chapter and section coordinates at the top of each page. The style of the book is clear and all details are given. All in all, the monograph is an important addition to the literature on topological semigroups and their harmonic analysis; in due time we will be in a better position to judge which parts of this material will most affect and stimulate further research.

REFERENCES

1. J. F. Berglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes in Math., vol. 42, Springer-Verlag, Berlin and New York, 1967. MR **36** #6531.

2. Mary Katherine Bennett, *Review of "Distributive Lattices* by R. Balbes and P. Dwinger," Bull. Amer. Math. Soc. **82** (1976), 246–249.

3. R. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970. MR **41** #8562.

4. H. Carruth, Hildebrant and R. J. Koch, *Topological semigroups* (to appear).

5. K. H. Hofmann, *Topological semigroups, history, theory, applications*, Jber. Deutsch. Math. Verein. **78** (1976), 9–59.

6. ———, *The duality of compact semigroups and C^*-bigebras*, Lecture Notes in Math., vol. 129, Springer-Verlag, Berlin and New York, 1970.

7. K. H. Hofmann, M. Mislove and A. Stralka, *The Pontryagin duality of compact* 0-*dimensional semilattices and its applications*, Lecture Notes in Math., vol. 396, Springer-Verlag, Berlin and New York, 1974. MR **50** #7398.

8. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Merrill, Columbus, 1966; errata in Semigroup Forum **5** (1973), 285–322. MR **35** #285; **47** #8757.

9. J. D. Lawson, *Additional remarks on continuity in semitopological semigroups*, Semigroup Forum **12** (1976), 265–280.

10. A. B. Paalman-de Miranda, *Topological semigroups*, Math. Centre Tracts, 11, Math. Centrum, Amsterdam, 1964. MR **31** #1663.

11. N. Rothman and M. W. Schuh, *Laplace transforms on vanishing algebras*, Semigroup Forum **8** (1974), 189–214.

12. J. L. Taylor, *Measure algebras*, C.B.M.S. Regonal Conf. Ser. in Math., no. 16, Amer. Math. Soc., Providence, R.I., 1973.

13. J. H. Williamson, *Harmonic analysis on semigroups*, J. London Math. Soc. **42** (1967), 1–41. MR **34** #8101.

KARL H. HOFMANN

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, March 1977

Pursuit games, by Otomar Hájek, Mathematics in Science and Engineering, vol. 120, Academic Press, New York, 1975, xii + 266 pp., $10.50.

From the beginnings of the differential calculus, through the calculus of variations to modern control theory, dynamical and optimization problems have always provided a stimulus for mathematical activity. A two person differential game is a generalization of a control system, and can be considered as a control system with two competing controllers or players. (The theory of differential games with more than two controllers is in an even more elementary state, basic problems being the possibilities of coalitions, how to model information flow, and all the other problems of von Neumann's discrete game theory, now in a dynamic setting.) Conversely, control theory can be considered as a special case of a differential game with just one player.

Pioneering work on differential games was undertaken by Rufus Isaacs in the 1950's, though his work was not generally available until his book *Differential games* (J. Wiley and Sons, New York, London), was published in

1965. In fact, ideas such as Isaacs' 'main equation' are generalizations of Bellman's equation of dynamic programming.

Control theory has two main branches. The first studies the problem of controllability, that is, whether a given objective can be attained. The second, optimal control theory, studies how a given objective can be attained in an optimal way, (usually meaning with minimal cost). The theory of two person differential games divides similarly into two areas, though the problems are now more involved. The first area discusses whether one controller (or player) can force the dynamical system to attain some objective whatever the other player does. Such a situation was called a 'differential game of kind' by Isaacs. The second, usually a two person zero sum game, discusses a real valued cost or payoff which is a function of how the game evolves, and which one player is trying to mimimize the other to maximize. Such games were called 'differential games of degree' by Isaacs.

Following Isaacs, the authors of other early papers in the subject often studied differential games through the so-called Isaacs-Bellman equation (see below). Unfortunately, this equation is nonlinear and in general it is not known whether it has a solution. Further development of the theory of differential games was closely related to progress in mathematically modelling the ideas of strategy and control. Certain approaches to these problems will now be briefly described.

