on separable ones. Roy's example removes all doubt. The reader will find it rewarding to work it carefully through.

There are also examples of the pathological behavior of dimension on bicompacta. An example due to V. V. Filippov is presented as well as ones due to Lokucievskii and Vopenka. Filippov constructed a bicompactum X which has $\dim X = 1$, $\operatorname{ind} X = 2$, and $\operatorname{Ind} X = 3$. This example and its modifications show that except for the inequalities $\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$, dim, ind, and Ind are independent variables on bicompacta.

One will find a section on local dimension containing Dowker's classic Example M of a completely regular space of covering dimension 1 and local dimension 0. There are sections detailing the results on dimension raising and lowering mappings, on the dimension of product spaces, on analytic dimension of algebras of continuous real-valued functions (reminiscent of Chapter 16 of Gillman and Jerison's *Rings of continuous functions*), and on dimension and bicompactification of completely regular spaces. There are extensive historical notes at the end of each chapter which help the reader put individual results in perspective. In these one is also introduced to still other advances and open problems.

The book does not treat homological dimension theory except in the historical notes. A separate volume would be necessary to handle this area with the same thoroughness. Rumor has it that such a volume is forthcoming from the Russian school. This would be a welcome addition. The theory for separable metric spaces is not treated with the thoroughness of Hurewicz and Wallman although there are some recent results appearing in the notes that bring one up to date. For those whose interest is separable metric spaces Hurewicz and Wallman remains the recommended text. One might criticize omission of some detail in this or that section of the book. Be reasonable! How long do you want the book to be? The historical notes and references will lead the reader to most additional results.

We have in Pears an excellent reference. The broad spectrum of recent advances is painted with a fine brush. Important (and complicated) examples are thoroughly examined. In a book of this level the statement of many theorems is necessarily technical. The novice may not appreciate the years of agonizing effort made by dimension theorists to weaken each hypothesis and make each theorem the paragon of precision. However, researchers who need the exact results of dimension theory for general spaces will find them here.

JAMES KEESLING

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Elementary calculus, by H. Jerome Keisler, Prindle, Weber and Schmidt, Boston, 1976, xviii + 880 + 61 (appendix) pp.

Our educational system contains many interesting paradoxes. We tell the students to get involved in the world, and the curriculum becomes increasingly abstract. Courses in sociology, anthropology, economics, et cetera are introduced, in which mathematical models are discussed. We describe the advantages of having a liberally educated citizenry, to students whom we make increasingly anxious and decreasingly able to think for themselves. We ask the students to understand, and we examine them on facts and technique.

We go to great lengths to see that students are given equal opportunities, and competition becomes more and more intense. The student who is able to achieve superlative grades will go to medical school, if he wishes, and perhaps elect to become a surgeon making hundreds of dollars an hour. Another student who does not take his courses quite so seriously may end up with a menial job that pays less than a hundred dollars a week, and be lucky to get it.

Mathematics now plays an important role in the process of determining who will get what, in part because it is considered to be especially difficult, especially objective, and especially useful to contemporary man. Tests of mathematical proficiency are regarded as a fair and efficient means of eliminating large numbers of superfluous aspirants to choice degrees. Unfortunately, in the process of testing larger and larger numbers of terrified candidates for success, we are telling our students that performance is the name of the game. If they emerge from their courses with any interest in mathematics at all, it will not be a thoughtful interest. It is bad form to ask what it all means. Since reality is so elusive, models are the order of the day, and truth is relative to the model, a kind of super-chess.

Contrary to the expressed intentions of some of its founders, the new math has contributed much to the mystification of students. From a primary concern with numbers and geometrical objects, the pre-college curriculum has moved on to open sentences, sets of sets, distinctions between numbers and numerals, and the like. The students quickly get the idea that they are not supposed to take it seriously: the teachers do not, do they?

Now the colleges have been more conservative, but a new book, *Elementary calculus* by H. Jerome Keisler, could change all that. To quote from the book: "In 1960 Robinson solved a three hundred year old problem by giving a precise treatment of infinitesimals. Robinson's achievement will probably rank as one of the major mathematical advances of the twentieth century." Again: "Recently, infinitesimals have had exciting applications outside mathematics, notably in the fields of economics and physics. Since it is quite natural to use infinitesimals in modelling physical and social processes, such applications seem certain to grow in variety and importance. This is a unique opportunity to find new uses for mathematics, but at present few people are prepared by training to take advantage of this opportunity."

No evidence of these claims is given in Keisler's book, but the students will not notice that. Those students who think that mathematics is about something will be disabused. To quote Keisler: "Do not be fooled by the name 'real number'. The real number system is a purely mathematical creation which may or may not give an accurate picture of a straight line in physical space." Again: "In discussing the real line we remarked that we have no way of knowing what a line in physical space is really like. It might be like the hyperreal line, the real line, or neither. However, in applications of the calculus it is helpful to imagine a line in physical space as a hyperreal line."

