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BOOK REVIEWS

A history of ancient mathematical astronomy, by O. Neugebauer, Studies in the History of Mathematics and Physical Sciences, Vol. 1, Springer-Verlag, New York, Heidelberg, Berlin, 1975, xxi + 555 pp. (Part One), pp. 556–1058 (Part Two), and pp. 1058–1457 (Part Three), $124.70.

It is pretentious that one with my credentials should sign–albeit jointly, with a historian of science–a review of a compendium of knowledge which has rarely, if at all, been surpassed during this century. But the editor of these reviews feels that there should be some statement of how "a sophisticated astronomer" of the present reacts to the astronomy of antiquity. Perhaps he had in mind a statement like the following which one reads in Hardy's well-known *A mathematician's apology*.

> Finally, as history proves abundantly, mathematical achievement, whatever its intrinsic worth, is the most enduring of all.
>
> We can see this even in semi-historic civilizations. The Babylonian and Assyrian civilizations have perished; Hammurabi, Sargon, Nebuchadnezzar are empty names; yet Babylonian mathematics is still interesting, and the Babylonian scale of sixty is still used in astronomy. But of course the crucial case is that of the Greeks.
>
> The Greeks were the first mathematicians who are still "real" to us today. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greeks first spoke a language which modern mathematicians can understand; as Littlewood said to me once, they are not clever schoolboys or 'scholarship candidates' but 'Fellows of another college.' So Greek mathematics is 'permanent', more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.

In a similar vein–exaggerated but not unfairly–what could a "real" astronomer of today say of ancient astronomy? Here there is a difficulty. Mathematical truths are indeed permanent; but ancient astronomy in which circular motions play a role comparable to a law of inertia can hardly claim the allegiance of a modern astronomer in the manner that Archimedes' method of determining the value of pi can claim the allegiance of a mathematician. But to say that is not to say that the demonstration of Apollonius, that an eccentric movement can always be replaced by an epicyclic motion where the center of the epicycle moves on a circle with the observer at its center and with the radius of the epicycle equal to the eccentricity, will not delight anyone with some feeling for mathematical elegance.

In another context Professor Neugebauer quotes Hilbert as having once expressed that the importance of a scientific work can be measured by the number of previous publications it makes superfluous to read. And it is a fact that Kepler and Newton have made the study of Ptolemaic and Babylonian astronomy superfluous. But at what loss? An astronomer of today is hardly even aware of what Professor Neugebauer has repeatedly emphasized: "Astronomy is the only branch of the ancient sciences which has a continuous recorded history, from antiquity, through the collapse of the Roman Empire, through the Renaissance, and to the present .... And further that astronomy is the most important force in the development of science since its origin some time around 500 B.C. to the days of Laplace, Lagrange, and Gauss."

And the loss is even greater: there are several aspects of current astronomical problems which have overtones in the astronomy of antiquity; more than overtones, indeed parallels. Let two examples suffice.

The basic problem that was undertaken by the scribes of Babylon consisted in predicting the positions of the moon and the planets over long periods of time with a precision beyond those of the individual observations subject to fluctuations and to gross errors. All these phenomena have periodic character with complicated fluctuations superposed.

The basic problem confronted by the Babylonian astronomers is one that is familiar to an astronomer of the present: it is to unravel a complicated periodic phenomenon as the superposition of a number of periodic effects. The method (as explained in greater detail below) probably originated in the theory of the moon. The times of the new moons could easily be found if the sun and the moon each moved with a constant velocity. One supposes that mean values arranged in periodic cycles, e.g. 19 years contain 235 months, may be used to describe this ideal movement. The actual movement will deviate from this mean in a periodic manner. These deviations may now be treated as a new periodic phenomenon–in Babylonian astronomy these deviations were represented by zigzag or step functions. Thus starting with mean values, one calculated the secondary corrections required to describe the periodic deviations. Continuing in this manner, the Babylonians were led to a very close description of the actual facts. Described in this manner, there is no difference between the way a Babylonian scribe sought to predict the time at which and duration for which the crescent of the moon would first be sighted after conjunction and the way in which a radio astronomer of the present determines the periods of pulsars correct to nine decimals when successive pulses are subject to large fluctuations both in intensity and in form. The recognition of this fact is a sobering thought.

Consider again the basic theorem of Apollonius to which we referred earlier. The main importance of that theorem lies in the fact that it freed astronomy from Eudoxus' concept of geocentric spheres and allowed for variable geocentric distance and for centers of rotation outside the earth. The freedom which one thus obtained for interchanging eccenters and epicycles implies an equivalence of all four vertices of the corresponding parallelograms as centers of rotation. It is this freedom which provides the basic kinematic equivalence of geocentric and heliocentric motions; a freedom which Copernicus was to use extensively some 1750 years later. This transformation of Apollonius underlies modern galactic astronomy in which orbits of the individual stars are described as epicycles about chosen local frames describing circular orbits; and indeed there is no difficulty in generalizing the transformation of Apollonius to obtain the laws of star streaming.

