10. A. A. Kirillov, *Elements of the theory of representations*, transl. by E. Hewitt, Springer-Verlag, Berlin, Heidelberg, and New York, 1976. Original Russian ed., Nauka, Moscow, 1972; French ed., transl. by A. Sossinsky, Mir, Moscow, 1974.

11. A. W. Knapp, *Review of Lie groups, Lie algebras, and their representations, by V. S. Varadarajan*, and *Compact Lie groups and their representations, by D. P. Želobenko*, Bull. Amer. Math. Soc. **81** (1975), 865–872.

12. S. Lang, $SL_2(\mathbf{R})$, Addison-Wesley, Reading, Mass., 1975.

13. G. W. Mackey, *Harmonic analysis as the exploitation of symmetry—a historical survey*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 543–698.

14. J.-P. Serre, *Linear representations of finite groups*, transl. by L. L. Scott, Springer-Verlag, Berlin, Heidelberg, and New York, 1977. Original French ed., Hermann, Paris, 1971.

15. V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

16. J. A. Wolf, *Spaces of constant curvature*, 4th ed., Publish or Perish, Berkeley, 1977.

17. D. P. Želobenko, *Compact Lie groups and their representations*, transl. by Israel Program for Sci. Transl., Transl. Math. Mono., vol. 40, Amer. Math. Soc., Providence, R.I., 1973. Original Russian ed., Nauka, Moscow, 1970.

JONATHAN ROSENBERG

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

0273-0979/83 $1.00 + $.25 per page

Rings that are nearly associative, by K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov and A. I. Shirshov, translated by Harry F. Smith, Academic Press, New York, 1982, xi + 371 pp., $6.00. ISBN 0-1277-9850-1

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics, and even today the main motivation for studying some problems in the area is the applications. However, most types of nonassociative algebras are now studied more for their own sake.

The first class of nonassociative algebras to be investigated systematically was the class of Lie algebras, which arose out of the study of Lie groups. A nonassociative algebra L with product $[\,,]$ is defined to be a Lie algebra if it satisfies the identities

$$[x, y] = -[y, x], \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

If A is any associative algebra, we define $A^{(-)}$ to be the algebra obtained from A by replacing the associative multiplication in A by the new multiplication $[\,,]$ defined in terms of the associative multiplication by $[a, b] = ab - ba$. Then $A^{(-)}$ is a Lie algebra, and so is any subalgebra of $A^{(-)}$ (that is, any subspace of A closed under $[\,,]$). Conversely, any Lie algebra over a field arises as a subalgebra of an algebra $A^{(-)}$ constructed from some associative algebra A. The methods which are used for studying Lie algebras come out of this special connection with associative algebras and are thus somewhat different from the methods that work best for other classes of nonassociative algebras. Also, partly because of the nature of the principal applications of Lie algebras, the research done in this area tends to involve problems and concepts which do not have good analogues in other classes of nonassociative algebras. For these reasons, Lie algebras are sometimes not included as part of the area of nonassociative algebras. Since the book under review here adopts this point of view, we shall also, and our remaining comments shall apply to the class of nonassociative algebras with Lie algebras deleted. (The reader wishing to know more about Lie algebras might look at [**8** or **9**].)

Dual to the class of Lie algebras in many ways is the class of Jordan algebras, which are defined by the identities

$$xy = yx, \qquad (xy)x^2 = x(yx^2).$$

If the multiplication in an associative algebra A of characteristic not 2 is replaced by the new multiplication $a \cdot b = \frac{1}{2}(ab + ba)$, the new algebra $A^{(+)}$ obtained in this way is a Jordan algebra, and clearly any subalgebra of $A^{(+)}$ is also a Jordan algebra. The Jordan algebras that can be constructed from associative algebras in this manner are called *special*. Contrary to what the duality with Lie algebras would suggest, not every Jordan algebra is special, although in some sense most of them are. Jordan algebras first arose in physics, and they are named for the German physicist Pasqual Jordan. The subject was launched by a paper of Jordan, von Neumann and Wigner [**13**] which classified certain Jordan algebras over the field of real numbers.

