3. I. C. Gohberg and M. G. Krein, *Theory of Volterra operators in Hilbert space and its applications*, Math. Monographs, no. 18, Amer. Math. Soc., Providence, R.I., 1970.

4. H. Flanders, *Infinite networks*. I. *Resistive networks*, IEEE Trans. Circuit Theory, **CT-18** (1971), 326–331.

5. IEEE Trans, *Circuits and systems*, Special Issue in Mathematical System Theory (ed. R. Saeks), **CAS-25** (Sept. 1978).

6. Journal of the Franklin Institute, Sesquicentennial Special Issue on *Recent Trends in System Theory* (ed. W. A. Porter), **301** (Jan. 1976).

7. R. W. Newcomb, *Operator of networks*: *A short exposition*, IEEE Circuits and Systems Newsletter, 7 (1974), 4–8.

8. W. A. Porter, *An overview of polynomic system theory*, IEEE Proc. **64** (1976), 18–23.

9. ______, *Operator theory of systems*, IEEE Circuits and Systems Newsletter, 7 (1974), 8–12.

10. Proc. IEEE, *Special Issue on Recent Trends in System Theory* (ed. W. A. Porter), **64** (Jan. 1976).

11. J. R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. 3 (1965), 61–83.

12 R. Saeks, *Resolution space, operators, and systems*, Springer-Verlag, Heidelberg, 1973.

13. J. C. Williams, *Analysis of feedback systems*, MIT Press, Cambridge, 1971.

14. A. H. Zemanian, *Infinite electrical networks*, IEEE Proc. **64** (1976), 6–17.

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Gaussian random processes, by I. A. Ibragimov and Yu. A. Rozanov, Applications of Math., volume 9, Springer-Verlag, New York-Heidelberg-Berlin, 1978, x + 276 pp., $24.80.

A Gaussian law (= probability measure) P on a finite-dimensional vector space V is of the form $dP(x) = \exp(-Q(x))\,dx_J$, where Q is a quadratic polynomial and dx_J is Lebesgue measure on a linear variety (affine subspace) J. Such laws, also called *normal*, are staples of multivariate statistics ([1], [34], [43]), along with their relatives such as Wishart distributions.

Let $EX = \int X\,dP$, the mean of the (vector or scalar) X. In the rest of this review *Gaussian laws will all have mean* 0 unless otherwise stated. If A, B, C and D are any four linear forms on V, then $E(ABCD) = E(AB)E(CD) + E(AC)E(BD) + E(AD)E(BC)$. So, $E(A^4) = 3E(A^2)^2$, the first of a sequence of identities which characterize Gaussian laws on $\mathbf{R}^1$.

Given a probability space $(\Omega, \mathfrak{B}, \Pr)$ and any set T, a *Gaussian process* is any real function X on $T \times \Omega$ such that for each finite set $F \subset T$, $\{X(t, \cdot)\}_{t \in F}$ has a Gaussian law on $\mathbf{R}^F$. Let $X(t) \equiv X(t, \cdot)$.

If T is a Hilbert space H, the *isonormal* Gaussian process L maps H isometrically into an $L^2(\Omega, \Pr)$, with $EL(x, \cdot)L(y, \cdot) = (x, y)$, the inner product; this fixes the laws of L. For any Gaussian process X, there is a Y with the same laws and $Y(t, \omega) = L(g(t), \omega)$, where g maps T into some Hilbert space H. So L is *the* Gaussian process [13]; it clothes a pristine Hilbert space in full Gaussian attire.