To turn to another aspect of a modern astronomer's reaction to the astronomy of antiquity, namely, astrology. The fact that astrology seems to represent the principal application of the elaborate analysis of the Ptolemaic system, enshrouds ancient astronomy with a superstitious element which repels him. But this is an altogether irrational reaction. As Professor Neugebauer has emphasized on several occasions:

> We should not forget that we must evaluate such doctrines against the contemporary background. To Greek philosophers and astronomers, the universe was a well defined structure of directly related bodies. The concept of predictable influence between these bodies is in principle not all different from any modern mechanistic theory. And it stands in sharpest constrast to the ideas of either arbitrary rulership of deities or of the possibility of influencing events by magical operations. Compared with the background of religion, magic, and mysticism, the fundamental doctrines of astrology are pure science. Of course, the boundaries between rational science and loose speculation were rapidly obliterated and astrological lore did not stem–but rather promoted–superstition and magical practices. The ease of such a transformation from science to humbug is not difficult to exemplify in our modern world.

There is one other aspect of the science of antiquity that has intrigued me. We often think of science as a part of culture; that in its pure form the pursuit and practice of science, like the pursuit and practice of any of the arts, have elements that elevate the human mind; and that this common feature in all creative activity results from a search for beauty. One can find many expressions of this thought scattered through the scientific literature. Professor Neugebauer often quotes from Hilbert; let this further quotation from Hilbert suffice.

> Our science, which we loved above everything, had brought us together. It appeared to us as a flowering garden. In this garden there were well-worn paths where one might look around at leisure and enjoy oneself without effort, especially at the side of a congenial companion. But we also like to seek out hidden trails and discovered many an unexpected view which was pleasing to our eyes; and when the one pointed it out to the other, and we admired it together, our joy was complete.

Similar expressions of appreciation for the beauty of science can be found in antiquity, and one epigram on this subject contained in manuscripts of the *Almagest* and in the *Greek Anthology* (IX, 577) is attributed to Ptolemy.

> I know that I am mortal and the creature of a day,
> But when I contemplate the intricate circling spirals of the stars,
> No longer do my feet touch earth, but beside Zeus himself
> I take my fill of the immortal nectar of the gods.

Let me conclude by expressing my own reaction to the astronomy of antiquity in the following way. Some twenty-five years ago, I met a colleague of mine emerging from the office of Enrico Fermi. He told me that he had been discussing physics with Fermi; and after a moment's pause asked, "Why am I doing physics? I should probably be a grocer". If Apollonius or Ptolemy had offices adjoining mine, I do not doubt that I should have a similar feeling emerging from their offices.

S. CHANDRASEKHAR

A History of Ancient Mathematical Astronomy is at once the most comprehensive and detailed history of ancient astronomy undertaken. It is of vast scope. From Meton of Athens in the fifth century B.C. and the unnamed scribes of Babylon, through Hipparchus and Ptolemy, to the shadowy figures of Olympiodorus and Stephanus in the early period of the Byzantine Empire, from primitive shadow tables and calendars of star phases, through Babylonian ephemerides and the *Almagest*, to the odd fragments preserved in late astrologers, the entire panorama of astronomy is set forth. Professor Neugebauer brings to this work the abilities of a mathematician, philologist, and historian. Perhaps the most striking application of these skills is in the reconstruction of complex mathematical procedures, arithmetical or geometrical, from fragmentary and sometimes distorted evidence. Such reconstructions are the foundation of our understanding of Babylonian astronomy and, if we except only the work of Ptolemy, which can (almost) speak for itself, of most Greek astronomy. The source material is by and large broken pieces of clay tablets, scraps of papyrus, scattered remarks and quotations by writers of varying degrees of (in-) competence. It is the task of the historian to reconstruct from this evidence a rigorous technical analysis and, where possible, history of ancient astronomy. From the fundamental structure to the smallest detail everything must be subject to the most careful analysis, and frequently the former can only be discovered after the latter has been understood. All this Professor Neugebauer has accomplished in a way little short of miraculous.

As is well known, the methods of Babylonian and Greek mathematical astronomy are quite different, the former using periodic arithmetical functions for position, velocity, and time, the latter first deriving an appropriate geometrical model with suitable parameters, and then using the model itself for trigonometric computation. But the differences run still deeper. The very information that one wishes to learn from mathematical astronomy, which to a great extent determines the invention and application of techniques, is not the same. The principal goal of Babylonian astronomy is to answer the question at what *time* will a certain characteristic phenomenon–first or last visibility of the moon or a planet, full moon or acronychal rising of a planet, planetary stations, lunar eclipses–take place. The reason, as we know from texts antedating by hundreds of years the development of a mathematical astronomy capable of predicting such occurrences, is that all these events have value as omens or, in the case of first visibility of the moon, are of calendarical significance. Greek astronomy, at least as we have it in the work of Ptolemy, is most frequently concerned with determining for a given time what *position* a certain heavenly body will have. The one important exception to this is the determination of when eclipses of the sun and moon will take place. In the *Almagest*, phenomena such as stations or first and last appearances are treated in a way that allows one to discover whether at a given time a planet will be direct, stationary or retrograde, visible or invisible, but not, without considerable difficulty, to predict when a planet will reach a station or first become visible after conjunction with the sun. Largely for chronological reasons, it is not clear whether horoscopic astrology, which is concerned primarily with position and only secondarily with characteristic phenomena, provided the motivation for this development or was itself the application of techniques invented for purely astronomical considerations.

