BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

TORSION IN THE HOMOLOGY OF H-SPACES

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Communicated by P. T. Church, February 22, 1977

The purpose of this note is to announce some consequences of lack of torsion in $H_*(\Omega X; \mathbf{Z})$ when (X, μ) is a 1-connected H-space of finite type. Using this hypothesis we can deduce certain restrictions on the occurrence of torsion in the ordinary homology of X as well as in its BP, MU, and K homology. Our motivation for this approach comes from finite H-space theory. Certain cases of our restrictions or of the absence of torsion in $H_*(\Omega X; \mathbf{Z})$ have been proven for finite H-spaces (see [6]) or, at least, for compact Lie groups (see [1], [3] and [7]). Our arguments tie these results together and, furthermore, show that the relations do not depend on the finiteness of the spaces involved.

For the rest of the paper let p be a fixed prime and Q_p the integers localized at p. Let $H_*(X) = H_*(X; \mathbf{Z}) \otimes_{\mathbf{Z}} Q_p$. Let (X, μ) be a 1-connected H-space of finite type such that $H_*(\Omega X)$ is torsion free.

THEOREM 1. *$H_*(X)$ has no higher p torsion.*

Now $BP_*(X)$ is a module over

$$\Lambda = BP_*(pt) = Q_p[v_1, v_2, \ldots] \quad (\deg v_s = 2p^s - 2).$$

Thus, besides p torsion, we can also speak of v_s torsion for $s \geqslant 1$. However, the various torsion submodules are interrelated. In particular they are all contained in the v_1 torsion submodule. For let $\Lambda(1) = \Lambda(1/v_1)$ and $BP_*(X; \Lambda(1)) = BP_*(X) \otimes_\Lambda \Lambda(1)$.

THEOREM 2. *$BP_*(X; \Lambda(1))$ is torsion free.*

We can also deduce results about the algebra structure of $BP_*(X; \Lambda(1))$. Let P and Q denote primitives and indecomposables respectively.

THEOREM 3. *$BP_*(X; \Lambda(1))$ is commutative (associative) if, and only if, $H_*(X) \otimes_{\mathbf{Z}} Q$ is commutative (associative). When $H^*(X) \otimes_{\mathbf{Z}} Q$ is an exterior algebra then $BP_*(X; \Lambda(1))$ is generated as an algebra by the image of the delooping map $\Omega_*: Q(BP_*(\Omega X; \Lambda(1))) \longrightarrow P(BP_*(X; \Lambda(1)))$.*

Furthermore, it is necessary to localize with respect to v_1 to obtain these types of results.

AMS (MOS) subject classifications (1970). Primary 57F25; Secondary 55B15, 55B20.
Key words and phrases. H-space, torsion, BP homology, bordism, K-theory.
[1] Supported in part by NRC Grants A7357 and A3026.

THEOREM 4. *$BP_*(X)$ is v_1 torsion free if, and only if, $H_*(X)$ is torsion free.*

Granted the above then the results for bordism and K-theory follow easily. The canonical inclusion $\Lambda \longrightarrow \Omega = MU_*(pt) \otimes_Z Q_p$ (see [8]) enables one to define $\Omega(1)$ and $MU_*(X; \Omega(1))$ by localizing with respect to v_1 as before.

THEOREM 5. *Theorems* 2, 3, *and* 4 *are true when we replace $BP_*(X)$ and $BP_*(X; \Lambda(1))$ by $MU_*(X) \otimes_Z Q_p$ and $MU_*(X; \Lambda(1))$ respectively.*

The Conner and Floyd relation (see [2]) can then be used to show

THEOREM 6. *$K_*(X) \otimes Q_p$ is torsion free. Also it is generated as an algebra by* Image Ω_*: $Q(K_*(\Omega X) \otimes Q_p) \longrightarrow P(K_*(X) \otimes Q_p)$ *when $H^*(X; Q)$ is an exterior algebra.*

The basic technique used in proving both Theorems 1 and 2 is to study the homology of ΩX and then pass to the homology of X using an Eilenberg-Moore spectral sequence. For details on the above see [4] and [5]. Also, see Petrie's work in [7]. His results motivated our work in BP, MU, and K homology.

REFERENCES

1. R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61. MR **21** #1589.

2. P. E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Lecture Notes in Math., vol. 28, Springer-Verlag, Berlin and New York, 1966. MR **35** #7344.

3. L. Hodgkin, *On the K-theory of Lie groups*, Topology **6** (1967), 1–36. MR **35** #4950.

4. R. Kane, *The BP homology of H-spaces*, Trans. Amer. Math. Soc. (to appear).

5. ———, *On loop spaces without p torsion*. II, Pacific J. Math. (to appear).

6. J. Lin, *Torsion in H-spaces*. II, Ann. of Math. (to appear).

7. T. Petrie, *The weakly complex bordism of Lie groups*, Ann. of Math. (2) **88** (1968), 371–402. MR **42** #2504.

8. D. G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR **40** #6565.

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