Trajectories. Probabilists like to pick an ω and follow the wandering path, or sample function, $t \to X(t, \omega)$ ([3], [13], [20], [48]). The speed at which $\exp(-x^2/2)$ goes to 0 as $x \to \infty$ lets us make (almost) all paths continuous if $g(T)$ in H is compact enough. If $T = \mathbf{R}$, the process X is called *stationary* if all its laws are preserved by translations $t \to t + h$. For a stationary X

restricted to a finite interval T, Fernique ([19], [20]) proved that "compact enough" can be exactly measured by Kolmogorov's metric entropy: if you need $N(\varepsilon)$ points to get within ε of all points of $g(T)$, then convergence of $\int_0^1(\log N(u))^{1/2}\,du$ characterizes path-continuity (and is *sufficient* also for nonstationary Gaussian processes [13]), provided g is continuous.

Sudakov [55] characterizes sample continuity in terms of a mixed volume of infinite-dimensional convex sets. For some other recent sample function results see, e.g., [11], [12], [48].

General parameters. As knowledge of $X(t)$ for real t becomes refined, attention turns toward multidimensional t ("random fields") and to linear processes $X(f, \cdot)$ on spaces of test functions f ("generalized random fields"), where the connecting idea is $X(f, \omega) = \int X(t, \omega)f(t)\,dt$. For one class of these, let $EN(f)N(g)^- = \int(\mathcal{F}f)(\mathcal{F}g)^-\,d\mu$ where $\mathcal{F}$ denotes Fourier transform and μ is a nonnegative tempered measure. If $d\mu(y) = dy/(m^2 + |y|^2)$ for some $m > 0$, N is called a *Nelson process*, studied in quantum field theory ([8], [21], [44], [45], [50]).

Since a Gaussian process X_t (with mean 0) is characterized by its covariance EX_sX_t, one can look for covariances preserved by groups of isometries of symmetric spaces [2].

Abstract Wiener spaces and reproducing kernels. The process L on a Hilbert space H is not of the form $L(h, \omega) = (h, M(\omega))$ with $M(\omega) \in H$ (L is not sample-continuous). But if we restrict h to a dense, but small enough Banach subspace, we can take $M(\omega) \in B$ for any large enough Banach space B which is the completion of H for a small enough norm. L. Gross named such norms *measurable*; the arrangement (H, B) is called an *abstract Wiener space*, and seems to provide the best available substitute for Lebesgue measure in doing analysis on infinite-dimensional spaces ([9], [10], [15], [22], [23], [24], [25], [37], [38], [40], [47]); notable is Gross' logarithmic Sobolev inequality [25]. Conversely, given B, there is an H: if P is a Gaussian law on a Banach space B, then there is a natural bounded linear map j of the dual B' into the Hilbert space $J = L^2(B, P)$. The adjoint j^* takes J onto a subspace $H \subset B \subset B''$. This H is the reproducing kernel Hilbert space $RKHS(P)$. These notions extend to spaces of sections of a vector bundle [4].

The Banach norms and spaces are, of course, not H-unitarily invariant. But one can think of the Gaussian measure "on H" as concentrated on an infinite-dimensional sphere (surface) of radius $\sqrt{\infty}$, equipped with a Laplacian, spherical harmonics, etc. [42].

Analysis of functionals. For an orthonormal basis $\{e_n\}$ of a Hilbert space H_1, the $L(e_n)$ are independent, identically distributed standard Gaussian variables X_n. Let $H := L^2(\Omega, P)$ be the space of all complex-valued functions $f = f(X_1, X_2, \ldots)$ with $E|f|^2 < \infty$ (equivalence classes of measurable functions, actually). Then H is a countable orthogonal direct sum $\bigoplus_{n=0}^{\infty} H_{(n)}$, where $H_{(n)} = K_{(n)} \ominus \bigoplus_{j=0}^{n-1} H_{(j)}$ and $K_{(j)}$ is the set of all jth degree (or less) polynomials in the $L(x)$, $x \in H$. Let $\mathcal{H}_n$ be the n-fold symmetric tensor product of H_1 with itself, spanned by elements

$$\operatorname{sym}(x_1 \otimes \cdots \otimes x_n) := (n!)^{-1} \sum_{\pi \in S(n)} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)},$$