A Greek mathematical astronomy capable of computing positions from a geometrical model seems to appear at about the time of Hipparchus in the middle of the second century B.C. Hipparchus, and perhaps some of his contemporaries (of whom he seems not to have thought very highly), employed geometrical models possibly invented but certainly worked out in detail by Apollonius perhaps half a century earlier. Prior to this formative period between Apollonius and Hipparchus, Greek mathematical astronomy was primitive by comparison with contemporary Babylonian achievements. What little we know of it, mostly calendarical cycles and fixed star phases used for weather prognostication, suggests the same interest in the determination of time rather than position characteristic of Babylonian astronomy, although on a far lower level of technical proficiency. Eudoxus's purely qualitative spherical models, which could not be used to predict anything (indeed, it is not clear whether a Greek of the fourth century B.C. would care to know the position of a planet), Aristarchus's attempt to determine the distances of the sun and moon as well as his curious notion about the motion of the earth, and the strange planetary distances attributed to Archimedes, seem an exception to this rule, apparently motivated more by cosmological considerations about the structure of the heavens than by the more practical applications of mathematical astronomy.

What made the floursihing of Greek astronomy possible in the time of Hipparchus was not only the geometrical models of Apollonius, but also the transmission of a substantial body of Babylonian observational records, numerical parameters, and computational procedures. Although the last was gradually, but not entirely, replaced by trigonometric computation from geometrical models, the use of arithmetical constant-difference tabulations forming linear zigzag functions can still be seen in later Indian, Arabic, and Byzantine astronomy, all of which are descendents of a Hellenistic science built upon a foundation carried over from Babylon. In the most sophisticated development of Greek astronomy, represented first and foremost by Ptolemy, Babylonian procedures have all but disappeared, although the use of Babylonian observations and parameters is still prominent. Ptolemy's methods are so different from those of Babylonian astronomy that it is difficult to compare the procedures for finding some one thing, say the longitude of the moon or the first visibility of Venus, without entering into a detailed analysis of both astronomical systems. This, of course is what Neugebauer's history does. Never before has so thorough an exposition or so penetrating an analysis of ancient astronomy (or any science?) been brought together.

Not the least part of this accomplishment is the fact that most of the contents of these three volumes represent the original discoveries and analyses of the author himself in the course of some forty years of research. The treatment of Babylonian astronomy is based, after the earlier work of Epping and Kugler, upon Neugebauer's fundamental publications from 1936–1938, *Astronomical Cuneiform Texts* (ACT) [1955], containing full publication of all sources known at that time, and a series of later articles by himself, A. Sachs, A. Aaboe, and B. L. van der Waerden, to name the principal investigators of this subject. The exposition of Greek astronomy, by far the most comprehensive written, comes largely from notes taken over a period of many years, very little of which has been published previously in any form. It goes without saying that until the publication of these volumes it has been next to impossible for a reader to learn as much about ancient astronomy on his own as he now can through a careful study of Neugebauer's history.

Perhaps the best way to get some understanding of the content and analysis of Neugebauer's work is, drawing from different parts of the history, to examine the Babylonian and Greek methods of finding the same thing. We choose for examples the true velocity of the moon and the length of the synodic month, largely because these are especially suitable to Babylonian arithmetical functions, but are rather more difficult to determine with the (essentially modern) computation from a geometrical model used by Ptolemy.

The purpose of Babylonian lunar ephemerides is the prediction of the time of first visibility of the new moon after conjunction with the sun, for the evening of first appearance marks the beginning of the month in the civil calendar. This is in fact difficult to determine, requiring knowledge of solar velocity and longitude, lunar velocity, longitude, and latitude, and conditions specific to the location of Babylon, that is, oblique ascension and length of daylight. A complete lunar ephemeris can contain as many as eighteen columns each tabulating a different arithmetical function necessary for solving the apparently simple problem of whether a month will be 29 or 30 days long. Parallax is the only pertinent factor not specifically considered in the ephemerides, although the parameters chosen for lunar latitude (unknowingly) include, in a rough way, the effect of parallax near the horizon where the first crescent is visible. The ephemerides fall into two classes, called System A and System B, characterized initially by the method of tabulating solar velocity and longitude, and generally by different parameters in all columns. System A is on the whole the more sophisticated of the two, but we shall concern ourselves here with System B since, perhaps by the fortuitous chances of transmission, its parameters have had the greater influence on the history of astronomy.

TABLE 1

T [restored] Date Year-Month		F Lunar Velocity in °/d	G Length of Synodic Month in ° over 29d
	X	[13; 36, 10]	[3, 9; 1, 40]
	XI	[14; 12, 10]	[2, 46; 31, 40]
	XII	[14; 48, 10]	[2,]24; 1, 40
3,6	I	[15;]8	2, 1; 31, 40
	II	[14;]32	2, 6; 7, 30
	III	[13; 5]6	2, 28; 37, 30
	IV	[13; 2]0	2, 5[1]; 7, 30
	V	[12; 44]	3, [13; 3]7, 30
	VI	[12; 8]	[3,]36; 7, 30
	VII	[11; 32]	[3, 5]8; 37, 30
	VIII	[11; 14, 10]	4, 21; 7, 30
	IX	[11; 50, 10]	[4,]15; 16, 4[0]
	X	[12; 26, 10]	[3, 52; 46, 40]
	XI	[13; 2, 10]	[3, 30; 16, 40]
	XII	[13; 38, 10]	[3, 7; 46, 40]
	XII_2	[14; 14, 10]	[2, 45; 16, 40]
3,7	I	[14; 50,]10	2, 22; 46, 40
	II	[15;]6	2, 0; 16, 40
	III	[14;]30	2, 7; 22, 30
	IV	[13;]54	2, 29; 52, 30
	V	[13;]18	2, 52; 22, 30

Table 1 shows a part of three columns from a System B ephemeris (ACT 121a). The transcription here is highly edited. Numbers in square brackets, which are broken off or illegible in the original tablet, have been restored, sexagesimal places have been separated by commas, a semicolon separating integers from fractions, and headings added giving the significance of each column. Column T is the date in years of the eastern form of the Seleucid Era (−310 Apr. 3) and months of the civil calendar indicated by Roman numerals. Column F gives the lunar velocity in degrees per day for the time of mean conjunction, and Column G the length of the synodic month over 29 days in degrees of time where one day equals 360°. Thus, sexagesimally, where $1^d = 6, 0° = 24^h$, so $1, 0° = 4^h$ and $1^h = 15°$.

