar{e}_i = x^i(x-1)\frac{d}{dx}, \qquad i \in \mathbf{Z},$$

such that $[\bar{e}_i, \bar{e}_j] = (j-i)(\bar{e}_{i+j} - \bar{e}_{i+j-1})$. There exists a family of one-dimensional $L(0,1)$-modules $M(\alpha,\beta)$; $\bar{e}_i$ acts on $M(\alpha,\beta)$ as multiplication by $\beta-\alpha$ if $i=1$ and by β if $i>1$. The significance of $M(\alpha,\beta)$ is the following. The commutator of $L(0,1)$ consists of vector fields on the line, which, together with their first derivative, vanish at $0,1$. Therefore the character of $M(\alpha,\beta)$ takes the value $\alpha f'(0)+\beta f'(1)$ on the vector field $f(x)d/dx \in L(0,1)$. Observe that $M(\alpha,\beta) = M(\alpha,0)\otimes M(0,\beta)$, where $M(\alpha,0)$ is the module on which the vector field $f(x)d/dx$ acts by multiplication on $f'(0)$, and $M(0,\beta)$ is the one on which it acts by multiplication on $f'(1)$. Recall that $H_i(L_1)$ is two-dimensional for $i>0$, and that the weight of the two homogeneous basis elements of $H_i(L_1)$ are

$$\frac{3i^2+i}{2} \quad \text{and} \quad \frac{3i^2-i}{2}.$$

This result is proved in [**7**]. Further, the cohomology of $L(0,1)$ is also two-dimensional in every positive dimension. In [**5**] it is proved that $L(0,1)$ is a deformation of L_1. Namely, there exists a Lie algebra family $L(0,t)$ with the basis $\bar{e}_i$ and the bracket $[\bar{e}_i,\bar{e}_j] = (j-i)(\bar{e}_{i+j} + t\bar{e}_{i+j-1})$. It is clear that $L(0,0) \cong L_1$. In [**4**] it is proved that the cohomology spaces for $t=0$ and $t\neq 0$ are isomorphic as graded vector spaces (although the multiplication and the Massey operations in them are different).

We now describe the algebra structure of $H^*(L(0,1))$. It turns out that $H^*(L(0,1))$ is free and is generated by three generators, two of degree 1 and one of degree 2. The one-dimensional generators correspond to the cochains $f(x)d/dx \to f'(0)$ and $f(x)d/dx \to f'(1)$. The two-dimensional generator corresponds to the cochain

$$f(x)\frac{d}{dx} \wedge g(x)\frac{d}{dx} \to \int_0^1 (f'(x)g''(x) - f''(x)g'(x))\,dx.$$

The cohomology space $H^*(L(0,1))$ can be computed in the following way. The algebra $L(0,1)$ is the intersection of two algebras of vector fields, $L(0)$ and $L(1)$. Here $L(0)$ and $L(1)$ consist of vector fields on the line with polynomial coefficients vanishing at 0 and 1 respectively. Their sum $L(0)+L(1) = W_1$ is the algebra of all polynomial vector fields. We get a diagram of inclusions

$$\begin{tikzcd} & L(0) \arrow[dr] & \\ L(0,1) \arrow[ur] \arrow[dr] & & W_1 \\ & L(1) \arrow[ur] & \end{tikzcd}$$

From this it follows that, as a differential algebra, the standard cohomology complex $C^*(L(0,1))$ is the tensor product of the differential algebras $C(L(0))$ and $C(L(1))$ over $C^*(W_1)$. (Observe that $C^*(L(0))$ and $C^*(L(1))$ are modules over $C^*(W_1)$ as there exist inclusions $L(0) \to W_1$ and $L(1) \to W_1$.) This means that there exists an Eilenberg-Moore spectral sequence (see **[9]**) whose second term is $\operatorname{Tor}_{H^*(W_1)}(H^*(L(0)), H^*(L(1)))$, and which converges to $H^*(L(0,1))$. Further, $H^*(L(0)) = 0$ for $i \neq 0,1$ and $H^0(L(0)) \cong H^1(L(0)) \cong \mathbf{C}$, $L(1) \cong L(0)$. For W_1 we have $H^i(W_1) = 0$ for $i \neq 0,3$ and $H^0(W_1) \cong H^3(W_1) \cong \mathbf{C}$. The action of $H^*(W_1)$ on $H^*(L(0))$ and on $H^*(L(1))$ is obviously trivial. This means that

$$\operatorname{Tor}_{H^*(W_1)}(H^*(L(0)), H^*(L(1))) \cong H^*(L(0)) \otimes H^*(L(1)) \otimes \operatorname{Tor}_{H^*(W_1)}(\mathbf{C}, \mathbf{C}).$$

According to **[9]**, $\operatorname{Tor}_{H^*(W_1)}(\mathbf{C}, \mathbf{C})$ is a free algebra with one two-dimensional generator. The differentials in the spectral sequence are zero and we get the desired result.

