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ON HURWITZ' "84(g − 1) THEOREM" AND PSEUDOFREE ACTIONS

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1. **Definitions.** An *admissible* space is a finitistic space cf. [B, p. 133] with finitely generated integral homology. Let a group G act on a topological space X, and $\pi: X \longrightarrow G\backslash X$ the corresponding projection onto the orbit space. Then $S = \bigcup_{g \in G-e} X^g$ is the *singular* set and $\pi(S)$ the *branch* set of the G-action. The G-action is said to be *free* resp. *semifree* resp. *pseudofree* if $S = \emptyset$ resp. $S = X^G$ resp. S is discrete. If X is admissible, G finite acting pseudofreely on X then S is finite. In this case for $P^* \in \pi(S)$ the order n_{p^*} of the isotropy subgroup of G at $P \in \pi^{-1}(P^*)$ is called the *branching index* of the action at P^*.

2. **An extension of Hurwitz' theorem.** Let Aut X denote the full group of automorphisms of a Riemann surface X. If X is closed and has genus $g \geqslant 2$ Hurwitz cf. [H] proved that $|\text{Aut } X| \leqslant 84(g-1)$ and that the action of Aut X on the space of holomorphic differentials is faithful. Now every nonidentity holomorphic selfmap of X has only *isolated* fixed points and also Aut X (since it leaves a Riemannian metric invariant) is a compact Lie group. This theorem then extends to arbitrary pseudofree actions as follows. Let $\mathbf{N} = \{1, 2, \cdots\}$ and χ denote the Euler characteristic.

(2.1) THEOREM. *There exists $h: \mathbf{N} \longrightarrow \mathbf{R}_{>0}$ with the following property. If X is admissible, $\chi(X) < 0$, $m(X) = \Sigma_{i\geqslant 0} \dim H_{2i}(X; Q)$ and G is a compact Lie group acting on X so that every finite subgroup acts pseudofreely. Then G is finite, $|G| \leqslant h(m(X))|\chi(X)|$ and the action of G on $H_*(X; Q)$ is faithful.*

EXAMPLES. One has $h(1) = 6, h(2) = 42, h(3) = 1806, \cdots$. In Hurwitz' theorem one has $m(X) = 2$, $\chi(X) = 2 - 2g$ and so $h(2)|\chi(X)| = 84(g-1)$. For all other closed or nonclosed surfaces with $\chi(X) < 0$ one has $m(X) = 1$ and the bound for G is $6|\chi(X)|$. In the case of a closed nonorientable surface U_h with $h \geqslant 3$ crosscaps this bound is $6(h-2)$. As in the classical case cf. [M], [S] it may be shown that this bound is attained for infinitely many h's. Concretely there exist pseudofree actions of the alternating groups A_4, A_5 and the

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Klein's simple group of order 168 on U_4, U_{12} and U_{30} resp.

3. **The case $\chi(X) \geqslant 0$.**

(3.1) THEOREM. *There exists a function $\mathcal{I}: \mathbf{N} \cup \{0\} \longrightarrow \mathbf{R}_{>0}$ with the following property. If X is admissible, $\chi(X) = 0$, $m(X) = \Sigma_{i\geqslant 0} \dim H_{2i}(X; Q)$ and G is a compact Lie group so that every finite subgroup of G acts pseudofreely, then the isotropy subgroups of G are finite with order bounded by $\mathcal{I}(m(X) - 1)$ and act faithfully on $H_*(X; Q)$.*

In case X is admissible, $\chi(X) > 0$ and G a finite group acting pseudofreely on X we say that G is *exceptional* if (i) G acts trivially on $H_*(X; Q)$, (ii) G does not act semifreely and (iii) no partial sum of the reciprocals of the branching indices is an integer.

(3.2) THEOREM. *There exists $\mathcal{J}: \mathbf{N} - \{1\} \longrightarrow \mathbf{R}_{>0}$ with the following property. Let X be admissible and $\chi(X) > 0$. Then exceptional groups exist only if $\chi(X) \geqslant 2$ in which case their order is bounded by $\mathcal{J}(\chi(X))$.*

4. **Pseudofree actions on cohomology manifolds.** We have classified finite groups admitting pseudofree actions on cohomology manifolds with some homology properties common with disks, spheres[2] and real and complex projective spaces. For the case of S^2 such result was proved by Smith cf. [**Sm**, pp. 407–409]. His proof unfortunately does not generalize to other cases. As a sample of these results I mention

(4.1) THEOREM. *Let X be a $\mathbf{Z}$-cohomology manifold of dimension $2d > 2$ with $\mathbf{Z}$-cohomology ring isomorphic to that of $\mathbf{P}_d(\mathbf{C})$ and G a finite group acting pseudofreely on X. Then* (1) *$d + 1 \neq$ a prime $\Rightarrow G$ is cyclic,* (2) *$d + 1 = p$, a prime $\Rightarrow G$ is either* (i) *cyclic or* (ii) $\mathbf{Z}_p \times \mathbf{Z}_p$ or (iii) $\mathbf{Z}_n \rtimes \mathbf{Z}_p$ *where $p^2 \nmid n$ and $\mathbf{Z}_p$ acts nontrivially on $\mathbf{Z}_n$.*

5. **Remarks on proofs of (2.1), (3.1) and (3.2).**

(5.1) THEOREM. *Let a finite group G act* (*not necessarily pseudofreely*) *on X and assume that for all subgroups H of G, X^H is admissible. Then*

(i) $\Sigma_{g\in G} \chi(X^g) = |G|\chi(G\backslash X)$,

(ii) $\chi(X) - \chi(S) = |G|\{\chi(G\backslash X) - \chi(G\backslash S)\}$.

For $X =$ a finite complex these equations may be proved directly. In the general case we use [**Bw**], [**Z**] allowing an interpretation of $\chi(X^g)$ as the Lefschetz number $L(g)$. Roughly speaking these equations generalize the classical Riemann Hurwitz formula. The rest is an understanding and refinement of Hurwitz' original argument in [**H**].

[2] J. Shaneson informed me that he and S. Cappell have also classified groups acting pseudofreely on spheres in a sequel to [CS].

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