The structure of columns F and G can best be understood by graphing them as is done in Figure 1. Except at maximum and minimum, each succeeding entry has a constant difference d such that entry $(n + 1) = n + d$ where d is taken as positive on an ascending branch and negative on a descending branch. When lines are drawn through a number of entries, they form alternate ascending and descending branches of what is called a linear zigzag function, intersecting at fixed maxima and minima which are probably not among the tabulated entries. When two entries are adjacent to a boundary, entry $(n + 1)$ will fall as far inside the boundary as $n + d$ would

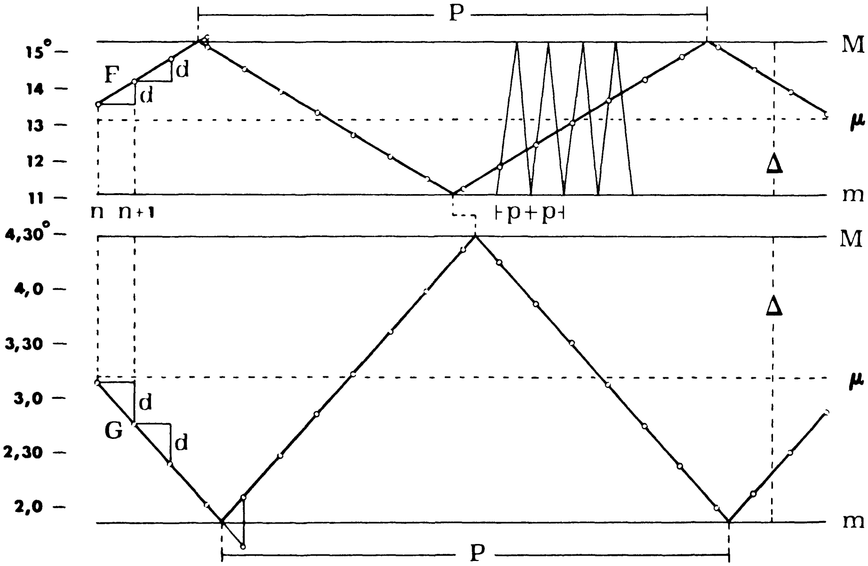

FIGURE 1

fall outside it, and thus

$$n + (n+1) = 2M - d \quad \text{at maximum,}$$
$$n + (n+1) = 2m + d \quad \text{at minimum,}$$

where M and m are the maximum and minimum values respectively. From the tabulated entries adjacent to the boundaries, M and m may be found from

$$M = \tfrac{1}{2}\left[n + (n+1) + d\right], \quad m = \tfrac{1}{2}\left[n + (n+1) - d\right].$$

We call $\mu = \frac{1}{2}(M + m)$ the mean value and $\Delta = M - m$ the amplitude of the function. Let the period P be defined as the number of steps (generally not an integer) of d in one oscillation of the zigzag function. Then

$$P = \frac{2\Delta}{d} = \frac{\Pi}{Z}$$

where Π and Z are the least integers representing an integral number of steps Π, called the number period, in an integral number of oscillations Z, called the wave number. Since $\Pi = PZ$, after Z periods containing Π entries, the series of entries will repeat. All of these parameters may be derived from the columns in Table 1, and we give them below:

	$F^{\circ/d}$	G°
d	0; 36, 0	22; 30
m	11; 5, 5	1, 52; 34, 35
M	15; 16, 5	4, 29; 27, 5
μ	13; 10, 35	3, 11; 0, 50
Δ	4; 11, 0	2, 36; 52, 30
$\frac{\Pi}{Z} = P$	$\frac{4,11}{18}$ = 13; 56, 40	$\frac{4,11}{18}$ = 13; 56, 40

Note that the periods of the two functions are identical while their ascending and descending branches on the graph have opposite phases with a displacement of about one-half an entry, that is, one-half a mean synodic month between corresponding opposite extrema of F and G. From the identical periods we conclude that Column G gives the length of the synodic month as a function of F, the lunar velocity at mean conjunction, that is, after the completion of the anomalistic month that is less than and contained by the synodic month falling between conjunctions $n - 1$ and n. Indeed, both the differences d and the amplitude Δ are related by

$$(d, \Delta)_G = 37, 30(d, \Delta)_F.$$

Thus, each increment of $0; 36^{\circ/d}$ in the lunar velocity will produce a change of $37, 30 \cdot 0; 36^\circ = 22; 30^\circ$ in the length of the synodic month. Paired functions of this kind are more common in System A, this being the only example in System B. The opposite phases of F and G follow from the obvious consideration that the length of the synodic month varies inversely with the lunar velocity in the interval after the completion of the anomalistic month. The displacement of the maxima of G behind the minima of F, which should be exactly one-half a synodic month but is a little over that, is because the entry in G for month n gives the length of month $n - 1$, but to clarify this we must consider the true function underlying F.

