mathematics (Kreisel); 4. Geometries in which straight lines give the shortest distance (Busemann); 6. Axioms for mathematical physics (Wightman); 7. Transcendence of a^b for a and b algebraic (Tijdeman); 9. General reciprocity (Tate); 12. Generalize Kronecker's Jugendtraum (Langlands); 13. Is every function of 3 variables expressible in terms of functions of 2 variables? (Lorentz); 14. Finite generation of subrings (Mumford); 15. Rigorous foundation of Schubert's enumerative calculus (Kleiman); 18. Crystalographic groups (Milnor); 22. Uniformizations (Bers).

With two further notes I conclude my task. The book reproduces the translation of Hilbert's address by Mary Winston Newson (1902, Bull. Amer. Math. Soc.). The German original is perhaps most readily available in the third volume of Hilbert's collected works [5]. In French there is in [4] a reprinting of the summary which appeared in the 1900 Congress Proceedings. To Paul Halmos we are indebted for 22 photographs (Conway is included and Matijasevic and Stampacchia are missing). I have cited so many integers in this review that I can't resist one more: there are 10 beards.

ERRATA. In the article entitled *Hilbert's tenth problem. Diophantine equations: Positive aspects of a negative solution*, by Martin Davis, Yuri Matijasevič and Julia Robinson, pages numbered 357–358 should precede pages numbered 355–356.

REFERENCES

1. L. Bieberbach, *Über den Einfluss von Hilberts Pariser Vortrag über "Mathematische Probleme" auf die Entwicklung der Mathematik in den letzten dreissig Jahren*, Naturwissenschaften **18** (1930), 1101–1111.

2. S. S. Demidov, *On the history of Hilbert's problems*, Istor.-Mat. Issled. No. **17** (1966), 91–121. (Russian) MR **35** #2696.

3. J. Fang, *Hilbert. Towards a philosophy of modern mathematics*. II, Paideia Press, New York, 1970, Chaps. VI, VII. MR **42** #2919.

4. D. Hilbert, *Problèmes mathématiques*, Enseignement Math. **2** (1900), 349–354.

5. ———, *Gesammelte Abhandlungen*, Bände I, II, III, Springer-Verlag, Berlin, 1932, 1933, 1935; reprinted, Chelsea, New York, 1965; 2nd. ed., Springer-Verlag, Berlin and New York, 1970. MR **32** #5489 a, b, c; **41** #8201.

6. *Hilbert's problems*, Izdat. "Nauka", Moscow, 1969. MR **40** #4036.

7. *Die Hilbertschen Probleme*, Akademische Verlagsgesellschaft, Leipzig, 1971. MR **49** #10488.

8. Hermann Minkowski, *Briefe an David Hilbert*, edited by L. Rudenberg and H. Zassenhaus, Springer-Verlag, Berlin and New York, 1973, pp. 126–131.

9. C. Reid, *Hilbert*, Springer-Verlag, Berlin and New York, 1970, pp. 65–85. MR **42** #5767.

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Decomposition of multivariate probabilities, by Roger Cuppens, Probability and Mathematical Statistics, Series, Vol. 29, Academic Press, New York, San Francisco, London, 1975, xv + 244 pp., $26.50.

In 1929, B. de Finetti introduced the class of infinitely divisible probability distributions on the reals. Such a distribution has the property that, for each positive integer k, the distribution has a factorization into a product of k probability distributions relative to the operation of convolution. Three years later, A. N. Kolmogorov found a representation of the logarithm of the Fourier transform of an infinitely divisible distribution under a finite second moment condition. The latter condition, as well as the restriction to one-dimensional Euclidean spaces, was removed by P. Lévy in 1937. A year earlier, H. Cramér had shown that the factors of an n-dimensional normal distribution must also be normal, possibly degenerate, and that an n-dimensional distribution is normal if and only if each of its projections onto one-dimensional subspaces is normal. These early works are the source of problems dealing with the factorization and structure of the factors of a probability distribution.

Cuppens' book deals with the factorization problem for many-variable distributions exclusively, referring frequently to E. Lukacs, *Characteristic functions*, 2nd ed., Griffin, London, 1970, for the one-dimensional case. His approach is mostly analytical and is carried out within the context of Fourier transforms of finite signed measures on n-dimensional Euclidean space R^n. Letting (t, u) denote the inner product of $t, u \in R^n$ and $\mathfrak{M}_n$ the class of finite signed measures on the Borel subsets of R^n, the Fourier transform $\hat{\mu}$ of $\mu \in \mathfrak{M}_n$ is given by $\hat{\mu}(t) = \int \exp[i(t, u)]\,\mu(du)$. In the first two chapters, properties of the Fourier transform are developed and used in the third chapter to prove H. Cramér's theorem on the normality of the factors of an n-dimensional normal distribution and H. Teicher's 1954 result which states that $p \in \mathfrak{P}_n$, the class of n-dimensional probability measures on the Borel subsets of R^n, is a Poisson distribution if and only if $\hat{p}$ is an entire function and certain one-dimensional marginals of p are Poisson. A similar characterization holds if p is a convolution of n-dimensional normal and Poisson distributions.

