independently by Henriksen, as Curtis points out. Proposition 3.30, credited to me alone, is joint work with Fine, as is stated explicitly in the expository paper from which the proof is taken. Contrary to the impression given in 4.43, Fine and I obtained a dense set of remote points, not just one. Proposition 7.2, attributed to me, is due to Comfort and Negrepontis.

Finally, I have some comments about a couple of proofs. It seems heavy-handed to prove that a countable completely regular space is normal by arguing (pp. 57 and 71) that it is regular and Lindelöf, hence paracompact, hence normal–as a simple direct proof is available [**GJ**, 3B.4,5 and, more generally, 3D.4]. Next, the author should note that the result obtained in 5.21, namely, that $\beta\mathbf{N}$ is the unique extremally disconnected compactification of $\mathbf{N}$, is an immediate corollary of problem 2J(4). Finally, it is a pity to omit a proof that X^* is an F-space for locally compact and σ-compact X on the grounds that the proof in [**GJ**] is algebraic (p. 36); the original, long proof was not algebraic, and, anyhow, Negrepontis came up with a short one [Proc. Amer. Math. Soc. **18** (1967), 691–694]: to show that a cozero-set A in X^* is C^*-embedded, note that since X is locally compact X is open in βX (and in $X \cup A$), and hence X^* is compact whence A is σ-compact; since X is σ-compact, $X \cup A$ is σ-compact and hence normal; consequently, A is closed in the normal space $X \cup A$ and is therefore C^*-embedded in $X \cup A$, hence in $\beta(X \cup A) = \beta X$, hence in X^*.

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Variation totale d'une fonction, by Michel Bruneau, Lecture Notes in Mathematics, vol. 413, Springer-Verlag, Berlin, 1974, 332 + xiv pp., $12.30.

Real analysis was an active research area at the beginning of the twentieth century when mathematicians were exploring the implications of Lebesgue theory. The recent appearance of H. Federer's extraordinary book, *Geometric measure theory*, shows that there is continuing interest in this field.

The book under review is an account of recent research on variations and measures associated with functions of a real variable. Since one of these, the Wiener pth power variation, is a little known concept with applications in various branches of analysis, it may be worthwhile to list some facts about it here.

Let $1 \leqslant p < \infty$ and let T denote a subset of the real line R^1. The pth *power variation* of a function f: $T \to R^1$ is defined as

$$V_p(f) = \sup\left(\sum_1^n |f(t_i) - f(t_{i-1})|^p\right)^{1/p},$$

where the supremum is taken over all finite partitions of T. The quantity $V_p(f)$ interpolates between the usual variation and the oscillation of f on T as p varies between 1 and ∞. The function f is said to be in the *Wiener space* W_p if $V_p(f) < \infty$. Minkowski's inequality shows that W_p is a linear space; Jensen's inequality implies that $W_p \subset W_q$ whenever $p < q$.

The spaces W_p were introduced by N. Wiener **[9]** when he derived his Fourier coefficient criterion for the continuity of functions with bounded variation. They were later studied by L. C. Young in a series of papers, some coauthored with E. R. Love. In **[6]** and **[10]** it was shown that the W_p spaces exhibit almost the same conjugacy relation enjoyed by the Lebesgue L^p spaces. For example, let $T = [a, b]$ and let CW_p denote the subspace of continuous functions f in W_p with norm $\|f\| = V_p(f) + |f(a)|$. If g is in W_q, then for each p with $1/p + 1/q > 1$, the integral $\int_a^b f\,dg$ exists and defines a linear functional on CW_p. Conversely if $p > 1$ and if $L(f)$ is a linear functional on CW_p, then there exists a g in W_q with $1/p + 1/q = 1$ such that $L(f) = \int_a^b f\,dg$ for all f in $\bigcup_{r<p} CW_r$. The proofs for these results depended upon Hölder type inequalities for triangular sums of the form $\sum_1^n (\sum_1^j a_i)b_j$.