where $S(n)$ is the symmetric group of all permutations of $\{1, \ldots, n\}$. Let $h \to :h:_{(n)}$ denote the orthogonal projection of H onto $H_{(n)}$. Then for each n, there is a map L_n such that for all $x_1, \ldots, x_n \in H$, $L_n(\text{sym}(x_1 \otimes \cdots \otimes x_n)) = :L(x_1) \ldots L(x_n):_{(n)}$. For some constant c_n, $c_n L_n$ is an isometry of $\mathcal{H}_n$ onto $H_{(n)}$. This structural theory, developed by Wiener [56], von Neumann, Kakutani [36], and Segal [51], is quintessentially Gaussian; for expositions and more recent work see Neveu [46, Chapter 7], Hida [31], [32], Linnik [39], Guichardet [26], and Gutmann [27].

For a bounded linear operator A from H_1 into itself and each n, $A \otimes \cdots \otimes A$ (n factors) maps $\mathcal{H}_n$ into itself. If $\|A\| \leqslant 1$, then these operators, via the above isometries, define a contraction $\Gamma(A)$ from H into itself. Nelson [45] proved a sharp inequality: if $1 \leqslant p \leqslant r \leqslant \infty$ and $\|A\| \leqslant ((p-1)/(r-1))^{1/2}$, then $\Gamma(A)$ is a contraction from $L^p(\Omega, P)$ into $L^r(\Omega, P)$.

Inequalities. Slepian [54] proved that if $EX_i^2 = EY_i^2$ and $EX_iX_j \leqslant EY_iY_j$ for all i, j then $\sup_i X_i$ is stochastically larger than $\sup_i Y_i$. Several inequalities relate Gaussian laws and convex sets ([6], [7], [49]). Pitt [49] proved $P(A \cap B) \geqslant P(A)P(B)$ for P Gaussian and A and B symmetric convex sets in $\mathbf{R}^2$ (for $\mathbf{R}^n$, it's a conjecture). Some inequalities follow from the logarithmic concavity of Gaussian densities (e.g. [7]); others, from rotational invariance (e.g. [16]).

Equivalence and singularity. Hájek [28] proved that two Gaussian laws P and Q on a vector space are either singular or equivalent (mutually absolutely continuous). Here P and Q are equivalent if and only if the "J-divergence" $(E_P - E_Q)\log(dP/dQ)$ is finite; it is the supremum of its finite-dimensonal analogues. Using our general representation of Gaussian processes, nondegenerate P and Q can be written as affine transformations of each other, say $dQ(x) = dP(Ax + m)$; Segal [51] showed for the isonormal process, and Feldman ([17], [18]) proved in general, that P and Q are equivalent if and only if $m \in J = RKHS(P)$, and $A = I + B$ where B restricted to J is a Hilbert-Schmidt operator into J, with -1 not in its spectrum. Then A is extended from J to the larger space by continuity. So to find the relations of infinite-dimensional Gaussian laws, it helps to be able to recognize Hilbert-Schmidt operators in specific Hilbert spaces. From $L^2(\mu)$ to $L^2(\nu)$ they are just given by $L^2(\mu \times \nu)$ integral kernels. For later work on singularity, equivalence, and Radon-Nikodym densities, see e.g. Shepp [53] and the book under review.

Prediction. A stationary Gaussian process $X(t, \cdot)$ gives a one-parameter unitary group U_h: $X(t) \to X(t+h)$, acting on the Hilbert space(s) of the process. There is then a finite measure μ on $\mathbf{R}$, called the spectral measure, such that there is a linear isometry of $L^2(\mathbf{R}, \mu)$ into $L^2(\Omega, P)$ taking $(x \to e^{itx})$ to $X(t)$. Prediction and filtering of such processes are concerned with the closed linear spans X_A of $\{X(t): t \in A\}$ for subsets A; or equivalently, with spans of $\{e^{itx}: t \in A\}$ in $L^2(\mathbf{R}, \mu)$: a matter of harmonic analysis. Classical prediction theory takes $A = [-\infty, s]$. Dym and McKean [14] treat this and other cases.