For both control and differential game problems it is usually supposed that the dynamics are described by a system of ordinary differential equations of the form

$$(1)\quad \dot{x} = f(t, x, p, q), \quad x(0) = x_0 \in R^n, \quad t \in [0, \infty), x \in R^n, p \in P, q \in Q,$$

where P and Q are, say, compact subsets of R^l. In the control situation there is just one control set and one control variable. The usual conditions required of a function appearing on the right-hand side of such a system, in order to ensure the existence and uniqueness of a solution trajectory, are that it is measurable in time t and satisfies some Lipschitz continuity condition in the state variable x. (See, for example, *The theory of ordinary differential equations*, McGraw-Hill, New York, 1955, by E. Coddington and N. Levinson.) This somewhat basic difference in the roles of time and space variables is not made clear by Hájek; on p. 36 he remarks that a time dependent (or, following Hájek, allonomous), system can be considered as time independent by labelling time a new state variable ξ such that $\dot{\xi} = 1$. The above standard existence conditions for the dynamics, however, have important implications for the kind of control functions that may be considered. A control function that is just a measurable map of time with values in the control set is called an open loop control. If, for example, the f occurring in the dynamics (1) is continuous in p and q and if $p(t)$ and $q(t)$ are open loop controls then, after substituting in f, we obtain

$$\dot{x} = f(t, x, p(t), q(t)).$$

That is, the right-hand side becomes a measurable function of time t, and so the system can be integrated.

However, a control function that is a function not only of time but also of the state is called a closed loop, or feedback, control. If $p(t, x)$ and $q(t, x)$ are feedback controls then, after substitution, we have

$$\dot{x} = f(t, x, p(t, x), q(t, x)).$$

It is clear that restrictive continuity conditions must be imposed upon f and the control functions to ensure that the right-hand side is Lipschitz continuous in the state x. Control theorists have circumvented these problems by considering feedback controls which, in time, are only piecewise Lipschitz continuous, or even piecewise constant. Consequently the theory of feedback controls is not satisfactory; this is particularly significant for two person zero sum differential games, where at any time each player is probably choosing his control value on the basis of what has happened to the system so far.

Fleming (J. Math. Anal. Appl., 1961), was among the first to consider the problem of strategies and controls in differential games rigorously. He considered piecewise constant controls and related strategies in the case when the control variables appear separated, e.g.:

$$f(t, x, p, q) = f_1(t, x, p) + f_2(t, x, q).$$

As the length of the intervals on which the controls are constant shrinks to zero, certain convergence problems arise, and in a sequel to the above paper discussing dynamics with a general f and published in the Ann. of Math. Studies volume **52** in 1964, Fleming had the excellent idea of introducing a random perturbation of the trajectory at each step to facilitate the proof of this convergence.

In a sequence of papers which culminated in his book *Differential games* (Wiley-Interscience, New York, London, 1971), Friedman considered control functions which were general measurable functions of time but approximated the idea of strategy by introducing partitions of the time interval. Corresponding to any such partition a lower strategy for a player is a function that selects a piece of control function to be used on the next time interval by that player, on the basis of the control functions chosen by both players so far; an upper strategy, in addition, depends on the choice of control functions selected by the opposing player for the next time interval. Again convergence problems arise as the length of the partition time intervals shrinks to zero; however, because Friedman is using measurable control functions these are more easily handled.

Another novel and interesting method of approximating strategies is introduced by Danskin in Bull. Amer. Math. Soc. **80** (1974), 449–455. Danskin considers partions of the time interval and piecewise constant controls; however, the controls are chosen by the players in an overlapping manner. For example, suppose both players have chosen initial control values to start the game; after a certain delay one player will change his control value, then after another delay the second player will choose a new control value, on the basis of what has happened so far. Play continues in this manner, the delay of each player being the same at each move. To establish convergence as the size of the partition intervals decreases to zero Danskin also introduces random perturbations of the trajectory. A. Friedman, N. Kalton and the reviewer also considered this method of play in J. Differential Equations in 1974.

A notion of strategy that does not involve partitions of the time interval was introduced by Roxin (J. Optimization Theory and Applications (1969)) and Varaiya and Lin (SIAM J. Control 1969). Write $\mathcal{U}$ (resp. $\mathcal{V}$) for the measurable functions on the time interval with values in P (resp. Q) (that is, the open loop control functions). Then, for example, an s-delay strategy ($s \geq 0$) for the first player is a map $\alpha\colon \mathcal{V} \to \mathcal{U}$ such that if for any t,

$$q_1(\tau) = q_2(\tau) \quad \text{a.e. } \tau \leq t,$$

where q_1 and q_2 are in $\mathcal{V}$, then

$$(\alpha q_1)(\tau) = (\alpha q_2)(\tau) \quad \text{a.e. } \tau \leq t + s.$$

Write Γ_s (resp. Δ_s) for the s-delay strategies for the first (resp. second) player. Clearly if $s_1 \leq s_2$ then, for example, $\Gamma_{s_2} \subset \Gamma_{s_1}$.

