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STABILITY AND SEMIPOSITIVITY OF REAL MATRICES[1]

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The purpose of this note is to describe the interrelations of various degrees of stability and semipositivity for real square matrices.

A matrix A is

(a) *stable* if there exists a positive definite matrix X such that

$$AX + XA^T \tag{1}$$

is positive definite [4], [5];

(b) *diagonally stable* if X in (1) may be taken to be diagonal [1]; and

(c) *semipositive* if there exists a positive vector x such that Ax is positive [6]. We denote by $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{S}$ the classes of stable, diagonally stable, and semipositive matrices, respectively.

With each of these classes, we associate two superclasses denoted by $\mathcal{WA}$, $\mathcal{WL}$, $\mathcal{WS}$ and $\mathcal{VWA}$, $\mathcal{VWL}$, $\mathcal{VWS}$. A matrix $A \in \mathcal{WL}$ ($\mathcal{VWL}$) if there exists a positive definite (nonzero positive semidefinite) matrix X such that (1) is positive semidefinite. The other four superclasses are defined similarly.

Let $\mathcal{I}$ denote any of these nine classes. With each $\mathcal{I}$ we associate subclasses using the following notation:

$A \in \mathcal{IS}$ if every principal submatrix of A is in $\mathcal{I}$;

$A \in \mathcal{DI}$ ($\mathcal{ID}$) if DA (AD) is in $\mathcal{I}$ for every positive diagonal matrix D. $\mathcal{ISD}$ and $\mathcal{DIS}$ are defined similarly.

We also consider the classes $\mathcal{P}$, $\mathcal{P}_0$, and $\mathcal{P}_0^+$ defined as follows:

$A \in \mathcal{P}$ if all its principal minors are positive;

$A \in \mathcal{P}_0$ if all its principal minors are nonnegative; and

$A \in \mathcal{P}_0^+$ if $A \in \mathcal{P}_0$ and has at least one positive principal minor of each order [2], [3].

A study of the above classes indicates that there exist 24 distinct ones. The inclusion relations between them are described by the following directed graph having classes as vertices in which there is a sequence of directed edges from vertex $\mathcal{X}$ to vertex $\mathcal{Y}$ if and only if $\mathcal{X}$ is contained in $\mathcal{Y}$.

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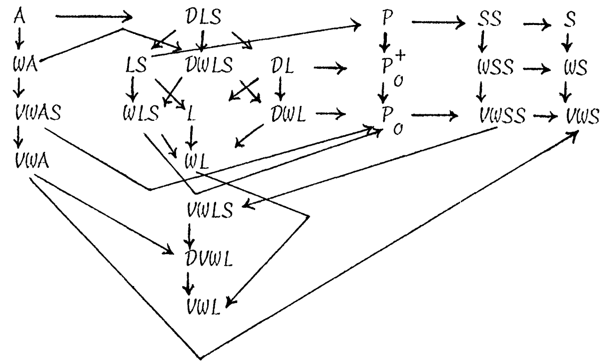

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