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*Spectral theory of linear operators*, by H. R. Dowson, London Math. Soc. Monographs, No. 12, Academic Press, London and New York, 1978, xii + 422 pp., $39.00.

It is a commonplace that many of the phenomena arising in pure and applied analysis can be described directly or indirectly by continuous linear operators acting on infinite-dimensional complex Banach spaces. One need only recall, for example, the Fourier-Plancherel transformation, or integral operators, or the subject of group representations. We shall call a continuous linear mapping of a complex Banach space into itself an operator (or a bounded operator). One of the most powerful tools for linking the algebraic behavior of an operator $T$ on $X$ with its spatial action is the *spectrum of* $T$, $\sigma(T)$, defined to be the set of all complex numbers $\lambda$ such that $(\lambda - T)$ fails to be invertible in the algebra of all operators on $X$. In the general setting the spectrum plays a role analogous to that of the set of eigenvalues in the finite-dimensional case, and is a nonvoid compact set. Moreover, because the resolvent function $(\lambda - T)^{-1}$ is an analytic function of $\lambda$ on the complement of $\sigma(T)$, Cauchy's integral formula can be used to define $f(T)$ whenever $f$ is a complex-valued function analytic on a neighborhood of $\sigma(T)$. This is the "functional calculus" of $T$ and illustrates one benefit of spectral theory–the infusion of the machinery of complex analysis into general operator theory. More broadly, spectral theory (the analysis of operators by way of their spectra) seeks to facilitate the study of operators through scalar considerations. Here are some further examples. The spectral theorem for a normal operator $N$ on a Hilbert space $H$ (i.e., $N^*N = NN^*$) asserts the existence of a measure $E(\cdot)$ on the Borel sets of $\sigma(N)$ having selfadjoint projections on $H$ for its values and such that, among other things, $N = \int \lambda E(d\lambda)$. In the infinite-dimensional setting, a close analogy with finite-dimensional operators is provided by *compact operators* (operators mapping bounded sets onto sets with compact closure). Compact operators, including many integral operators, have a spectral theory which reflects their kinship with finite-dimensional operators. For instance every nonzero point in the spectrum of a compact operator is an eigenvalue with finite-dimensional eigenmanifold, and the spectrum is countable with no accumulation point except possibly the origin. Spectral theory is an extensive subject which blends with all aspects of operator theory, and the foregoing brief discussion is intended only to indicate some of its flavor.

Dowson's *Spectral theory of linear operators* enables readers with a basic understanding of Banach spaces to learn the spectral theory of bounded Banach spaces operators from the ground up, and to reach the frontiers of knowledge for three important classes of operators with a rich spectral theory (the Riesz, prespectral, and well-bounded operators). This is accomplished by careful organization of the material and an overview of the machinery of spectral theory; tne tenor of the book is decidedly toward unification and cohesion of concepts. The book is divided into five main parts, which are, in order of presentation: general spectral theory, Riesz operators, hermitian operators, prespectral operators, and well-bounded operators. Each part ends with a section of extensive notes and comments. The text of the first part consists of one chapter on the basics of spectral theory, including the spectral radius formula, subdivisions of the spectrum, the functional calculus, idempotents, spectral sets, the minimal equation theorem, and ascent and descent.

Part two begins with a chapter on compact operators, treated both for their own sake and as a prelude to Riesz operators. After attending to the basic structure theory, the treatment of compact operators moves on to the Hilden-Lomonosov "ping pong" proof that a nonzero compact operator on an infinite-dimensional Banach space has a proper hyperinvariant subspace. This sets the stage for taking up Ringrose's theory of superdiagonal forms for compact operators, in which "simple" nests of invariant subspaces can be used to obtain a type of multiplicity theory for compact operators–in analogy with superdiagonal forms for operators on finite-dimensional spaces. The text of Part 2 concludes with a chapter on Riesz operators, whose spectral theory is like that of compact operators. Various characterizations of Riesz operators are given. The following one illustrates the parallel with compact operators: an operator is a Riesz operator if and only if every nonzero point of its spectrum is a pole of its resolvent operator, and the spectral projection corresponding to each nonzero point in its spectrum has finite-dimensional range. The Riesz operators on a Banach space $X$ can also be described as the operators whose image in the Calkin algebra of $X$ is quasinilpotent. T. T. West's characterization of the Riesz operators on a Hilbert space is shown. This states that every Riesz operator on a Hilbert space is a compact perturbation of a quasinilpotent, and makes use of an appropriate "superdiagonalization process." It is an open question whether West's characterization is valid for the Riesz operators on an arbitrary Banach space.