The period of the lunar velocity is the anomalistic month which is less than the synodic month by about 2 days. Thus, F is in fact the tabulation function of a true function with a period p equal to the anomalistic month and less than the synodic month. For F to have its greatest effect on G, either positive or negative, the opposite extreme of the true function should occur in the middle of the interval between conjunctions $n - 1$ and n. Hence the extrema of F occurring about half a month before the opposite extrema of G. Since the period p of the anomalistic month is only slightly less than the line interval 1 of the synodic month, thus

$$\frac{p}{1} = \frac{P}{P + 1} = \frac{2\Delta}{2\Delta + d}.$$

In this way, we find for column F,

$$p = \frac{4, 11}{4, 11 + 18} = \frac{4, 11}{4, 29} = 0; 55, 59, 6 \ldots.$$

Since the line difference 1 represents the mean synodic month, this tells us that

$$251 \text{ syn. mo.} = 269 \text{ anom. mo.},$$

a relation that is of some interest in that it was later used by Hipparchus and is noticed, and slightly corrected, by Ptolemy. The mean synodic month of Column G,

$$29^d\, 3,11;0,50^\circ = 29;31,50,8,20^d = 29^d\, 12;44,3,20^h$$

was used without modification by Hipparchus and Ptolemy, is the mean length of the month in the Jewish calendar, and is found frequently in medieval astronomical works. The mean lunar velocity of column F, 13; 10, 35°/d, a fairly crude parameter, is later found in Greek, Indian, and medieval sources.

Once the parameters of these columns have been derived, computing an ephemeris for any number of months or years is a simple matter. Given an initial value n, one forms the sum $(n + 1) = n + d$, the sign of d depending upon whether one is on an ascending or descending branch of the zigzag. Whenever a boundary M or m would be passed, one computes

$$(n + 1) = (2M - d) - n \quad \text{at maximum,}$$

$$(n + 1) = (2m + d) - n \quad \text{at minimum.}$$

In this way the lunar velocity at the time of mean conjunction and the corresponding length of the synodic month can easily be found. But thus far we have considered only the effect of lunar velocity on the month while the length of the synodic month also depends (directly this time) upon the velocity of the sun. In order to account for the effect of solar velocity, there are two further columns, H and J. H is a linear zigzag function with a period of about one-half a year that is used to form the differences of J, which is thus a function of constant second differences having a period of one year. The extrema for J are

$$M = -m = 32;\ 28, 6^\circ.$$

The values of J must be added to G, column J indicating whether an entry is positive or negative, in order to form Column K, that is,

$$\mathrm{K} = \mathrm{G} + \mathrm{J},$$

and K gives the final corrected length of the synodic month. The extrema for K are thus

$$\mathrm{K}_{\max} = \mathrm{G}_{\max} + \mathrm{J}_{\max} = 4, 29;\ 27, 5^\circ + 32;\ 28, 6^\circ = 5, 1;\ 55, 11^\circ$$

and

$$\mathrm{K}_{\min} = \mathrm{G}_{\min} + \mathrm{J}_{\min} = 1, 52;\ 34, 35^\circ - 32;\ 28, 6^\circ = 1, 20;\ 6, 29^\circ.$$

The ease of computation shows the great advantage of purely arithmetical procedures since finding these quantities from a geometrical model is considerably more complicated.

This can be seen by examining Ptolemy's methods for solving related problems that occur in Book VI of the *Almagest*, which is on the computation of eclipses. Ptolemy never specifically considers either the true length of the synodic month or the first visibility of the new moon, probably because he does not use a lunar calendar. However, in computing eclipses it is necessary to find the true velocity of the moon and sun in order to determine the time and longitude of true syzygy. Further, in deriving the intervals of ecliptic syzygies, those at which an eclipse is possible, Ptolemy sets out procedures sufficient to find the true length of the synodic month if one wishes to do so.

Ptolemy employs two lunar models, a simpler one developed in Book IV that is accurate only near syzygy and a more complicated model in Book V that is usable at any elongation of the moon from the sun. Since Book VI is concerned with eclipses, which are of course restricted to syzygies, Ptolemy reverts to the simple model of Book IV. This model can be represented in two forms, eccentric or epicyclic, and here we shall use the epicyclic form which is

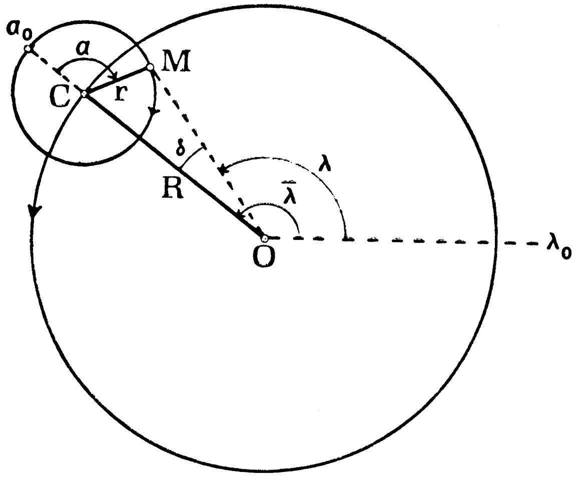

FIGURE 2

shown in Figure 2. We let the center of the earth be O and draw a circle of radius $R = OC$. The center C of the epicycle has moved through the mean motion in longitude $\bar{\lambda}$ from λ_0 at the rate $\bar{v} \approx 13; 10, 35^{\circ/d}$. Describe the epicycle of radius $CM = r = 5; 15$ where $R = 1, 0$, and let the moon at M move through the mean anomaly α measured from the apogee α_0 in the direction opposite to the motion of C about O at the rate $\bar{\alpha} \approx 13; 3, 54^{\circ/d}$. The equation of the anomaly δ is the difference between the mean longitude $\bar{\lambda}$ and the true longitude λ, that is $\lambda = \bar{\lambda} \pm \delta$. δ can be found trigonometrically, and is tabulated directly by Ptolemy.

