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FACTORIZATION INDICES FOR MATRIX POLYNOMIALS

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Let Γ be a rectifiable simple closed contour in the complex plane $\mathbf{C}$ bounding the domain F^+. The notation F^- will be used for the complement to $F^+ \cup \Gamma$ in $\mathbf{C} \cup \infty$. It will always be assumed that $\lambda = \infty \in F^-$.

A polynomial $L(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ with $n \times n$ matrix coefficients A_j and with $\det L(\lambda) \neq 0$ for $\lambda \in \Gamma$ is said to admit a (right standard) *factorization* relative to the contour Γ in case

$$L(\lambda) = L_+(\lambda)D(\lambda)L_-(\lambda), \tag{1}$$

where $L_+(\lambda)$ is a matrix polynomial with $\det L_+(\lambda) \neq 0$ for $\lambda \in F^+ \cup \Gamma$, $L_-(\lambda)$ is a matrix polynomial in the variable $(\lambda - a)^{-1}$ for some $a \in F^+$ and $\det L_-(\lambda) \neq 0$ for $\lambda \in F^-$, and

$$D(\lambda) = \operatorname{diag}((\lambda - a)^{\kappa_1}, (\lambda - a)^{\kappa_2}, \ldots, (\lambda - a)^{\kappa_n})$$

with some nonnegative integers $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$. Such a factorization of $L(\lambda)$ is not unique, but the numbers $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$ which are called the (*right*) *partial indices* are uniquely determined by $L(\lambda)$ (see [1], [5]). An analogous definition of left standard factorization is possible. Here we deal only with right factorization.

The factorization indices play an important role in the theory of systems of singular integral equations (see [1], [5]), Wiener-Hopf equations, partial differential equations, and the classification of holomorphic vector bundles on the Riemann sphere. There exists an algorithm for computing the indices which is described in [1].

In this paper we obtain some explicit formulas for the partial indices. As a main tool, we use the ideas of spectral analysis of matrix polynomials developed in [2], [3], [4].

To begin with, consider the linear case $L(\lambda) = A - \lambda I$. Let $A = \operatorname{diag}(A_1, A_2)$ be a block representation where the eigenvalues of A_1 are inside F^+ and the eigenvalues of A_2 are inside F^-. Then (assuming that $\lambda = 0$ is inside F^+)

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the equality

$$A - \lambda I = \operatorname{diag}(I, A_2 - \lambda I)\operatorname{diag}(\lambda I, I)\operatorname{diag}(\lambda^{-1}A_1 - I, I)$$

is a factorization of $A - \lambda I$ and the indices are $(0, 0, \ldots, \dot{0}, 1, 1, \ldots, 1)$ where the number of ones is exactly equal to rank $A_1 (=$ rank Im $\int_\Gamma (A - \lambda I)^{-1}\, d\lambda)$. For a matrix polynomial, this result can be generalized as follows.

THEOREM 1. *Let* $L(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ *be a matrix polynomial with* $\det L(\lambda) \neq 0$ *for* $\lambda \in \Gamma$. *Then for the factorization indices* $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$ *of* $L(\lambda)$ *the following equalities hold*

$$\kappa_i = |\{j | n + r_{j-1} - r_j \leqslant i - 1, j = 1, 2, \ldots, m\}| \quad (i = 1, 2, \ldots, n),$$

where

$$r_j = \operatorname{rank}\begin{pmatrix} B_{-1} & B_{-2} & \cdots & B_{-m} \\ B_{-2} & B_{-3} & \cdots & B_{-m-1} \\ \cdots & \cdots & \cdots & \cdots \\ B_{-j} & B_{-j-1} & \cdots & B_{-j-m-1} \end{pmatrix}, \qquad B_{-j} = \frac{1}{2\pi i}\int_\Gamma \lambda^{j-1}L^{-1}(\lambda)\, d\lambda$$

and $|\Omega|$ *denotes the number of elements in the set* Ω.

The same formulas for κ_j hold in case Γ is the unit circle and the B_j are the Fourier coefficients of $L^{-1}(\lambda)$:

$$L^{-1}(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j B_j \qquad (|\lambda| = 1).$$

THEOREM 2. *Suppose that all the eigenvalues of* $L(\lambda)$ *are inside* F^+ *and* $L(0) = I$. *Then*

$$\kappa_j = |\{i | n + q_{i-1} - q_i \leqslant j - 1, i = 1, 2, \ldots, n\}|$$

where

$$q_i = \operatorname{rank}\begin{pmatrix} PC^\nu \\ PC^{\nu+1} \\ \vdots \\ PC^{\nu+i-1} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & I \\ -A_m & -A_{m-1} & \cdots & -A_1 \end{pmatrix},$$

$\nu \geqslant 0$ *is the minimal integer for which* rank $C^\nu =$ rank $C^{\nu+1}$ *and* P *is the projector on the first* n *coordinates.*

If not all the eigenvalues of $L(\lambda)$ are inside F^+, then a similar result holds upon replacing C by $C\widetilde{P}$, where $\widetilde{P} = (2\pi i)^{-1}\int_\Gamma (\lambda I - C)^{-1}\, d\lambda$.

We now briefly sketch the main steps in the proofs.

At first without loss of generality in Theorem 1, we assume also that all the eigenvalues of $L(\lambda)$ are in F^+. This reduction is justified by some results from

[4]. Now assuming that all the eigenvalues of $L(\lambda)$ are in F^+, let J_F denote an $r \times r$ matrix in Jordan form, which has the same eigenvalues and the same elementary divisors as $L(\lambda)$. Let X_F denote an $n \times r$ matrix with the eigenvectors and generalized eigenvectors of $L(\lambda)$ as columns arranged in correspondence with J_F. These eigenvectors can be chosen in a canonical way such that [4]

$$\operatorname{rank} K_m = nm, \text{ where } K_j = \begin{pmatrix} X_F \\ X_F \; J_F \\ \vdots \\ X_F \; J_F^{j-1} \end{pmatrix} \qquad j = 1, 2, \ldots, m.$$

The next step contains the proof of the equalities $q_j = r_j = \operatorname{rank} K_j$ for $j = 1, 2, \ldots, m$.

The main step is the calculation of the indices κ_j via the numbers $\operatorname{rank} K_j$. It is based on an investigation of the subspaces $\operatorname{Ker} K_j^*$. Using the structure of these subspaces, we are able to construct a matrix polynomial $M(\lambda)$ with the same eigenvalues, eigenvectors and generalized eigenvectors as $L(\lambda)$ and with an additional important property: for some integers $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n$ the product $(\lambda^{\nu_j}\delta_{jk})M(\lambda)$ is a matrix polynomial with an invertible leading coefficient. The connection between the numbers ν_j and $\operatorname{rank} K_j$ allows us to complete the proof.

The full proofs together with generalization for matrix valued functions and other results on indices will appear elsewhere.

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