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MAXIMAL FUNCTIONS: A PROOF OF A CONJECTURE OF A. ZYGMUND

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In $\mathbf{R}^n$ let us consider the family B_n of parallelepipeds with sides parallel to the coordinate axes. We may ask for conditions upon the locally integrable function f in order that

$$[*] \qquad \lim_{\substack{x\in R\in B_n \\ \operatorname{diam}(R) \to 0}} \frac{1}{\mu\{R\}} \int_R f(y)\, d\mu(y) = f(x)$$

a.e. x., where μ = Lebesgue measure in $\mathbf{R}^n$.

In 1935 B. Jessen, J. Marcinkiewicz and A. Zygmund [1] showed that [*] holds so long as $f \in L(1 + (\log^+ L)^{n-1})(\mathbf{R}^n)$ locally. Furthermore this result is the best possible in the following sense: if $\psi(t)$ is an Orlicz's space defining function such that $\psi(t) = o(t(\log t)^{n-1})$, $t \to \infty$, then statement [*] is false for a typical L_ψ-function (typical in the sense of Baire's category). Of course the case $n = 1$ was known before as Lebesgue's Differentiation theorem.

The following natural problem was proposed by A. Zygmund: given a positive function Φ on $\mathbf{R}^2$, monotonic on each variable separately, let us consider the differentiation basis B_Φ in $\mathbf{R}^3$ defined by the two parameters family of parallelepipeds whose sides are parallel to the rectangular coordinate axis and whose dimensions are given by $s \times t \times \Phi(s, t)$, s, t positive real numbers. For which locally integrable functions f is statement [*] true with respect to the family B_Φ?

In general the differentiation properties of B_Φ must be, at least, not worse than B_3, the basis of all parallelepipeds in $\mathbf{R}^3$ whose sides have the direction of the coordinate axes, and, of course, not better than B_2. A. Zygmund conjectured after his 1935 paper that B_Φ behaves like B_2. This conjecture is now a theorem with applications to a.e. convergence of Poisson Kernels associated to certain symmetric spaces.

THEOREM. (a) *B_Φ differentiates integrals of functions which are locally in $L(1 + \log^+ L)(\mathbf{R}^3)$, that is*

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$$\lim_{\substack{R \Rightarrow x \\ R \in B_\Phi}} \frac{1}{\mu\{R\}} \int_R f(y)\, d\mu(y) = f(x), \quad a.e.\ x$$

so long as f is locally in $L(1 + \log^+ L)(\mathbf{R}^3)$, where μ denotes Lebesgue measure in $\mathbf{R}^3$.

(b) *The associated maximal function*

$$M_\Phi f(x) = \operatorname*{Sup}_{\substack{x \in R \\ R \in B_\Phi}} \frac{1}{\mu\{R\}} \int_R |f(y)|\, d\mu(y)$$

satisfies the inequality

$$\mu\{M_\Phi f(x) > \alpha > 0\} \leqslant C \int_{\mathbf{R}^3} \frac{|f(x)|}{\alpha} \left\{1 + \log^+ \frac{|f(x)|}{\alpha}\right\} d\mu(x)$$

for some universal constant $C < \infty$.

The proof is based on the following geometric argument:

COVERING LEMMA. *Let B be a family of dyadic parallelepipeds in $\mathbf{R}^3$ satisfying the following monotonicity property: if $R_1, R_2 \in B$ and the horizontal dimensions of R_1 are both strictly smaller than the corresponding dimensions of $\mathbf{R}^2$, then the vertical dimension of R_1 must be not bigger than the vertical dimension of R_2.*

Then the family B satisfies the exponential type covering property, that is: Given $\{R_\alpha\} \subset B$ one can select a subfamily $\{R_j\} \subset \{R_\alpha\}$ such that,

(i) $\mu\{\bigcup R_\alpha\} \leqslant C\mu\{\bigcup R_j\}$,

(ii) $\int_{\bigcup R_j} \exp(\Sigma \chi_{R_j}(x))\, d\mu(x) \leqslant C\mu\{\bigcup R_j\}$

for some universal constant $C < \infty$.

APPLICATION. Consider

$$\mathbf{R}^3 = \{X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}, \text{ real, symmetric, } 2 \times 2\text{-matrices}\},$$

and the cone $\Gamma = \{X \in \mathbf{R}^3$, positive definite$\}$. Then T_Γ = tube over Γ = Siegel's upper half-space = $\{X + iY,\ Y$ positive definite$\}$.

For each integrable function f in $\mathbf{R}^3$ we have the "Poisson integral",

$$u(X + iY) = P_Y * f(x), \qquad Y \in \Gamma,$$

where

$$P_Y(X) = C\, [\det Y]^{3/2} / |\det(X + iY)|^3$$

and we may ask the following question: for which functions f is it true that $u(X + iY) \longrightarrow f(X)$, a.e. X, where $Y \longrightarrow 0$?

It is a well-known fact that if $Y = cI = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \longrightarrow 0$, then $u(X + iY) \longrightarrow f(X)$, a.e. X for integrable functions f. On the other hand if $Y \longrightarrow 0$ without any restriction than a.e. convergence fails for every class $L^p(\mathbf{R}^3)$, $1 \leqslant p \leqslant \infty$.

Here we can settle the case

$$Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \longrightarrow 0$$

because an easy computation shows that

$$Mf(X) = \mathrm{Sup}_Y \, |u(X + iY)|$$

where

$$Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$$

is majorized, in a suitable sense, by $M_\Phi f$ with $\Phi(s, t) = (s \cdot t)^{1/2}$. Therefore we have convergence for $L(1 + \log^+ L)(\mathbf{R}^3)$ and, since $Mf \geqslant CM_\Phi f$ is also true for some $c > 0$, $L(1 + \log^+ L)(\mathbf{R}^3)$ is the best class for which almost everywhere convergence holds.

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