The text of Part 3 consists of a brief chapter concerning hermitian operators on Banach spaces. A bounded operator $T$ on a complex Banach space $X$ is called *hermitian* provided its numerical range (in a certain appropriate sense) is real. Equivalently, $T$ is hermitian if and only if the one-parameter group (continuous in the uniform operator topology) generated by $iT$ consists of isometries of $X$. Such operators coincide with the usual selfadjoint ones in Hilbert space, and, in the general Banach space setting, retain a limited, but nevertheless surprising, number of the pleasant properties of selfadjoint operators. For example the hermitian operators on a Banach space form a closed real linear manifold, the spectrum of a hermitian operator is real, and, though it is not at all apparent, the spectral radius and norm of a hermitian operator are equal. On the other hand, the square of a hermitian operator need not be hermitian, and the spectral theorem is not valid for hermitian operators in the general Banach space setting. These facts, among others, are taken up in Part 3, which neatly develops key results needed for applications to prespectral operators. The link between the hermitian and prespectral operators stems from the fact that a bounded Boolean algebra of projections can be made to consist of hermitians by a suitable equivalent renorming of the underlying Banach space.

Part 4, the largest part of the book, comprised of ten chapters, deals with prespectral operators. Let $X$ be a Banach space, and $\Gamma$ a total linear manifold in the dual space $X^*$. A bounded operator $T$ on $X$ is called *prespectral of class* $\Gamma$ provided there is a bounded, $\Gamma$-countably additive spectral measure $E(\cdot)$ on the Borel subsets of the complex plane (whose values are projections on $X$) such that $T$ commutes with all values of $E(\cdot)$, and for each Borel set $\delta$ the restriction of $T$ to $E(\delta)X$ has spectrum contained in the closure of $\delta$. $E(\cdot)$ is then called a *resolution of the identity of class* $\Gamma$ *for* $T$. $T$ is called a *spectral operator* provided the above holds with $\Gamma = X^*$, in which case $E(\cdot)$ is necessarily strongly countably additive. Early in the study of prespectral operators Fixman gave an example, reproduced in the book, which shows that a prespectral operator can have two distinct resolutions of the identity of different classes. For about twenty years it was not known whether a prespectral operator of class $\Gamma$ had a unique resolution of the identity of class $\Gamma$. This was settled affirmatively by Dowson in 1973. A streamlined proof, using hermitian operators, is given near the beginning of Part 4, thereby freeing subsequent formulations of circumlocutions that were once necessary. It should perhaps be pointed out here that the adjoint of any prespectral operator on $X$ is prespectral of class $X$, and the operator in the Fixman example is the adjoint of a spectral operator, but is not spectral. This illustrates the desirability of going beyond spectral operators to the study of prespectral operators. The next notion is a natural generalization of normal operator. A *scalar-type operator of class* $\Gamma$ is a prespectral operator $S$ of class $\Gamma$ whose resolution of the identity $E(\cdot)$ of class $\Gamma$ satisfies $S = \int_{\sigma(S)} \lambda E(d\lambda)$. An operator $T$ is prespectral of class $\Gamma$ if and only if $T$ can be written $T = S + N$, where $S$ is scalar-type of class $\Gamma$ and $N$ is a quasinilpotent commuting with the resolution of the identity of class $\Gamma$ for $S$. Thus, via the Jordan form, prespectral operators can be viewed as a generalization of operators on finite-dimensional spaces. The foregoing provides a glimpse at the early stages of Part 4, which covers a variety of topics too numerous to discuss here in detail. These topics include: the functional calculus for prespectral operators, the single-valued extension property, Boolean algebras of projections, spectral operators, logarithms of prespectral operators, compact prespectral operators, point spectrum of a prespectral operator, and restrictions of spectral and prespectral operators. Two chapters in Part 4 deal with normal operators. The first of these develops the basic properties, and gives Whitley's neat and rapid proof of the spectral theorem for a normal operator. This proof is Banach-algebra-minded, but avoids Banach algebra theory. Rosenblum's elegant proof of Fuglede's theorem is presented afterwards. The second chapter on normal operators centers on property (P) (a normal operator is said to have property (P) if each of its invariant subspaces is reducing). Among other things, Sarason's theorem that a normal operator is reflexive is shown here, and this leads to the theorem that a normal operator $A$ has property (P) if and only if $A^*$ is in the weakly closed algebra generated by $A$ and the identity operator. Part 4 includes a proof of Wermer's result that the sum and product of commuting spectral operators on a Hilbert space are spectral, and presents McCarthy's example of two commuting scalar-type spectral operators on a separable, reflexive Banach space whose sum and product both fail to be spectral.