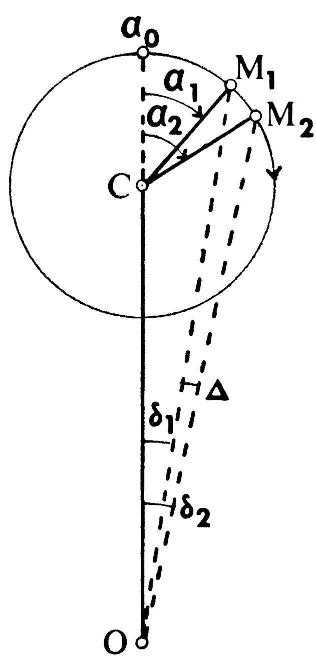

FIGURE 3

In *Almagest* VI, 4 Ptolemy explains how to use this model to find the true velocity of the moon in degrees per hour. Figure 3 shows the moon at two positions of anomaly α_1 and α_2 with the corrections δ_1 and δ_2. In the interval between α_1 and α_2 the lunar equation will change by

$$\Delta = \delta_{2(\alpha_2)} - \delta_{1(\alpha_1)}.$$

Let $\alpha_2 = \alpha_1 + 1°$; then

$$\Delta = \delta_{(\alpha_1 + 1°)} - \delta_{(\alpha_1)}.$$

Since $\bar{\alpha} \approx 0;\ 32,\ 40^{°/\mathrm{h}}$, the hourly increment in Δ will be

$$\Delta^{°/\mathrm{h}} = 0;32{,}40\ \Delta.$$

Since the mean motion $\bar{v} \approx 0;\ 32,\ 56^{°/\mathrm{h}}$, the true velocity

$$v = \bar{v} \pm \Delta = 0;\ 32,\ 56^{°/\mathrm{h}} \pm \Delta^{°/\mathrm{h}}.$$

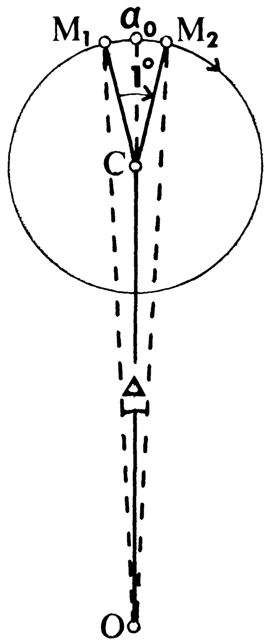

FIGURE 4

We shall solve the problem numerically for minimum and maximum velocity which, as is clear from Figure 2, occur respectively at α_0 and α_{180}. The configuration for α_0 is shown in Figure 4. To find $v_{\min}$ we interpolate in Ptolemy's correction table to find δ for $\alpha_0 \pm 0;\ 30$, and find

$$\Delta = \delta_2 - \delta_1 = -0;\ 2,\ 30° - (+0;\ 2,\ 30°) = -0;\ 5°.$$

Thus

$$\Delta^{°/\mathrm{h}} = 0;\ 32,\ 40\ (-0;\ 5°) = -0;\ 2,\ 43,\ 20^{°/\mathrm{h}},$$

and

$$v_{\min} = 0;\ 32,\ 56^{°/\mathrm{h}} - 0;\ 2,\ 43,\ 20^{°/\mathrm{h}} = 0;\ 30,\ 12,\ 40^{°/\mathrm{h}}.$$

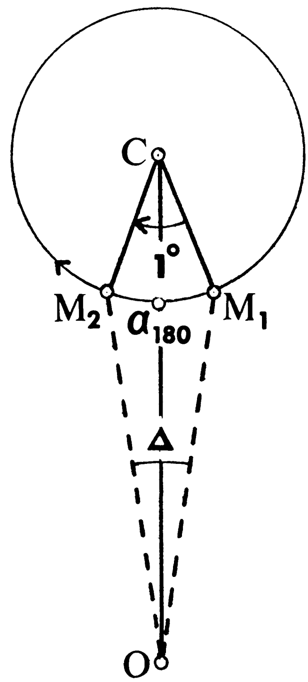

FIGURE 5

The maximum velocity, at α_{180}, is shown in Figure 5. Again finding δ by interpolation for $\alpha_{180} \pm 0; 30°$, we have

$$\Delta = \delta_2 - \delta_1 = +0;\ 3° - (-0;\ 3°) = +0;\ 6°.$$

Now

$$\Delta^{°/h} = 0;\ 32,\ 40\ (+0;\ 6°) = 0;\ 3,\ 16^{°/h},$$

and thus

$$v_{max} = 0;\ 32,\ 56^{°/h} + 0;\ 3,\ 16^{°/h} = 0;\ 36,\ 12^{°/h}.$$

Dividing the corresponding values from column F by 24, we can make the following comparison to two places:

Velocity	Column F	Ptolemy
v_{min}	0; 27, 42°/h	0; 30, 13°/h
$\bar{v}$	0; 32, 56	0; 32, 56
v_{max}	0; 38, 10	0; 36, 12

The zigzag function, as is often the case, has a greater amplitude than the sinusoidal curve of the trigonometrically derived values following from Ptolemy's procedures and parameters.