Given a control function $q(t) \in \mathcal{V}$ and a strategy $\alpha \in \Gamma_s$ a control function $(\alpha q)(t) \in \mathcal{U}$ is determined, and so, by solving the dynamical equations, a trajectory is obtained.

Consider for the moment a two person zero sum differential game and suppose that the payoff has the form

$$g(x(T)) + \int_0^T h(t, x, p, q)\, dt.$$

Here T is the time the game terminates; T may be determined in advance so that the differential game has fixed duration, or it may be the 'time of capture'. g and h are real valued and satisfy suitable continuity and measurability conditions. In this situation, corresponding to $q \in \mathcal{V}$ and $\alpha \in \Gamma_s$ a trajectory $x(t)$, and so a payoff

$$\prod(\alpha q, q) = g(x(T)) + \int_0^T h(t, x(t), \alpha q(t), q(t))\, dt$$

is determined. Player J_1 controlling p is trying to maximize this quantity whilst player J_2 controlling q is trying to minimize. Consequently, if J_1 uses strategy α the worse outcome for him would be the quantity

$$u(\alpha) = \inf_{q \in \mathcal{V}} \prod(\alpha q, q),$$

and the best outcome for J_1 if he uses s-delay strategies is

$$U(s) = \sup_{\alpha \in \Gamma_s} u(\alpha).$$

Similarly for $p \in \mathcal{U}$ and $\beta \in \Delta_s$, we define the quantities

$$\nu(\beta) = \sup_{p \in \mathcal{U}} \prod(p, \beta p), \qquad V(s) = \inf_{\beta \in \Delta_s} \nu(\beta).$$

Clearly $U(s)$ (resp. $V(s)$) is monotonic increasing (resp. decreasing). If U and V denote their respective limits as s tends to 0 then N. Kalton and the reviewer established in Memoir 126 of the American Mathematical Society that U (resp. V) is equal to the upper (resp. lower) value obtained in the approximating procedures of both Fleming and Friedman, and then $U = V$ if, for all t and $x \in R^n$, $y \in R^n$

$$\min_{q \in Q} \max_{p \in P} H(t, x, y, p, q) = \max_{p \in P} \min_{q \in Q} H(t, x, y, p, q).$$

Here $H(t, x, y, p, q)$ is the Hamiltonian $y \cdot f(t, x, p, q) + h(t, x, p, q)$, and the above condition is called the Isaacs condition. If the Isaacs condition is not satisfied it is shown in the Memoir that the upper and lower values are equal if the players are allowed to use relaxed controls. This idea is similar to von Neumann's introduction of mixed strategies into a two person zero sum matrix game. A relaxed control selects at each time t a probability measure on the space P (resp. Q) of control values and the functions f and h are evaluated at this measure by integration–not in the probabilistic sense of selecting a particular control value according to this measure. In an interesting paper to appear in the J. of Differential Equations, E. N. Barron has shown that if the players control functions start at known values, and the players must then use controls which are Lipschitz continuous in time, for fixed Lipschitz constants, then the upper and lower values are equal, even if the Isaacs condition is not satisfied.

Suppose the trajectory now starts at x_0 at time t_0. If the functions f, g and h satisfy suitable Lipschitz continuity conditions it can be shown that the value function $U(t_0, x_0)$ is Lipschitz continuous in the initial conditions. From a classical result of Denjoy it is, therefore, differentiable almost everywhere and at points of differentiability it can be shown to satisfy the 'Isaacs-Bellman' equation:

$$\frac{\partial U}{\partial t} + \min_q \max_t \; H(t, x, \nabla U, p, q) = 0$$

with boundary condition

$$U(T, x) = g(x).$$

It was Fleming in a paper in the J. Math. Mech. in 1964 who had the nice idea of obtaining solutions for certain nonlinear equations of the form

$$\frac{\partial u}{\partial t} + G(t, x, \nabla u) = 0$$

by constructing differential games whose Hamiltonians were equal to G. N. Kalton and the reviewer further extended this idea in papers in the J. Math. Anal. and Appl. in 1974, and in the Proceedings of the First Kingston Conference on Differential Games and Control Theory, published by Marcel Dekker, New York, 1974. Because of the use of approximating differential games with noise this technique is close to some singular perturbation methods. Analogies with solving certain boundary value problems by taking the expected value of functions of diffusions are also apparent.