The review. Ibragimov and Rozanov's book actually treats three topics on stationary Gaussian processes (cf. also [30]): 1) singularity and equivalence, and calculation of densities (Radon-Nikodym derivatives) in case of equivalence; 2) in prediction, to find spectral measures μ for which X is "regular" or "completely nondeterministic" in the sense that $\bigcap_n X_{]-\infty,-n]} = \{0\}$, and to study "mixing rates" for such processes; 3) in statistics, to estimate the mean $f(t)$ of a process $X(t, \cdot) + f(t)$ ("filtering", cf. [35]). The list of references at the end of the book contains 28 items, mostly standard textbooks in analysis; 23 papers are cited in footnotes scattered through the volume.

Bits. Electrical engineering and information theory have, since Wiener's fruitful intermediation, been in contact with Gaussian processes; recently flourishing related work is surveyed in [5] (level crossings), [35] (filtering), and [57].

A goodly number of functional limit theorems give Gaussian processes as limits–but that's another story.

Reviews and bibliography. So far, authors of books and surveys have not tried to encompass the whole subject. Neveu [46] gave what is still the largest Gaussian bibliography, as far as I know, with some 600 items. Jain [33] and Marcus [41] gave courses. Each annual index of Mathematical Reviews currently lists between 50 and 100 papers on Gaussian processes (60G15). Of the 57 references below, 20 are themselves surveys or monographs, many of which have extensive bibliographies.

References

1. T. W. Anderson, *An introduction to multivariate statistical analysis*, Wiley, New York, 1958.
2. R. Askey and N. H. Bingham, *Gaussian processes on compact symmetric spaces*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete **37** (1977), 127–143.
3. A. Badrikian and S. Chevet, *Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussiennes*, Lecture Notes in Math., vol. 379, Springer-Verlag, Berlin and New York, 1974, x + 383 pp.
4. P. Baxendale, *Gaussian measures on function spaces*, Amer. J. Math. **98** (1976), 891–952.
5. I. F. Blake and W. C. Lindsey, *Level-crossing problems for random processes*, IEEE Trans. Information Theory **IT19** (1973), 295–315.
6. C. Borell, *Gaussian Radon measures on locally convex spaces*, Math. Scand. **38** (1976), 265–284.
7. H. J. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), 366–389.
8. J. T. Cannon, *Continuous sample paths in quantum field theory*, Comm. Math. Phys. **35** (1974), 215–233.
9. R. Carmona, *Measurable norms and some Banach space valued Gaussian processes*, Duke Math. J. **44** (1977), 109–127.
10. ______, *Potentials on abstract Wiener spaces*, J. Functional Analysis **26** (1977), 215–231.
11. S. Chevet, *Un resultat sur les mesures gaussiennes*, C. R. Acad. Sci. Paris, Sér. A **284** (1977), A441–444.
12. B. S. Cirel'son, I. A. Ibragimov and V. N. Sudakov, *Norms of Gaussian sample functions*, Proc. Third Japan-USSR Sympos. Probability Theory (Tashkent, 1975), Lecture Notes in Math., vol. 550, Springer-Verlag, Berlin and New York, 1976, pp. 20–41.
13. R. M. Dudley, *Sample functions of the Gaussian process*, Ann. Probability **1** (1973), 66–103.
14. H. Dym and H. P. McKean, Jr., *Gaussian processes, function theory, and the inverse spectral problem*, Academic Press, New York, 1976, xi + 335 pp.
15. C. M. Elson, *An extension of Weyl's lemma to infinite dimensions*, Trans. Amer. Math. Soc. **194** (1974), 301–324.
16. C. Fefferman, M. Jodeit, Jr. and M. D. Perlman, *A spherical surface measure inequality for convex sets*, Proc. Amer. Math. Soc. **33** (1972), 114–119.