In the following proposition we state the result about the homology of $L(0,1)$ with coefficients in $M(\alpha,\beta)$. It will be more convenient for us to describe the structure of the dual cohomology space. It is clear that $H_*(L(0,1), M(\alpha,\beta))$ is dual to $H^*(L(0,1), M(-\alpha,-\beta))$. The Lie algebra $L(0,1)$ can be embedded in the topological Lie algebra of all vector fields on the line. That means we can define the continuous cohomology

$$H_c^*(L(0,1), M(-\alpha,-\beta)).$$

PROPOSITION 1. *The space $H^*(L(0,1), M(-\alpha,-\beta))$ is different from zero only in the case where there exist two nonnegative integers k, l such that*

$$\alpha = \frac{3k^2 \pm k}{2}, \qquad \beta = \frac{3l^2 \pm l}{2}.$$

The space

$$H^*\left(L(0,1), M\left(-\frac{3k^2 \pm k}{2}, -\frac{3l^2 \pm l}{2}\right)\right)$$

is a free module over $H^*(L(0,1))$, with one generator of degree $k+l$.

The proof is a standard exercise in continuous cohomology theory. The generator can be obtained as follows. Let $L(p)$ be the Lie algebra of vector fields on the line vanishing at $p \in \mathbf{R}$, and let $\overline{M}(\alpha)$ be the module on which $f(z)d/dz$ acts as multiplication by $f'(p)$. The cohomology space $H^*(L(p), \overline{M}(\alpha))$ is known, see e.g. **[6]**. It is nontrivial only if $\alpha = (3k^2 \pm k)/2$ where k is a nonnegative integer, and in that case $H^*(L(p), \overline{M}(\alpha))$ is a free module over $H_c^*(L(p))$ with one generator of degree k. Further, $L(0,1)$ can be embedded in $L(0)$ and in $L(1)$. The restrictions of $\overline{M}(\alpha)$ and $\overline{M}(\beta)$ give the modules $M(\alpha,0)$ and $M(0,\beta)$. The product of the restricted classes of H_c^k and H_c^l ($\alpha = (3k^2 \pm k)/2, \beta = (3l^2 \pm l)/2$) gives the generator of H_c^{k+l}. Finally, one can show that $H^*(L(0,1), M(-\alpha,-\beta))$ is isomorphic to $H_c^*(L(0,1), M(-\alpha,-\beta))$.

The standard complex

$$C_*(L_1, \mathcal{F}_{\lambda,\mu}) = \Lambda^* L_1 \otimes \mathcal{F}_{\lambda,\mu},$$

which yields $H_*(L_1, \mathcal{F}_{\lambda,\mu})$, is naturally graded, as L_1 is a graded algebra and $\mathcal{F}_{\lambda,\mu}$ is a graded module over L_1 (e.g. the chain $e_1 \otimes f_{-1}$ has degree zero). Let $C^0_*(L_1, \mathcal{F}_{\lambda,\mu})$ be the subcomplex of the elements of degree zero. It is clear that $C_*(L_1, \mathcal{F}_{\lambda,\mu}) \cong \bigoplus_k C^0_*(L_1, \mathcal{F}_{\lambda,\mu+k})$, where the sum is taken over all natural numbers k. That means that it is enough to compute the cohomology of $C^0_*(L_1, \mathcal{F}_{\lambda,\mu})$.

PROPOSITION 2. *The complex $C^0_*(L_1, \mathcal{F}_{\lambda,\mu})$ is isomorphic to the complex $M(\alpha,\beta) \otimes \Lambda^* L(0,1)$, which yields the homology of $L(0,1)$ with coefficients in $M(\alpha,\beta)$, where $\alpha = -\mu - \lambda$, $\beta = \lambda - 1$.*

PROOF. Observe that $L(0,1)$ can be realized as the subalgebra of $\mathcal{L}$ with basis $\bar{e}_i = e_i - e_{i-1}$, $i = 1, 2, \ldots$. Put $e'_i = \bar{e}_i - e_0$; on $M(\alpha,\beta)$, e'_i induces multiplication by $i\beta - \alpha$. Let z be a generator in $M(\alpha,\beta)$. The differential $M(\alpha,\beta) \otimes \Lambda^* L(0,1)$ acts in the following way:

$$
\begin{aligned}
&d(z \otimes e'_{i_1} \wedge \cdots \wedge e'_{i_k}) \\
&\quad = \sum_{r,s} (-1)^{r+s}(i_r - i_s) z \otimes e'_{i_r+i_s} \wedge \cdots \wedge \hat{e}'_{i_s} \wedge \cdots \wedge \hat{e}'_{i_r} \wedge \cdots \wedge e'_{i_k} \\
&\quad + \sum_s (-1)^s (i_1 + \cdots + i_k - i_s) \cdot z \otimes e'_{i_1} \wedge \cdots \wedge \hat{e}'_{i_s} \wedge \cdots \wedge e'_{i_k} \qquad (1)\\
&\quad + \sum_s (-1)^{s+1} (i_s\beta - \alpha) \cdot z \otimes e'_{i_1} \wedge \cdots \wedge \hat{e}'_{i_s} \wedge \cdots \wedge e'_{i_k}.
\end{aligned}
$$