Ptolemy never specifically takes up the problem of finding the length of the synodic month as a function of lunar and solar velocity, but he does solve a somewhat more complex problem in *Almagest* VI, 6 in finding the number of synodic months that must elapse between eclipses. His method allows the determination of the maximum and minimum synodic months as we show here:

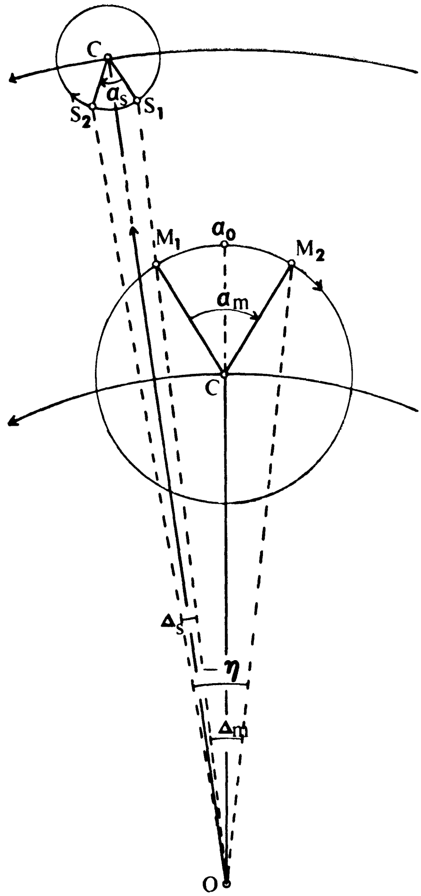

FIGURE 6

The configuration for the longest month is shown in Figure 6. Since the synodic month exceeds the anomalistic month, we let α_0 bisect the arc α_m of the lunar anomaly during the excess, and in this way the moon's true velocity during the excess will be the slowest possible. Thus, if we have a true conjunction at M_1, then one mean synodic month later the moon will be at M_2 and will not yet have reached true conjunction. In the same way, we assume that the sun is located near perigee such that α_{180} bisects the arc α_s of its anomaly during a mean synodic month so that the sun's motion will be as fast as possible. Using Ptolemy's tables in VI, 3, we find that $\alpha_m = 25; 49°$ and $\alpha_s = 29; 6°$. Taking half of each, we have $\frac{1}{2}\alpha_m \approx 12; 55°$ and $\frac{1}{2}\alpha_s = 14; 33°$ on either side of apogee and perigee respectively. From the equation tables in Books III and IV it follows that

$$\Delta_m = \delta_2 - \delta_1 = -1; 2° - (+1; 2°) = -2; 4°$$

and

$$\Delta_s = \delta_2 - \delta_1 = +0; 38° - (-0; 38°) = +1; 16°.$$

Therefore

$$\eta = \Delta_m - \Delta_s = -2; 4° - (+1; 16°) = -3; 20°$$

is the elongation of the moon from the sun at the end of one mean synodic month. Now, says Ptolemy, assuming for this interval a constant solar and lunar velocity, we must add $1/12\eta$ to η to account for the additional motion of the sun until the moon overtakes it at true conjunction. This approximation follows from observing that the velocity of the sun is about $1/13$ the velocity of the moon, so that for a given η the moon will reach the sun after moving

$$\eta' = \eta + 1/13\eta + (1/13)^2\eta + \ldots = \eta + 1/12\eta.$$

In this case true conjunction will occur when the moon has moved $\eta' = 1; 5\eta = 3; 36, 40°$. Ignoring the change in anomaly while the moon crosses this interval, we use the moon's true velocity at M_2 which we compute by the previous method to be

$$v = 0; 32, 56^{\circ/\mathrm{h}} - 0; 2; 32^{\circ/\mathrm{h}} = 0; 30, 24^{\circ/\mathrm{h}},$$

and so

$$\frac{\eta'}{v} = \frac{3; 36, 40°}{0; 30, 24^{\circ/\mathrm{h}}} = 7; 7, 37, 53 \ldots {}^{\mathrm{h}}$$

is the excess of the longest month over the mean synodic month. Of course, the various approximations used in its derivation make any places beyond the minutes meaningless, and even the minutes are suspect.

