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COVERING SPACES OF 3-MANIFOLDS

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Let M be a closed P^2-irreducible 3-manifold with infinite fundamental group. It is a long-standing conjecture that the universal cover $\tilde{M}$ of M must be homeomorphic to R^3. Waldhausen [**Wa**] proved that this is the case when M is Haken. The first result of this announcement is the following generalization of Waldhausen's result.

THEOREM 1. *Let M be a closed P^2-irreducible 3-manifold. If $\pi_1(M)$ contains the fundamental group of a closed surface other than S^2 or P^2 then the universal cover of M is homeomorphic to R^3.*

We will say that a 3-manifold is *almost compact* if it can be obtained from a compact manifold N by removing a closed subset of ∂N. Then Theorem 1 is equivalent to the assertion that the universal covering of M is almost compact. A natural way to attempt to generalize Theorem 1 is to show that other coverings of M are almost compact. It was conjectured by Simon [**Si**] that if M is any compact P^2-irreducible 3-manifold and if M_1 is a covering of M with finitely generated fundamental group then M_1 must be almost compact. Simon verified this conjecture for the case when $\pi_1(M_1)$ is the fundamental group of a boundary component of M. Jaco [**J**] generalized this to the case when $\pi_1(M_1)$ is a finitely generated peripheral subgroup of $\pi_1(M)$. More recently, Thurston [**Th**] showed that if M admits a geometrically finite complete hyperbolic structure of infinite volume then Simon's conjecture is true. Finally, Bonahon [**B**] showed that any hyperbolic 3-manifold with finitely generated fundamental group is almost compact provided that $\pi_1(M)$ is not a free product.

The second result of the announcement is the following.

THEOREM 2. *Let M be a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains a subgroup A isomorphic to $Z \times Z$. Then the covering M_A of M with $\pi_1(M_A) = A$ is almost compact.*

If M is Haken then the conclusion of the theorem follows immediately from Simon's work [**Si**]. For the crucial condition needed in [**Si**] is that if H is any finitely generated subgroup of $\pi_1(M)$, then $H \cap A$ is also finitely generated, and this obviously holds, as A is isomorphic to $Z \times Z$. If M is not Haken, then the version of the Torus Theorem proved by Scott [**Sc**] shows that some infinite cyclic subgroup of A must be normal in $\pi_1(M)$. Now the following is another long-standing conjecture.

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CONJECTURE 3. *If M is a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is a Seifert fiber space.*

If M is a Seifert fiber space, then Theorem 2 is easily proved. Thus the interest of Theorem 2 is that we do not assume that M is either Haken or a Seifert fiber space. We hope that our result will be a step towards proving Conjecture 3.

Now we give a brief outline of the arguments for proving Theorem 1. Suppose that F is a closed surface, not S^2 or P^2, such that $\pi_1(M)$ contains $\pi_1(F)$. Then there is a map $f\colon F \to M$ inducing the inclusion of fundamental groups and we choose f to be a least-area map in its homotopy class. This can be done in the smooth category by choosing a Riemannian metric on M and then applying the existence results of Schoen-Yau [**S-Y**], or it can be done in the P-L category using work of Jaco and Rubinstein [**J-R**]. Now in the universal cover $\tilde{M}$ of M, the complete pre-image Π of $f(F)$ consists of planes as f injects $\pi_1(F)$ into the fundamental group of M, and these planes are embedded by results of Freedman, Hass, and Scott [**F-H-S**]. Further, the intersection of two planes never contains a circle. These planes define a division of M into 3-dimensional chambers, each of which covers one of the compact chambers in M obtained by cutting M along $f(F)$. We show that each of the chambers in $\tilde{M}$ is simply connected and that each chamber in M is π_1-injective. We also show that if M is not Haken then each chamber in M is a handlebody. It follows that the closure of each chamber in $\tilde{M}$ is almost compact. We show that if a finite collection Σ of the planes in Π are removed, then the new bigger chambers formed by cutting $\tilde{M}$ along $\Pi - \Sigma$ have the same property. As a compact set in $\tilde{M}$ can meet only finitely many planes of Π, it follows immediately that any compact subset of $\tilde{M}$ lies in the interior of a 3-ball. This shows that M is homeomorphic to R^3 as required.

The proof of Theorem 2 uses similar ideas. Let T denote the 2-torus and let $f\colon T \to M$ be a map of least area such that $f_*(\pi_1(T)) = A$. The preimage in M_A of $f(T)$ is a collection of possibly singular tori and annuli. If they were all embedded in M_A then we would argue much as in the proof of Theorem 1. This need not be the case, even when M is a Seifert fiber space, but we show that M_A must have a finite cover in which the preimage of $f(T)$ consists of embedded tori and annuli. This result completes the proof of Theorem 2.

REFERENCES

[**B**] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) **124** (1986), 71–158.

[**F-H-S**] M. H. Freedman, J. Hass and P. Scott, *Least area incompressible surfaces in 3-manifolds*, Invent. Math. **71** (1983), 609–642.

[**J**] W. Jaco, *Lectures on 3-manifold topology*, CBMS Regional Conf. Ser. in Math., no. 43, Amer. Math. Soc., Providence, R.I., 1980.

[**J-R**] W. Jaco and H. Rubinstein, *P-L minimal surfaces in 3-manifolds*, preprint.

[**Sc**] P. Scott, *A new proof of the annulus and torus theorems*, Amer. J. Math. **102** (1980), 241–277.

[Si] J. Simon, *Compactification of covering spaces of compact 3-manifolds*, Michigan Math. J. **23** (1976), 245–356.

[S-Y] R. Schoen and S. T. Yau, *Existence of incompressible minimal surfaces and the topology of 3-dimensional manifolds with non-negative scalar curvature*, Ann. of Math. (2) **110** (1974), 127–142.

[Th] W. Thurston, *Geometry and topology of 3-manifolds*, Lecture Notes, Princeton University, 1978–1979.

[Wa] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

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