The first two sums correspond to the bracket with e'_{i_r} and e'_{i_s} (we remark that $[e'_i, e'_j] = (j-i)e'_{i+j} + ie'_i - je'_j$), while the last one corresponds to the action of e'_j on z. Now we determine the differential in $C^0_*(L_1, \mathcal{F}_{\lambda,\mu})$. The elements $f_{-j} \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}$, $j = i_1 + i_2 + \cdots + i_k$, form a basis of $C^0_*(L_1, \mathcal{F}_{\lambda,\mu})$. Then

$$
\begin{aligned}
(2)\quad &d(f_{-j} \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) \\
&= \sum (-1)^{r+s} f_{-j} \otimes (i_r - i_s) e_{i_r+i_s} \wedge \cdots \wedge \hat{e}_{i_s} \wedge \cdots \wedge \hat{e}_{i_r} \wedge \cdots \wedge e_{i_k} \\
&\quad + \sum (-1)^{s+1} (\lambda(i_s+1) + \mu - j) f_{-j+i_s} \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_s} \wedge \cdots \wedge e_{i_k}.
\end{aligned}
$$

Observe that (1) becomes (2) if $\alpha = -\lambda - \mu$, $\beta = \lambda - 1$.

Combining Propositions 1 and 2, we get a new method for computing $H^*(L_1, \mathcal{F}_{\lambda,\mu})$. Finally, we have the following result.

THEOREM. $H_*(L_1, \mathcal{F}_{\lambda,\mu}) \cong \bigoplus_k H_*(L(0,1), M(-\lambda - \mu + k, \lambda - 1))$, $k \in \mathbf{Z}$.

REFERENCES

1. I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Differential operators on the basic affine space and a study of $\mathfrak{g}$-modules*, Lie Groups and Their Representations, Summer school of the Bolyai Janos Math. Soc. (Hungary) (I. M. Gelfand, ed.), pp. 21–64, New York: Division of Wiley and Sons, Halsted Press, 1975.

2. B. L. Feigin and A. Fialowski, *On the cohomology of nilpotent algebras of flows*, Dokl. Akad. Nauk SSSR **271** (1983), 813–816; English translation: Soviet Math. Doklady **28** (1983), No. 1, 178–181.

3. B. L. Feigin and D. B. Fuks, *Homology of the Lie algebra of vector fields on the line*, Funktsional Anal. Prilozhen. **14** (1980), No. 3, 45–60; English translation: Funct. Anal. Appl. **14** (1980), 201–212.

4. B. L. Feigin and V. S. Retakh, *On the cohomology of certain Lie algebras and superalgebras of vector fields*, Uspekhi Mat. Nauk **37** (1982), No. 2, 233–234; English translation: Russian Math. Surveys **38** (1982), No. 2, 251–252.

5. A. Fialowski, *Deformations of the Lie algebra of vector fields on the line*, Uspekhi Mat. Nauk **38** (1983), No. 1, 201–202; English translation: Russian Math. Surveys **38** (1983), No. 1, 185–186.

6. D. B. Fuks, *Cohomology of infinite dimensional Lie algebras*, Contemporary Soviet Math., Consultants Bureau, New York, 1986. §2.4.

7. L. V. Goncharova, *Cohomology of Lie algebras of formal vector fields on the line*, Funktsional Anal. Prilozhen. **7** (1973), No. 2, 6–14; English translation: Funct. Anal. Appl. **7** (1973), 91–97.

8. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil Theorem*, Ann. of Math. (2) **74** (1951), 329–387.

9. S. Mac Lane, *Homology*, Springer-Verlag, 1963.

INSTITUTE OF SOLID-STATE PHYSICS OF THE SOVIET ACADEMY OF SCIENCES, 142432 CHERNOGOLOWKA (MOSCOW DISTRICT), USSR (Address of B. L. Feigin)

INSTITUTE OF MATHEMATICS OF THE HUNGARIAN ACADEMY OF SCIENCES, 1053 BUDAPEST, REÁLTANODA U. 13–15, HUNGARY

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139