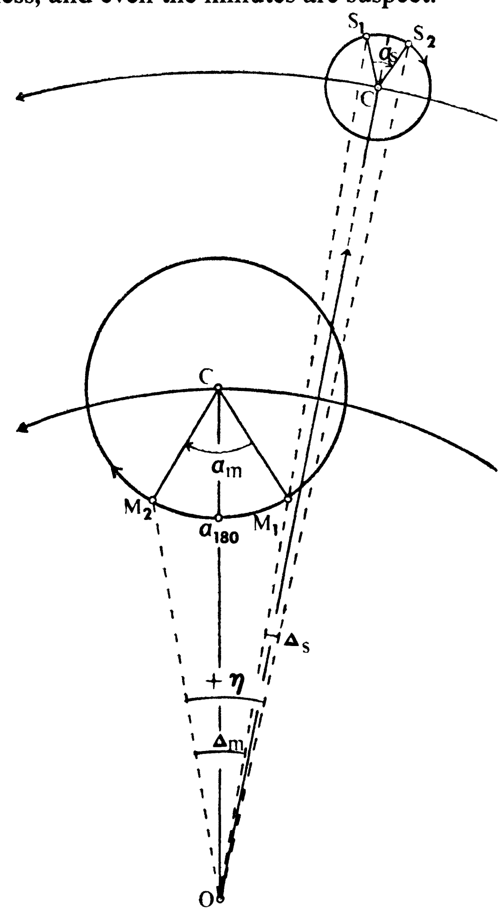

FIGURE 7

The configuration for the shortest month is shown in Figure 7. This time α_{180} bisects the lunar anomaly so that the moon's motion will be as fast as possible and α_0 bisects the solar anomaly so the sun will move at its slowest velocity. Now at one mean synodic month after a true conjunction at M_1 the moon will have reached M_2, having already passed true conjunction. Using the previous values of α, we now have at $\alpha_m = \alpha_{180} \pm 12; 55°$

$$\Delta_m = \delta_2 - \delta_1 = +1; 14° - (-1; 14°) = +2; 28°$$

and at $\alpha_s = \alpha_0 \pm 14; 33°$

$$\Delta_s = \delta_2 - \delta_1 = -0; 34° - (+0; 34°) = -1; 8°.$$

Thus

$$\eta = \Delta_m - \Delta_s = 2; 28° - (-1; 8°) = 3; 36°$$

is the elongation at the end of one mean synodic month. True conjunction will have been passed by $\eta' = 1; 5\eta = 3; 54°$. Since the true lunar velocity at M_2 is

$$v = 0; 32, 56^{°/h} + 0; 2, 54^{°/h} = 0; 35, 50^{°/h},$$

therefore

$$\frac{\eta'}{v} = \frac{3; 54°}{0; 35, 50^{°/h}} = 6; 31, 48, 50 \ldots {}^{h}$$

is the deficit of the shortest month from the mean synodic month.

Finally, we add the excess of the longest month and the deficit of the shortest month to the mean synodic month, and tabulate the values along with those of Column K converted to standard hours where $1° = 0; 4^h$.

Month $29^d +$	Column K	Ptolemy
m	5; 20, 25, 56^h	6; 12, 14, 3^h
μ	12; 44, 3, 20	12; 44, 3, 20
M	20; 7, 40, 44	19; 51, 41, 13

Again the amplitude of the zigzag function is greater. However, M and m are extrema that can seldom occur, so in general the differences between the two computations will be far smaller. It should be noted especially that using the *Almagest* procedures to find the length of a particular synodic month is always as complicated as these examples (although ways to shorten the computation for purposes of tabulation could be devised) while the Babylonian ephemerides require only the addition of a few numbers. Clearly the latter are more suitable to practical use.

The examples given here are, of course, but the smallest fraction of the subjects treated in Neugebauer's history. We hope nevertheless through this fairly technical exposition to have given some idea of the weeks and months of discovery that await the readers who, in the author's words, "are willing to penetrate the jungle of technical details and become fascinated by the kaleidoscopic picture which I have tried to unfold here of the history of the first and oldest natural science".

One can only hope that a future historian will be able to accomplish as much when the astronomy of the twentieth century has itself been reduced to a few odd books and some handfuls of fragments.

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Examples of groups, by Michael Weinstein, Polygonal Publishing House, Passaic, N.J., 1977, 307 pp.

"Why study examples?" asks the author as he opens his preface to this curious volume. Why, indeed. This is a question which I think many of today's graduate students and more than a few of their instructors could ponder profitably. The author gives us three reasons:

(1) to motivate new theorems,
(2) to illustrate and clarify old theorems,
(3) to obtain counterexamples.

While all of this is well put and certainly true, it seems to me that the main reason for studying examples is simply that we can't do without them. What, after all, is a theorem if it is not a simultaneous assertion about some properties of a large class of examples? What better way to understand what a theorem says than to apply it to some concrete examples? Everyone appreciates the power and desirability of generalization. Studying examples is just the reverse process of going from the general to the specific. Mathematics without examples would become the uninteresting exercise in formal deduction which it is sometimes mistaken for.

Unfortunately, the study of examples is seldom given the status which it deserves, particularly in some modern texts, and the present book is an admirable attempt to rectify this situation, as it pertains to the theory of discrete groups. How well does it succeed?

The author presents us with a rather long list of specific discrete groups; finite and infinite, abelian and nonabelian. In each case, a number of properties are obtained. For example, turning (at random) to p. 194, we see: "Result 5.11.5. G is not an M_1 group. Result 5.11.8. G is Hopfian." There is also a section of comments ("notes") following each example (e.g. "G shows that the class of cohopfian groups is not quotient closed") and a number of exercises. The first example appears on p. 101 and is preceded by an entire section devoted to some abstract construction techniques (e.g. direct, central, semidirect, and wreathed products) and some elementary facts about free groups and matrix groups. Additional elementary results appear in a series of ten appendices. All arguments are given in a very careful and complete manner, but the price we pay for this is a rather pedantic and heavy style:

"If k is a natural number such that $2 < k$, then 2 and k are distinct divisors of $k!$, and hence $2k \leqslant k!$. Also $2 < k$ implies $1 \leqslant k$ so $1 + k \leqslant 2k$. Thus $1 + k \leqslant k!$ for all k such that $2 < k$."