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RESIDUES AND CHARACTERISTIC CLASSES OF FOLIATIONS

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In this note we announce results and construct examples which show that a large number of characteristic classes for real foliations vary linearly independently. This generalizes the result of Thurston on the variation of the Godbillon-Vey invariant [T]. The method used is a special case of the general theory of residues of singular foliations due to Baum and Bott [BB].

DEFINITION. Let τ be a codimension q foliation on a manifold M. A vector field X on M is a Γ vector field for τ if $[X, Y]$ is tangent to τ whenever Y is tangent to τ. The singular set of X is the set of points where X is tangent to τ.

Let τ be an oriented codimension q foliation on an oriented manifold M. Let X be a Γ vector field for τ and assume the singular set of X consists of a single compact leaf N of τ. On $M - N$, τ and X span a foliation $\hat{\tau}$ of codimension $q - 1$. Let $\alpha^*: H^*(WO_{q-1}) \longrightarrow H^*(M - N; R)$ be the natural map associated to $\hat{\tau}$. Each element $\hat{\phi}$ of $H^{2q-1}(WO_{q-1})$ determines in a natural way an element ϕ of $H^{2q}(BU_q; R)$. Choose an embedded normal sphere bundle S of N in M and let $i: S \longrightarrow M - N$ be the inclusion. Denote by $\sigma: H^{2q-1}(S; R) \longrightarrow H^q(N; R)$ integration over the fiber of the sphere bundle S. On M, τ and X span a singular foliation with singular set N. Applying the theory of [BB], $\phi \in H^{2q}(BU_q; R)$, τ and X determine a cohomology class $\mathrm{Res}_\phi(\tau, X, N) \in H^q(N; R)$. We have

THEOREM 1. *For M, N, τ, and X as above and $\hat{\phi} \in H^{2q-1}(WO_{q-1})$,*

$$\sigma(i^*\alpha^*(\hat{\phi})) = \mathrm{Res}_\phi(\tau, X, N).$$

Let $\phi \in H^{2q}(BU_{q-1}; R)$. Then ϕ and $\hat{\tau}$ determine an element $S_\phi(\hat{\tau}) \in H^{2q-1}(S; R/Z)$, the Simons' character of $\hat{\tau}$, [ChS]. The element ϕ determines in a natural way an element ϕ in $H^{2q}(BU_q; R)$. We have

THEOREM 2. $S_\phi(\hat{\tau})[S] = \mathrm{Res}_\phi(\tau, X, N)[N] \bmod Z$, *where $[S]$ and $[N]$ are the homology classes determined by S and N.*

We give some examples which show that these residues are nontrivial and in fact vary linearly independently.

EXAMPLE 1. Denote by G the product of k copies of the special linear group SL_2R. Let K be a maximal compact subgroup of G and Γ a uniform dis-

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crete subgroup of G so that $\Gamma\backslash G/K$ is a compact manifold. Let M be the flat R^{2k} bundle $M = (G/K) \times_\Gamma R^{2k}$ with the natural flat foliation τ. Choose k nonzero numbers $\mu_1, \ldots, \mu_k$ and let X be the vector field on R^{2k}

$$X_\mu = \sum_{i=1}^{k} \mu_i(x_{2i-1}\partial/\partial x_{2i-1} + x_{2i}\partial/\partial x_{2i}).$$

The natural action of G on R^{2k} preserves X_μ and so X_μ induces a Γ vector field X_μ on M with singular set the zero section $N = \Gamma\backslash G/K$. For $\phi \in H^{4k}(BU_{2k}; R)$ we compute

$$\mathrm{Res}_\phi(\tau, X_\mu, N) = \frac{\pi^k\phi(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_k, \mu_k)\mathrm{vol}}{(\mu_1\mu_2 \cdots \mu_k)^2}.$$

Here vol is a fixed volume form on N and $\phi(\mu_1, \ldots, \mu_k)$ is ϕ, thought of as an invariant polynomial on the lie algebra of the unitary group U_{2k}, applied to the diagonal matrix $\mathrm{diag}(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_k, \mu_k)$.

EXAMPLE 2. Let G and K be as in Example 1. We let $G \times R$ act on $R^{2k+1} = R^{2k} \times R$ by the natural action of G on R^{2k} and by the action of R on R^{2k+1} defined as follows. Let ω be a smooth, even, nonnegative function on R such that

(i) $0 < \omega(x) \leqslant 1$ for all $x \neq 0$.