The last part of the book deals with well-bounded operators. For $J = [a, b]$, a compact interval of the real line $R$, and $f$ a complex-valued function of bounded variation on $J$, let $|||f||| = |f(b)| + \operatorname{var}(f, J)$. An operator $T$ on a Banach space is said to be well bounded if there are a compact interval $J$ and a constant $K$ such that $\|p(T)\| \leq K|||p|||$ for every polynomial $p$ on $J$ having complex coefficients (in particular, it follows that $\sigma(T) \subseteq J$). To illustrate how such operators can arise, we mention Gillespie's work showing that if $G$ is a locally compact abelian group, then the translation operators on $L^p(G)$, $1 < p < \infty$, possess logarithms which are well-bounded operators multiplied by $i$. (This example is worked out in the last chapter.) Ringrose's "spectral theorem" for well-bounded operators is demonstrated. This theorem states that if $T$ is an operator on a Banach space $X$, then well boundedness of $T$ is equivalent to the existence of a compact interval $[a, b]$ and a one-parameter family $\{E(t)\}$, of projections on $X^*$ satisfying certain natural (but in part technically involved) requirements, with $\langle Tx, y\rangle = b\langle x, y\rangle - \int_a^b\langle x, E(t)y\rangle\, dt$ for all $x$ in $X$ and $y$ in $X^*$. The family $\{E(t)\}$ is called a *decomposition of the identity for $T$*, and need not be unique. Despite its *formal* similarity with the spectral theorem for selfadjoint operators (i.e., formally we integrate by parts in the latter), Ringrose's "spectral theorem" has a quite different proof (which is not short). The proof provides useful side-results including a necessary and sufficient condition for a well-bounded operator to have a unique decomposition of the identity. If $T$ is well bounded on $X$, and there exists a family $\{F(t)\}$, $t \in R$, of projections on $X$ such that $\{F(t)^*\}$ is a decomposition of the identity for $T$, then $T$ is called *decomposable* in $X$. It is shown that in this case $T$ has a unique decomposition of the identity, and hence the family $\{F(t)\}$, $t \in R$, is unique. By requiring that $F(\cdot)$ be right continuous on $R$ with respect to the strong operator topology we arrive at the definition of type (A) operator. If $T$ is of type (A), and, at each real number, $F(\cdot)$ has a left-hand limit in the strong operator topology, then $T$ is said to be of type (B). An operator $T$ of type (A) has an operational calculus based on the left-continuous functions of bounded variation on some interval $[a, b]$; moreover, $T = \int_{a-}^b t\, dF(t)$, where the integral exists as a strong limit of Riemann sums. An operator of type (B) has an operational calculus based on the functions of bounded variation. In addition to further properties of operators of types (A) and (B), the last part of the book also examines the relationships between well-bounded and prespectral operators on Hilbert and Banach spaces (in particular, it is shown that the class of well-bounded operators on $l^2$ is strictly larger than the class of scalar-type spectral operators on $l^2$ with real spectrum).

Being of moderate size, this book is not a comprehensive treatise on all aspects of spectral theory. There are certainly some advanced topics not covered (e.g., general multiplicity theory is only mentioned, the Brown-Douglas-Fillmore results on essentially normal operators are absent). What the book does do is provide a fund of valuable up-to-date information, much of it not heretofore available in one place. It covers some important classes of operators which are not treated in any detail in the three volumes of Dunford and Schwartz: Riesz operators, generalized hermitian operators, prespectral (as opposed to spectral) operators, and well-bounded operators. The exposition is well knit, and there are numerous examples. Dowson's book is a fine contribution to the literature, and should benefit both experts and novices. If there is any shortcoming, it would be the absence of exercises.

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*Bessel polynomials*, by Emil Grosswald, Lecture Notes in Math., vol. 698, Springer-Verlag, Berlin-Heidelberg-New York, 1978, xiv + 182 pp., $9.80.

Many mathematicians, of whom I am one, find orthogonal polynomials fascinating. I was introduced to the Legendre polynomials by O. D. Kellogg, in a course on potential theory, almost half a century ago. At the time, I was entranced more by their elegant formal properties than by their applications. Later, I encountered other orthogonal polynomials. One of the ways in which they arise is as eigenfunctions of differential equations, where the boundary condition is just that of *being* a polynomial, and so involving only finitely many parameters. Perhaps if high-speed computers had been invented earlier, the computational advantages of polynomial solutions would have seemed less compelling, but it is hard to imagine that the so-called classical polynomials (Laguerre, Hermite, Jacobi–Legendre and Chebyshev are special cases) could have escaped notice for long.

I expect (without having actually investigated their history) that all the named systems had been studied by predecessors of the mathematicians they are named for. The Bessel polynomials, however, are exceptional: they appear not to have been studied by Bessel (although they are related to Bessel functions), and were named by Krall and Frink [2] in 1949. They had, in fact, been more or less known at least since 1873, and had occurred in connection with the irrationality of $\pi$, statistics, and the wave equation; and were introduced (independently) at about the same time in electrical engineering. Such an ubiquitous set of polynomials surely deserves not only a name but more than the casual mention it got in the Bateman Project volumes [1] in 1953.

The paper by Krall and Frink was actually the first systematic study of the Bessel polynomials; since objects of mathematical discourse, like continents, are so often named for those who popularize them rather than for those who discover them, it is only because of Krall and Frink's good taste that we do not now know these polynomials as the Krall-Frink polynomials. Some of the subsequent active research on Bessel polynomials seems to have been inspired by Krall and Frink's calling attention to the orthogonality of the polynomials–in the complex plane rather than on the real intervals where the classical polynomials are orthogonal.

Grosswald's bibliography lists 116 titles dealing with Bessel polynomials. The book is a quite detailed survey. It describes not only the analytic properties such as one finds for the classical orthogonal polynomials in Szegö's book [3], but also algebraic properties (irreducibility, the Galois group). Grosswald has also provided abstracts of many results that he could