What are we to make of these statements? Is Keisler describing mathematics as we know it, and the world as we have come to perceive it? The answer would appear to be "no", but perhaps we have not kept pace. If not, and his statements are true, the evidence should be somewhere in the book. So let us examine the book.

Keisler gets down to business on p. 25 by defining the average slope between two points on a curve in the usual way. Then he computes the average slope $2x_0 + \Delta x$ between two points on the parabola $y = x^2$. Reasoning nonrigorously, as he calls it, he then neglects the Δx (because it is very small) and gets the value $2x_0$ for the slope at (x_0, y_0). The argument is repeated, this time for velocities. The trouble with these intuitive arguments, he says, is that it is not clear when something is to be neglected. "What is needed is a sharp distinction between numbers which are small enough to be neglected and numbers which aren't. Actually, no real number except zero is small enough to be neglected."

Since Keisler wants to "neglect" the Δx (and gives the students the impression that we *need* to neglect the Δx), he would seem to have reached an impasse: On the one hand Δx represents a nonzero real number, and on the other he has told us that no real number except zero is small enough to be neglected. The impasse is broken by forgetting that Δx is a real number, calling it something else (an infinitesimal), and telling us that it is all right to neglect it.

Actually the presentation is much more complicated than that. We are not told what an infinitesimal Δx is, or what $f(x + \Delta x)$ means. Instead, the matter is treated axiomatically. Perhaps the intuitive content is intended to be supplied by our imagining a line in physical space as a hyperreal line.

It is not until p. 298 that Keisler relates his development of calculus to the usual one, and puts everything in what to him is its proper place. The conventional definition of limit is grudgingly given. He tells the student that "Indeed, the whole point of our infinitesimal approach to calculus is that it is easier to define and explain limits using infinitesimals".

This claim deserves examination. Of course, it is all in the axioms. I sometimes tell mathematicians whose only concern is to deduce theorems from axioms to add the axiom "$0 = 1$". They are outraged, ostensibly because that axiom would be inconsistent. What really bothers them is that it would make mathematics too easy, and put them out of business.

In the sense that Keisler has developed limits from a supposedly consistent system of axioms, they have been explained. But he has not explained the axioms. They are mere conveniences for generating proofs, whose intuitive content will certainly excape the students. If you do not believe this then read them, axioms V* and VI* in particular.

Of course, the usual notions all get defined, sooner or later, in the usual way, because calculus is about the real numbers. The book offers no evidence that the hyperreal numbers are anything except a device for proving theorems about the real numbers. They are not even an efficient device, depending as they do on axioms V* and VI*, among other things.

The technical complications introduced by Keisler's approach are of minor importance. The real damage lies in his obfuscation and devitalization of those wonderful ideas. No invocation of Newton and Leibniz is going to justify developing calculus using axioms V* and VI*–on the grounds that the usual definition of a limit is too complicated!

Although it seems to be futile, I always tell my calculus students that mathematics is not esoteric: It is common sense. (Even the notorious ε, δ definition of limit is common sense, and moreover is central to the important practical problems of approximation and estimation.) They do not believe me. In fact the idea makes them uncomfortable because it contradicts their previous experience. Now we have a calculus text that can be used to confirm their experience of mathematics as an esoteric and meaningless exercise in technique.

ERRETT BISHOP

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Basic linear partial differential equations, by François Treves, Academic Press, New York, 1975, xvii + 470 pp., \$29.50.

How, and why, would one write 470 pages on "basic" linear PDE, a subject which advanced calculus texts purport to treat in 50 or 60 pages? It is not because Treves has enlarged the stock of basic equations: the standard problems and their immediate generalizations essentially fill the book. It is not because of space spent on preliminaries: distribution theory and basic functional analysis are assumed. The answer may be found by considering another question: How does one approach a typical basic problem in a modern way?

Consider a simple "mixed initial-boundary value problem" for the heat equation. The object is, given a function $u_0(x)$, $x \in [-1, 1]$, to find a function u defined on $[-1, 1] \times [0, \infty)$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = u_0(x), \qquad u(\pm 1, t) \equiv 0. \tag{1}$$

Let us look at (1) as an ordinary differential equation for a vector-valued function. We denote by A the linear operator $(d/dx)^2$, with domain a suitable space of functions on $[-1, 1]$ which vanish at the endpoints. We let X be a space of functions containing the domain of A and the initial value u_0, and look for u: $[0, \infty] \to X$ such that

$$\frac{du}{dt} = Au, \qquad u(0) = u_0. \tag{2}$$