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THE NUMBER OF SOLUTIONS TO THE CLASSICAL PLATEAU PROBLEM IS GENERICALLY FINITE

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0. Introduction. The question of how many solutions there are to the classical problem of Plateau has been open for roughly a century. Existence of at least one solution was proved in 1931 independently by T. Rado [7] and J. Douglas [3]. Courant, in his book *Dirichlet's principle, conformal mappings and minimal surfaces* [2], outlines an argument which suggests that there may exist rectifiable curves in $\mathbf{R}^3$ bounding on uncountable number of solutions. It has been believed for some time that for all sufficiently nice curves there are only a finite number of surfaces of mean curvature zero which they bound. In this note we state that there exists an open dense set of curves which bound a finite number of classical minimal surfaces of the type of the two disc. This result essentially is a synthesis of the ideas of [1], [9], [10].

I. Formulation of results. Let $\alpha: S^1 \longrightarrow \mathbf{R}^n$ be a C^∞ embedding of S^1 into $\mathbf{R}^n$, where S^1 denotes the boundary of the open disc $\mathcal{D}$ in $\mathbf{R}^2$. Let $\Gamma^\alpha = \alpha(S^1)$ denote its image.

DEFINITION. A classical solution to Plateau's problem for α is a map u from $\overline{\mathcal{D}}$ into $\mathbf{R}^n$ satisfying the following properties.

(i) $u \in C^0(\overline{\mathcal{D}}) \cap C^\infty(\mathcal{D})$,

(ii) $\Delta u = 0$,

(iii) $\partial u/\partial x \cdot \partial u/\partial y = 0 \ \forall (x, y) \in \mathcal{D}$,

(iv) $\|\partial u/\partial x\| = \|\partial u/\partial y\| \ \forall (x, y) \in \mathcal{D}$,

(v) $u: S^1 \longrightarrow \Gamma^\alpha$ homeomorphically.

REMARK. 1. By well-known regularity results for minimal surfaces first proved by Hildebrandt [5], and then later improved by Nitsche [6], Heinz and Tomi [4], and Tomi [8], $\alpha \in C^\infty$ implies $u \in C^\infty(\overline{\mathcal{D}})$.

Let $\mathcal{A}$ denote the space of all C^∞ embeddings of S^1 into $\mathbf{R}^n$ with the C^∞ topology.

THEOREM 1. *There exists an open and dense subset $\mathcal{A}_0 \in \mathcal{A}$ such that for all $\alpha \in \mathcal{A}_0$ there are only finitely many classical solutions of the Plateau problem.*

REMARK 2. These finitely many surfaces are nondegenerate in the sense described in [1].

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REMARK 3. There is an H^k Sobolev space version of Theorem 1. As a direct corollary of this fact we obtain that for $\alpha \in A_0$ the set of solutions are differentiable functions of α. This immediately implies the stability of the number of solutions in A_0.

REMARK 4. The equations in the above definition are invariant under the action of the conformal group of $\mathcal{D}$. In Theorem 1 surfaces equivalent under the action of this group are identified.

DEFINITION. A branch point $p \in \overline{\mathcal{D}}$ of a minimal surface u is a point where u fails to be an immersion.

THEOREM 2. *If $\alpha \in A_0$ it bounds no minimal surface with branch points on the boundary, and if $n > 3$ it bounds no minimal surface with boundary or interior branch points.*

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