The class of alternative algebras is defined by the two identities

$$(1) \qquad x^2y = x(xy), \qquad (yx)x = yx^2,$$

and the most important (and motivating) example of this class is the algebra of octonions. The latter algebra is a nonassociative algebra of dimension 8 over the field which is its center. The specific form of the octonions first studied is a division algebra over the reals which arises from the quaternions by the same doubling construction that gives rise to the quaternions out of the complex numbers. As a basis for the octonions one can choose $1, e_1, e_2, \ldots, e_7$ where 1 acts as the identity element and where for each $i \in \{1, 2, \ldots, 7\}$

$$e_i^2 = -1, \qquad e_ie_{i+1} = e_{i+3} = -e_{i+1}e_i,$$
$$e_{i+1}e_{i+3} = e_i = -e_{i+3}e_{i+1}, \qquad e_{i+3}e_i = e_{i+1} = -e_ie_{i+3}.$$

One verifies easily that if the subscripts here are interpreted modulo 7, every possible product of basis elements is defined exactly once by these equations. It is also clear that for each value of i the elements $1, e_i, e_{i+1}, e_{i+3}$ span a subalgebra which is isomorphic to the quaternions. Once the octonions have been defined over the real numbers, the definition can be extended in the obvious way to define an octonion algebra over an arbitrary field.

The octonions have been used in physics and in various other parts of mathematics. They also play a role in Jordan theory, since the prototype of a Jordan algebra which is not special is the 3×3 Hermitian matrices over the octonions. For an exposition of the early history of alternative and Jordan algebras, the reader is referred to [**16**].

The coming of age of nonassociative algebras occurred in the later 1940s when Albert developed a structure theory for finite-dimensional Jordan algebras over an algebraically closed field of characteristic not 2 [**1, 2**]. In the following two decades, Albert and various other people investigated many identities and the nonassociative algebras satisfying them. The model which was mostly used here was the theory of finite-dimensional associative algebras as developed by Wedderburn. Following this model, the researcher investigating a finite-dimensional algebra satisfying a particular set of identities needed to define a maximal ideal which was nilpotent or something close to nilpotent, and to show that modulo this ideal the algebra was a direct sum of a finite number of simple algebras. Then he wanted to classify the simple algebras. The methods used were very reminiscent of Wedderburn—for example, Peirce decompositions, calculations with an appropriately chosen basis, induction on the dimension.

During this period most authors were not interested at all in applications or connections with other areas, but rather in generating more structure theory to help put together the picture of what the area of nonassociative algebras should look like. Part of the picture that emerged is that for almost any set of identities the simple finite-dimensional algebras satisfying these identities are either much too numerous to classify or else they turn out to be Lie algebras, Jordan algebras, octonion algebras or one other class of algebras constructed from associative algebras (the quasiassociative algebras). This result tended to emphasize the importance of the big three (Lie, Jordan, and alternative) and to discourage research on the other classes.

Even while the Wedderburn theory was being used as the principal model in nonassociative theory, everyone knew that the more modern associative ring theory would prove to be a better model once it was understood how to emulate it. A few pieces of such a more general theory for arbitrary nonassociative algebras have been around for a long time (see [7] for example), but progress has been slow even for alternative and Jordan algebras. In the case of alternative algebras, the first steps were taken by Artin and Zorn, later big advances were made by Kleinfeld and then by Slater, and most of the final part of the known theory was done by the four authors of the book under review. The theory of alternative rings as found in this book is pretty complete, not only in the sense that just about everything known about alternative rings is covered, but also in the sense that most of what is known about associative ring theory and seems worth carrying over to alternative rings has been carried over.

Five years ago, when this book was published in Russian, Jordan theory had not progressed significantly beyond the theory of Jordan algebras satisfying a descending chain condition which was developed principally by Jacobson. In the meantime, a very nice general theory has been developed by Zel′manov. Thus Jordan theory has now mostly caught up to alternative theory, but not in time for this book.

