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# THE CELL-LIKE MAPPING PROBLEM

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A major unresolved issue in topology is whether or not there is a dimension raising cell-like mapping. A *cell-like map* $f: X \longrightarrow Y$ is a proper mapping between metrizable spaces such that $f^{-1}(y)$ has the shape of a point for each $y \in Y$ (i.e., every map of $f^{-1}(y)$ to a polyhedron is null-homotopic or, equivalently for finite dimensional $X$, $f^{-1}(y)$ admits a cellular embedding in a Euclidean space).

During the period of his pioneering work on cellular decompositions of 3-space, Bing asked if the quotient space of such a decomposition is an ANR **[Bi]**. In the mid-sixties, Armentrout asked if the quotient space of a cellular decomposition of $E^3$ is finite dimensional **[Ar]**. It was shown in **[Koz-1]** that these two questions are equivalent and both are settled affirmatively by the following theorem.

THEOREM. *If $f: X \longrightarrow Y$ is a cell-like mapping defined on a subset of a 3-manifold, then* $\dim Y \leqslant 3$.

The theorem is best understood by considering the case with $X$ a compact subset of $E^3$. The Sphere Theorem easily implies that (1) a cell-like subset of $E^3$ has arbitrarily small aspherical open neighborhoods and (2) each component of the intersection of two aspherical open subsets of $E^3$ is aspherical. In **[Koz-2]**, the cell-like mapping problem is reduced to an extension problem for maps into polyhedra; this extension problem is solvable for maps into aspherical polyhedra. These facts are combined to prove the theorem by producing an $\epsilon$-map of $Y$ into $E^3$ for each $\epsilon > 0$.

It was known previously that cell-like maps on 1-dimensional spaces do not raise dimension. This is proved by using the Vietoris Mapping Theorem to conclude that the image has (integral) cohomological dimension at most 1 and then appealing to the fact that covering dimension agrees with cohomological dimension in this case (both are "classified" by extensions of maps into $S^1$). The precise connection between the cell-like mapping problem and the classical question of whether or not covering dimension and cohomological dimension agree for compacta was recently established by R. D. Edwards **[Ed-1]**; he announced that a compactum with cohomological dimension $\leqslant n$ is the cell-like image of a compactum with covering dimension $\leqslant n$.

It seems appropriate to sketch two reductions of the cell-like mapping problem for such maps on ANR's and manifolds. First, the result of Bothe **[Bo]**

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that each $n$-dimensional compactum imbeds in a $(n + 1)$-dimensional AR and the result of Sieklucki [Si] that a collection of pairwise disjoint $n$-dimensional closed subsets of an $n$-dimensional ANR is countable can be used to show that the cell-like mapping problem for $n$-dimensional compacta is equivalent to the cell-like mapping problem for $(n + 1)$-dimensional ANR's. Second, using techniques from [Ed-2], R. Daverman has shown that, for $n \geqslant 4$, if there is a cell-like dimension raising mapping on an $n$-manifold, then there is a cell-like dimension raising mapping on an $(n - 2)$-dimensional compactum. Starting with a cell-like mapping $f\colon M^n \longrightarrow Y$ $(n \geqslant 4)$, Daverman constructs a cell-like mapping $\bar{f}\colon M^n \longrightarrow Y$ with $\bar{f}$ one-to-one on an "infinite 1-skeleton" of $M^n$; if $\dim Y = \infty$, then there is a compact subset $Y' \subseteq Y$ with $\dim Y' = \infty$ and with $\dim \bar{f}^{-1}(Y') \leqslant n - 2$.

The following illustrate that even with substantial additional hypotheses there are currently few results. In [Dy] Dyer showed that if $f\colon X \longrightarrow Y$ is an open mapping with $\dim X = n$ and with each $f^{-1}(y)$ a point or an arc, then $\dim Y = n$ or $n - 1$ provided $Y$ is assumed to be finite dimensional. In [Bor] Borsuk pointed out that a consequence of Smale's Vietoris mapping theorem for homotopy [Sm] is that if a space $Y$ is the image of ANR $X$ under a proper map $f\colon X \longrightarrow Y$ with each $f^{-1}(y)$ an AR, then $Y$ is an ANR provided $Y$ is assumed to be finite dimensional. Both Dyer and Borsuk asked whether or not the assumption of finite dimensionality is necessary; these questions remain unsettled.

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