The study of decompositions of n-dimensional probability measures commences in the fourth chapter of the book. A probability measure p is indecomposable if $p = p_1 * p_2$, with $p_1, p_2 \in \mathfrak{P}_n$, implies p_1 or p_2 is degenerate, a concept introduced by P. Lévy and A. Khintchine in 1937–1938. According to a 1962 result of K. R. Parthasarathy, R. Ranga Rao, and S. R. Varadhan, there is an abundance of indecomposable probability measures since the set of indecomposable purely atomic n-dimensional probabilities is dense in $\mathfrak{P}_n$ relative to complete convergence. The only condition given for determining if a probability measure is indecomposable is an algebraic condition on its support; that is, if the support of $p \in \mathfrak{P}_n$ is linearly independent over the rationals, then p is indecomposable. This condition is used to show that a probability measure p concentrated on a sphere in R^3 is indecomposable, a 1972 result due to Ju. V. Linnik and I. V. Ostrovskiĭ. Going to the other extreme, $p \in \mathfrak{P}_n$ is infinitely divisible if for each positive integer k, there is a $p_k \in \mathfrak{P}_n$ such that $p = p_k^{*k}$, the k-fold convolution of p_k. A probability measure p is said to have an (α, P, μ)-Lévy-Khintchine representation if there is an $\alpha \in R^n$, a nonnegative definite function P on R^n, and finite signed measure μ such that

$$\log \hat{p}(t) = i(\alpha, t) - P(t) + \int \left(e^{i(t,u)} - 1 - \frac{i(t,u)}{1 + \|u\|^2} \right) \frac{1 + \|u\|^2}{\|u\|^2} \mu(du)$$

for $t \in R^n$. The first term on the right determines a degenerate component of p, the second term a normal component, and the last term a Poisson component. The signed measure μ is called the Poisson measure of p and the support $S(\mu)$ of μ is called the Poisson spectrum of p. The results of Kolmogorov, Lévy, and Khintchine referred to above state that $p \in \mathcal{P}_n$ is infinitely divisible if and only if p has an (α, P, μ)-Lévy-Khintchine representation where μ is a measure with no atom at the origin of R^n.

In 1937, A. Khintchine proved that each $p \in \mathcal{P}_n$ has a decomposition $p = p_1 * p_2$ in which p_1 and p_2 can be degenerate, p_1 has no indecomposable factor, and p_2 is a vague limit of finite convolutions of indecomposable probabilities. This result justifies the introduction of a class I_0^n of probability distributions with no indecomposable factor, the subject matter of the remaining chapters of the book. The structure of a probability measure p with an (α, P, μ)-Lévy-Khintchine representation appears to be related to the algebraic structure of its Poisson spectrum $S(\mu)$. The algebraic structure being that of a Linnik set. A subset of the reals is a Linnik set if for any two distinct elements having the same sign, one of the two is an integral multiple of the other; in the $n \geqslant 2$ case, a subset of R^n is a Linnik set if it is the span of Linnik sets on each of the n coordinate axes. Such sets arose in a 1960 paper by Ju. V. Linnik in which he proved that the Poisson measure of a probability measure $p \in I_0^1$ with a nondegenerate normal factor is concentrated on a Linnik set. It is not known if this result holds in the $n \geqslant 2$ case. Going the other way, if $p \in \mathcal{P}_1$ has an (α, P, μ)-Lévy-Khintchine representation, μ is concentrated on a Linnik set, and the μ measure of the complement of a ball decays exponentially with the squared radius, then p has no indecomposable factor. Sufficient conditions for this result to hold in the $n \geqslant 2$ case are given which involve additional restrictions on the Poisson spectrum $S(\mu)$, namely that $S(\mu)$ is a Linnik set and that each point of $S(\mu)$ is accessible from certain types of cones in the complement of $S(\mu)$.

In the preface to the book, Cuppens states that a complete report on multivariate characteristic functions would be beneficial, taking up where Lukacs and others stop. He has been successful in this respect, the above results being only a sampling of the more easily stated results. He is to be commended for providing the mathematical community with this compilation of widely scattered results.

It is unlikely that this book will prove useful to graduate students. The exposition is minimal and usually poorly done. The density of errors is tolerable except when an incorrect reference to a previous theorem is involved. It is not likely that the nonspecialist will want to own a personal copy of this book.

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