It is gratifying that both alternative algebras and Jordan algebras have theories which are close to the usual structure theories of associative rings. In both cases the generalizations to nonassociative algebras of the most common types of radicals in associative theory can be shown to be pretty well behaved. In particular, these radicals turn out to be *hereditary*—that is if I is an ideal of an algebra A which is either alternative or Jordan and if *rad* denotes one of the more common notions of radical, then $rad(I) = I \cap rad\, A$. As an example of one approach, an element u in an algebra A is called *quasiregular* if there exists $v \in A$ with $u + v = uv$, and in both alternative and Jordan theory there exists in each algebra a maximal ideal all of whose elements are quasiregular. Modulo this ideal (called the Jacobson radical) the algebra is a subdirect sum of primitive algebras. An alternative algebra A is called *primitive* if it contains a maximal right ideal which contains no nonzero two-sided ideal of A, and a Jordan algebra is *primitive* if it contains a maximal inner ideal which contains no nonzero ideal of A. The notion of inner ideal (which will not be defined here) plays the role in Jordan theory of the one-sided ideals in associative or alternative theory. As an example of a second approach, the smallest ideal N of an algebra A such that A/N has no nonzero trivial ideals is called the *lower nil radical*, and it can be shown that A/N is a subdirect sum of prime algebras where an algebra is called *prime* if it does not contain two nonzero ideals whose product is zero.

The nicest feature of the alternative and Jordan theories is that one can characterize quite well the structure of the primitive and prime algebras. For alternative algebras a primitive algebra is either associative or else is an octonion algebra (over an appropriate field), and a prime algebra is either associative or a Cayley-Dickson ring or a nil algebra of characteristic 3. By a *Cayley-Dickson ring* we mean an algebra which becomes an octonion algebra when it is tensored with the quotient field of its center. It is not yet settled whether the prime nil algebras of characteristic 3 exist. For Jordan algebras the primitive algebras either arise from associative algebras using several constructions or are simple exceptional Jordan algebras of dimension 27 over their centers. Similarly any prime Jordan algebra can be constructed from an associative algebra in one of several precise ways or is an *Albert algebra*, where the latter is an algebra which becomes a central simple exceptional Jordan algebra of dimension 27 when it is tensored with the quotient field of its center.

We note that in each of the alternative and Jordan theories there is a sharp dichotomy in the structure of the primitive or prime algebras between those that are associative or derived from associative algebras, and those that are derived in an obvious way from a single example. Thus, modulo associative theory, the structure theory for primitive or prime alternative or Jordan algebras is quite sharp. This contrasts with associative ring theory where one cannot classify primitive or prime rings in any real way.

Some remarks are in order on the general methods that are needed to develop the alternative and Jordan structure theories just described. First of all, it has been necessary in both cases to establish various more specialized identities which are implied by the identities defining the class. Secondly, in both theories it has been necessary to develop sophisticated machinery for representing elements in a certain canonical form in terms of the generators, and for proving things by induction on the length of the form. Thirdly, in both cases it has proved crucial in certain proofs to define just the right subset to make the proof work. For example, the annihilator of a subset has to be defined just the right way. Fourthly, free algebras and algebras satisfying a polynomial identity both play an essential role in alternative and Jordan structure theory. Fifthly, the ring of operators plays a more important role than in associative theory. It is sometimes necessary to show that the ring of operators is an integral domain and then to tensor the algebra with the quotient field of the ring of operators in order to deduce what the structure of the original algebra is. Finally, we note that module theory is not used in either alternative or Jordan algebras to get the structure theory. There is a notion of right module for alternative algebras which can be tied up with the notion of primitive ring, as in the associative case, but which is not used in an essential way. For Jordan algebras, module theory has not yet been connected to the structure theory.

There are two parts of Jordan theory which we did not mention earlier because they did not fit into the comparison that we were making between the alternative and Jordan structure theories modeled on associative ring theory. The first part is the theory of Jordan bimodules developed by Jacobson and referred to at the end of the last paragraph. The second is the study of quadratic Jordan algebras which was developed by McCrimmon to allow inclusion of the characteristic 2 case which is excluded in the study of the usual Jordan algebras (hereafter called *linear* Jordan algebras). In the theory of linear Jordan algebras, there is a certain quadratic operation which is important, and the quadratic theory arises by taking this quadratic operation as the basic operation. When the ring of operators contains $\frac{1}{2}$, the original (linear) multiplication can be reclaimed by linearizing the quadratic operator. There are also two more generalizations of Jordan algebra which have been made, namely to Jordan pairs and Jordan triple systems. The motivation for these generalizations comes from the use of Jordan algebras in the classification of symmetric spaces.