(ii) For all x, $|x| > ½$, $\omega(x) = 1$.

(iii) ω and all its derivatives are zero at $x = 0$.

On the lie algebra level R acts on R^{2k+1} by $\partial/\partial t \longrightarrow |x_{2k+1}|\omega(x_{2k+1})\partial/\partial x_{2k+1}$. Choose a uniform discrete subgroup Γ of $G \times R$ so that $\Gamma\backslash(G \times R)/K$ is a compact manifold. Set $M = (G \times R)/K \times_\Gamma R^{2k+1}$ and let τ be the natural flat foliation on M. Choose nonzero real numbers $\mu_1, \ldots, \mu_{k+1}$ and let X_μ be the vector field on R^{2k+1}.

$$X_\mu = \left(\sum_{i=1}^{k} \mu_i(x_{2i-1}\partial/\partial x_{2i-1} + x_{2i}\partial/\partial x_{2i})\right) + \mu_{2k+1}x_{2k+1}\omega(x_{2k+1})\partial/\partial x_{2k+1}.$$

The action of $G \times R$ on R^{2k+1} preserves X_μ and so X_μ induces a Γ vector field X_μ on M with singular set the zero section $N = \Gamma\backslash(G \times R)/K$. For $\phi \in H^{4k+2}(BU_{2k+1}; R)$ we compute

$$\mathrm{Res}_\phi(\tau, X, N) = \frac{2\pi^k\phi(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_k, \mu_k, 0)\mathrm{vol}}{(\mu_1\mu_2 \cdots \mu_k)^2\mu_{k+1}}.$$

As before vol is a fixed volume form on N and $\phi(\mu_1, \ldots, \mu_k, 0)$ is ϕ applied to the diagonal matrix $\mathrm{diag}(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_k, \mu_k, 0)$.

Let $R[\sigma_1, \ldots, \sigma_n]$ be the algebra of symmetric polynomials on the variables $\mu_1, \ldots, \mu_n$ and denote by R^n_m the subalgebra generated by the elements $\sigma_i(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_m, \mu_m, 0, \ldots, 0)$, $i = 1, \ldots, n-1$. Let R^n_{m0} be the ideal in R^n_m generated by the elements $\sigma_i(\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_m, \mu_m, 0, \ldots, 0)$ where $i = 1, 3, 5, \ldots, 2t + 1$ and $2t + 1 = n - 2$ or $n - 1$. Now set $a(m, n) =$ the dimension of the space of elements of degree n in R^n_{m0}, and $b(m, n) =$ the dimension of the space of elements of degree n in R^n_m. Finally set $a(2k + 1) = a(k, 2k + 1)$, $a(2k) = a(k, 2k) - 1$, $b(2k + 1) = b(k, 2k + 1)$ and $b(2k) = b(k, 2k) - 1$. The characteristic classes mentioned in Theorems 1 and 2 come from universal characteristic classes in the R and R/Z cohomology of $B\Gamma_q$, the classifying space for codimension q foliations. Combining the examples with Theorems 1 and 2 we see that $B\Gamma_q$ has many cohomology classes which vary linearly independently. In particular we may view these classes as maps from the homology of $B\Gamma_q$ to R or R/Z and so obtain

THEOREM 3. *$H_{2q+1}(B\Gamma_q; Z)$ admits epimorphisms onto $R^{a(q+1)}$ and $(R/Z)^{b(2q+1)}$.*

In Example 1, the foliation $\hat{\tau}_\mu$, spanned by τ and X_μ on $M - N$ is transverse to the sphere bundle $M^0 = G/K \times_\Gamma S^{2k-1}$ provided all the μ_i are close to 1. We lift this foliation to the bundle over M^0, $P = G/K \times_\Gamma SO_{2k}$ (actually the bundle $(\Gamma\backslash G) \times_K SO_{2k}$) obtaining a foliation with trivial normal bundle. The projection map $\pi: P \longrightarrow M^0$ is injective in cohomology in dimension $4k - 1$. Thus we have

THEOREM 4. *Let $F\Gamma_q$ be the classifying space for codimension q real foliations with trivial normal bundle. Then $H_{4k-1}(F\Gamma_{2k-1}; Z)$ admits an epimorphism onto $R^{b(2k)}$.*

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