Finally, it is probably worth indicating why the theory of the control of stochastic differential systems and stochastic differential games is in some ways easier; this is because a new concept of solution developed from a result of Girsanov allows the almost unrestricted use of feedback controls. To describe Girsanov's theorem consider a stochastic differential equation in one dimension on the unit time interval:

$$dx = f(t, x)\, dt + dw, \qquad x \in R, t \in [0, 1], x(0) = 0.$$

Here w_t is a Brownian motion. To obtain a stochastic process which can be considered a solution of the above system proceed as follows: Suppose B_t is a Brownian motion on a probability space (Ω, μ). Then the stochastic process

$$w_t = B_t - \int_0^t f(s, B_s)\, ds$$

is in general no longer a Brownian motion on (Ω, μ). However, replace μ by $\bar{\mu}$, where $d\bar{\mu}/d\mu = \exp \xi(f)$, and $\xi(f) = \int_0^1 f(s, B_s) dB_s - \frac{1}{2}\int_0^1 f^2(s, B_s) ds$. (Note the first integral is a stochastic integral.) Girsanov's theorem states that, if f satisfies quite weak measurability and growth conditions in t and x, then w_t is a Brownian motion on $(\Omega, \bar{\mu})$, i.e. on $(\Omega, \bar{\mu})$, $dB = f(t, B) + dw$, so the original Brownian motion B_t is a solution to the stochastic differential equation under the new measure $\bar{\mu}$. Girsanov's result extends to more general stochastic systems. Because only measurability conditions are required it is particularly useful in control and stochastic differential games. It was first used in control by Benes (SIAM J. Control 1971), and Davis and Varaiya (SIAM J. Control 1973), and in differential games by the reviewer (SIAM J. Control 1976). The same measure transformation method enables solutions to be found when at time t, f depends on the part of the trajectory up to time t, or, in particular, f depends on the trajectory at some time just before t (so the equation is a stochastic 'delay' equation). The application of Girsanov's result to the solution of stochastic differential equations should be more widely known. Girsanov's original paper appears in English translation in the Theor. Probability Appl. in 1960.

Having sketched the development of differential games, consider now the book by Professor Hájek under review. As the title indicates, the book is concerned with games of kind, and in particular with pursuit games. The discussion, therefore, is about whether capture can be effected at all, rather than whether there is some minimum time for capture (though this latter problem is briefly touched upon in Chapter IV).

In spite of the hardback cover, characteristic of this series, the text is photocopied from typescript. The first chapter presents well-known examples of pursuit games, such as the Homicidal chauffeur and the Lion and Man game (which has many unusual features). The second chapter discusses necessary conditions for the existence of solutions of the dynamical equations. The control sets P and Q are compact subsets of R^l as above and, appropriately, denote the space of control values for the pursuer and quarry respectively. The definition of strategy used by the author is that of Roxin and Varaiya and Lin described above.

Chapters III to VI discuss aspects of the possibility of capture in linear pursuit games. Included in Chapter III is the interesting idea of considering the nonlinear problem of bringing a pendulum to rest in minimum time as a linear problem in game theory, by treating the nonlinear terms as the controls of a fictional opponent. Although the notion of Pontrjagin difference is used from §3.1 onwards it is not defined until §3.8. Nonlinear pursuit games are discussed briefly in Chapter VII and the compactness of, and other results for, strategies presented in Chapter VIII. There is no bibliography; any references are included in the text. Fortunately there are few typographical errors; only the statement at the top of p. 30 is initially confusing. There are large sections of exercises throughout the text and the mathematical background required for understanding the book is quite modest; probably only a course in ordinary differential equations is a necessary prerequisite. The writing has an idiosyncratic style, but as a set of lecture notes the book is, on the whole, quite readable and even enjoyable.

The theory of differential games is still at an early stage of development. Though it asks interesting and important questions, only time will tell whether they are presently being approached in the best way, whether this approach will make significant contributions to mathematics, and whether the models are realistic and the solutions obtained helpful in solving real problems.

ROBERT J. ELLIOTT