In addition to the developments in alternative and Jordan theory just described, some other fundamental changes have taken place in nonassociative theory in the last two decades. There is less interest today in many of the remaining identities that were studied by Albert and by others in the 1950s and 1960s. However, interest has endured in right alternative algebras (defined by the second identity in (1)) and one or two other generalizations of alternative. The class of *Lie-admissible* algebras, characterized as those nonassociative algebras A such that the derived algebra $A^{(-)}$ with product $[a, b] = ab - ba$ is a Lie algebra, were originally defined by Albert. But not much was done with these algebras for some years, since the methods that worked for most other classes of algebras did not work here. More recently the interest of some physicists in these algebras and the introduction of new methods have spurred much more activity in this area (see [**4** and **5**], for example). The class of right symmetric algebras, defined by the identity

$$(xy)z - x(yz) = (xz)y - x(zy),$$

has also been of interest in recent years since right symmetric algebras arise in the study of symmetric spaces in analysis. However, no one knows yet how to prove much about these algebras without the added assumption that $(xx)x = x(xx)$ for all elements x.

We turn now to a more specific discussion of the book under review. As we commented earlier, this book contains a very complete account of the theory of alternative algebras. This is also the only book to contain the results proved in the last two decades on alternative algebras. Jordan algebras are approached here in much the same spirit as the alternative algebras, namely in a manner modeled after the theory of associative rings. Only the first part of this theory of Jordan algebras is in this book, since the last part of the theory had not been done by Zel'manov when the book was written. As a consequence of the point of view taken here, quadratic Jordan algebras and Jordan bimodules are not considered in the book, nor does the book include a proof of the structure of simple Jordan algebras with descending chain condition. For a complete account of the descending chain condition theory and of Jordan module theory, all done for linear Jordan algebras, the reader is referred to Jacobson [**10**]. Quadratic Jordan algebras can be found in [**11** or **12**]. The latter reference also includes some of Zel'manov's results.

In addition to alternative and Jordan theory, the book under review contains good chapters on varieties of algebras, radical theory, and right alternative algebras. The material in this book is well-organized and well integrated, and the proofs are often easier mathematically than the original proofs or else made easier to read by the manner in which they are broken into lemmas. The book is quite readable, but also quite concentrated. The expert will find this book indispensable, partly because the material is so well integrated, and partly because it makes much more accessible many of the results of the Russian mathematicians working in this area. The nonexpert with some knowledge of associative rings will find this book a good introduction to nonassociative theory. For the nonexpert who wishes to learn more gradually, we recommend reading Schafer [**15**] first and then this book.

We end with a warning to the reader that this book contains several typographical errors which could cause confusion. Perhaps the most confusing one is on line 4 of p. 272 where it is stated that a free alternative ring on nine or more generators has no nonzero nilpotent elements. In fact, at this point in the text it has just been shown that nonzero nilpotent elements exist in such a ring.

REFERENCES

1. A. A. Albert, *On Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. **59** (1946), 524–555.

2. ______, *A structure theory for Jordan algebras*, Ann. of Math. (2) **48** (1947), 546–567.

3. ______, *Studies in modern algebra*, Studies in Math., Vol. 2, Math. Assoc. Amer.; distributed by Prentice-Hall, Englewood Cliffs, N. J., 1963.

4. G. M. Benkart, *The construction of examples of Lie-admissible algebras*, Hadronic J. **5** (1982), 431–493.

5. G. M. Benkart and J. M. Osborn, *Flexible Lie-admissible algebras*, J. Algebra **71** (1981), 11–31.

6. H. Braun and M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin and New York, 1966.

7. B. Brown and N. H. McCoy, *Prime ideals in nonassociative rings*, Trans. Amer. Math. Soc. **89** (1958), 245–255.

8. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, Berlin and New York, 1972.

9. N. Jacobson, *Lie algebras*, Interscience, New York and London, 1962.

10. ______, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., vol. 39, Amer. Math. Soc., Providence, R. I., 1968.

11. ______, *Lectures on quadratic Jordan algebras*, Tata Inst. of Fundamental Research, Bombay, 1969.

12. ______, *Structure theory of Jordan algebras*, Univ. of Arkansas Lecture Notes in Math., Fayetteville, Arkansas, 1981.

13. P. Jordan, J. von Neumann, and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. (2) **36** (1934), 29–64.

14. J. M. Osborn, *Varieties of algebras*, Advances in Math. **8** (1972), 163–369.

15. R. D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York, 1966.

16. M. L. Tomber, *A short history of nonassociative algebras*, Hadronic J. **2** (1979), 1252–1387.

J. MARSHALL OSBORN