17. J. Feldman, *Equivalence and perpendicularity of Gaussian processes*, Pacific J. Math. **8** (1959), 699–708; Correction, ibid. **9** (1960), 1295–1296.

18. ______, *Absolute continuity of stochastic processes*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970, pp. 71–86.

19. X. Fernique, *Des resultats nouveaux sur les processus gaussiens*, C. R. Acad. Sci. Paris, Sér. A **278** (1974), A363–A365.

20. ______, *Régularité des trajectoires des fonctions aléatoires gaussiennes*, École d'Été de Probabilités de Saint-Flour IV-1974, Lecture Notes in Math., vol. 480, Springer-Verlag, Berlin and New York, 1975, pp. 1–96.

21. J. G. Glimm, *The mathematics of quantum fields*, Advances in Math. **16** (1975), 221–232.

22. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statistics and Probability (1965), vol. 2, part 1, Univ. Calif. Press, Berkeley and Los Angeles, 1967, pp. 31–40.

23. ______, *Potential theory on Hilbert spaces*, J. Functional Analysis **1** (1967), 123–182.

24. ______, *Abstract Wiener measure and infinite dimensional potential theory*, Lectures in Modern Analysis and Applications II, Lecture Notes in Math., vol. 140, Springer-Verlag, Berlin and New York, 1970, pp. 84–116.

25. ______, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1976), 1061–1083.

26. A. Guichardet, *Symmetric Hilbert spaces and related topics: infinitely divisible positive definite functions, continuous products and tensor products, Gaussian and Poissonian stochastic processes*, Lecture Notes in Math., vol. 261, Springer-Verlag, Berlin and New York, 1972.

27. S. Gutmann, *Correlations of functions of normal variables*, J. Multivariate Analysis **8** (1978), 573–578.

28. J. Hájek, *A property of J-divergences of marginal probability distributions*, Czechoslovak Math. J. **8** (83) (1958), 460–463.

29. ______, *On a property of normal distribution of any stochastic process*, Czechoslovak Math. J. **8** (83) (1958), 610–618 = Selected Translations in Mathematical Statistics and Probability, no. 1, Amer. Math. Soc., Providence, R. I., (1961), 245–252.

30. T. Hida, *Stationary stochastic processes*, Princeton Univ. Press and Univ. Tokyo, Princeton, 1970, iv + 161 pp.

31. ______, *Analysis of Brownian functionals*, Carleton Univ. Math. Lecture Notes, no. 13, Carleton Univ., Ottawa, 1976, iv + 61 pp.

32. ______, *Generalized multiple Wiener integral*, Proc. Japan Acad. A **54** (3) (1978), 55–58.

33. N. Jain, *Notes on Gaussian processes*, Univ. Minnesota School of Math., 1973, iii + 110 pp.

34. N. L. Johnson and S. Kotz, *Distributions in statistics*: vol. 4, *Continuous multivariate distributions*, Wiley, New York, 1972.

35. T. Kailath, *A view of three decades of linear filtering theory*, IEEE Trans. Information Theory **IT20** (1974), 146–181.

36. S. Kakutani, *Determination of the spectrum of the flow of Brownian motion*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 219–323.

37. D. Kölzöw, *A survey of abstract Wiener spaces*, Stochastic Processes and Related Topics, Proc. Summer Research Inst. on Statist. Inference for Stochastic Processes, Indiana Univ., 1974, vol. 1, Academic Press, New York, 1975, pp. 293–315.

38. Hui-Hsiung Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin and New York, 1975.

39. Yu. V. Linnik, *A certain application of algebraic number theory to mathematical statistics*, Trudy Mat. Inst. Steklov **112** (1971), 22–29, 386. (Russian)

40. P. Malliavin, *Hypoellipticité avec dégénérescence et analyse stochastique en dimension infinie*, C. R. Acad. Sci. Paris, Sér. A **284** (1977), A1455–1456.

41. M. Marcus, *Sample paths of Gaussian processes*, Northwestern Univ., Dept. of Math., Lecture Note Series, no. 1, 1977.

42. H. P. McKean, *Geometry of differential space*, Ann. Probability **1** (1973), 197–206.

43. K. S. Miller, *Multidimensional Gaussian distributions*, Wiley, New York, 1964.

44. E. Nelson, *Construction of quantum fields from Markoff fields*, J. Functional Analysis **12** (1973), 97–112.

45. ______, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227.

46. J. Neveu, *Processus aléatoires gaussiens*, Les Presses de l'Université de Montréal, 1968.

47. M. A. Piech, *Smooth functions on Banach spaces*, J. Math. Anal. Appl. **57** (1977), 56–67.
48. V. I. Piterbarg, *Some directions in the investigation of properties of trajectories of Gaussian random functions*: *Supplement-survey*, Stochastic Processes: Sample Functions and Intersections, Matematika: novoe v zarubezhnoĭ nauke, vol. 10, Mir, Moscow, 1978, pp. 258–280. (Russian)
49. L. D. Pitt, *A Gaussian correlation inequality for symmetric convex sets*, Ann. Probability **5** (1977), 470–474.
50. M. Reed and L. Rosen, *Support properties of the free measure for Boson fields*, Comm. Math. Phys. **26** (1974), 123–132.
51. I. E. Segal, *Tensor algebras over Hilbert spaces*, Trans. Amer. Math. Soc. **81** (1956), 106–134.
52. ______, *Distributions in Hilbert space and canonical systems of operators*, Trans. Amer. Math. Soc. **88** (1958), 12–41.
53. L. A. Shepp, *Radon-Nikodym derivatives of Gaussian measures*, Ann. Math. Statist. **37** (1966), 321–354; correction, Ann. Probability **5** (1977), 315–317.
54. D. Slepian, *The one-sided barrier problem for Gaussian noise*, Bell System Tech. J. **41** (1962), 463–501.
55. V. N. Sudakov, *Geometrical problems in the theory of infinite-dimensional probability distributions*, Trudy Mat. Inst. Steklov. **141** (1976), Nauka, Leningrad (Russian); English transl., Proc. Steklov Inst. Math. (1979).
56. N. Wiener, *The homogeneous chaos*, Amer. J. Math. **60** (1939), 897–936.
57. E. Wong, *Recent progress in stochastic processes*: *a survey*, IEEE Trans. Information Theory **IT19** (1973), 262–275.

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Foundations of mechanics, Second Edition, Revised and enlarged, by Ralph Abraham and Jerrold E. Marsden, The Benjamin/Cummings Publishing Company, Reading, Mass., 1978, xxii + 806 pp., $39.50.

1. This excellent book is one of several superb books on mechanics which have appeared in the past decade, such as those of Souriau [**10**], Siegel-Moser [**9**], Arnold [**2**] and Thirring [**13**], indicating a revitalized interest in the venerable subject of classical mechanics. Actually, there have been at least three sources of revitalization in the past forty years. The first came from the solution of the "small divisor problem" in celestial mechanics. The breakthrough here was achieved by Siegel in a mathematical tour de force, and then a new powerful general principle was discovered by Kolmogorov and developed in the hands of Arnold and Moser into a major analytical tool. The second came from the study of geometric properties of mappings and flows, especially in their "generic" behavior. The guiding philosophy had come from the foundational work in differential topology of Whitney and Thom, and was developed by Smale, Anosov, Sinai and their schools. More recently, there has been an influx of new ideas coming from group theory, from the work of Kirilov and Kostant in representation theory, and of Souriau, in rethinking the physical and geometrical principles underlying mechanics. As the bulk of the material added in the second edition deals with this last topic, I will concentrate my attention on it.

Much, but not enough, has been written about the philosophical problems