The Wiener spaces have been studied by other authors. A characterization of the moments $\int_0^1 x^n\,df$ for f in W_p was given in **[4]**, while the interpolation aspect of these spaces was used in **[2]** to define a scale of integrals spanning those of Lebesgue and of Denjoy-Perron. When $T = \mathbf{Z}$, the p-convergent series with nth partial sum in W_p interpolate between absolutely and conditionally convergent series. Summability methods and Tauberian theorems for such series are considered in **[8]**. Finally the Wiener variation for mappings f of T into euclidean n-space R^n is useful in studying the behavior of a Brownian motion in R^n. See, for example, **[7]**.

The first three chapters of the present book are devoted to problems connected with the W_p spaces. In Chapter I the author lists some preliminary results established in his thesis, and in Chapter II he studies the extremal functions f in W_p for which there exist no nonconstant functions g such that $V_p(f \pm g) \leqslant V_p(f)$. These extremals exhibit quite irregular local behavior, and an interesting analytic characterization for them is obtained in Theorem 7. This result yields relations between the extremals and other subspaces of W_p. One of these is the space of p-α fine functions; when $p > 1$ these are the functions f such that

$$(1) \qquad \limsup_{y \to x} \frac{|f(y) - f(x)|^p}{|y - x|} = \sup_{y,z \in T} \frac{|f(z) - f(y)|^p}{|z - y|} = \alpha < \infty$$

for almost all $x \in T$. The space of such functions with period 1 is studied in Chapter III. There the exceptional set of $x \in [0, 1]$ for which (1) fails to hold is related to the set of numbers $y \in [0, 1]$ which are "badly approximated" modulo 1 by increasing sequences $\{\lambda_k\}$ for which $\lambda_k \to \infty$ and

$$|f(\lambda_k) - f(\lambda_{k-1})|^p \sim \alpha(\lambda_k - \lambda_{k-1}).$$

The next three chapters deal with measures on R^1. In Chapter IV the author introduces two general measures by means of interval functions, and shows when each has the intermediate value property. In particular, Theorem 2 extends a result of A. S. Besicovitch on the existence of subsets with finite positive measure. See also [**3**] and [**5**]. The object of Chapter V is to associate with each function f: $R^1 \to R^1$ measures which extend in a natural way the Lebesgue-Stieltjes measure and the increasing, decreasing and total variation measures associated with f, when f has locally bounded variation. A relatively complete solution for this problem is obtained in Theorems 18 and 20 after about seventy pages of analysis. The author gives several characterizations for these measures and indicates that they are equivalent to measures defined in a different way by B. H. Browne [**1**]. In Chapter VI the author presents an extension of the Lebesgue decomposition theorem for arbitrary positive measures. The book then concludes with a generalization of the Denjoy-Young-Saks theorem on derivates in Chapter VII and four Appendices of comments on earlier topics.

This book appears in the Springer-Verlag Lecture Notes in Mathematics, a series which aims at quick publication of new developments in research and exposition. The present volume falls in the former category. It is an account ofpersonal research, most of which has not been published in detail elsewhere. As such it cannot offer the integrated view of an area such as one finds in the attractive books of M. de Guzmán, *Differentiation of integrals in* R^n, or of H. M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation*, both of which appeared recently in the same series.

The reviewer found the present book difficult to follow, partly because of the large number of concepts introduced, partly because proofs of many lemmas were omitted, and partly because of ambiguities and errors. For example in Chapter IV, the classes of functions h for determining the measures of Carathéodory and of Besicovitch appear to coincide, even though they are defined by means of different conditions. Concerning Chapter V, the reviewer has learned from the author that Lemma 4, Theorem 1 and Theorem 16 are false, and that the statements in Definitions 4 and 5 require additional clauses. The serious reader should request a list of the corrigenda which the author kindly provided in a copy of his book sent to the reviewer.

These notes contain interesting and original ideas. The author has also included an impressive list of forty-seven open problems on the topics he has treated. His book should prove stimulating to researchers in the area.

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Commutative formal groups, by Michel Lazard, Lecture Notes in Mathematics, vol. 443, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 236 pp., \$9.90.

This is the book we have been waiting for ever since P. Cartier's pair of notes in the Comptes Rendus of 1967. In these, Cartier sketched a thoroughgoing extension of the Dieudonné theory that had already classified commutative formal groups over a perfect field of characteristic p, in terms of modules over a certain noncommutative ring. But Cartier left the job of exposition unfinished, and Lazard has done us the service of organizing the material, filling in all the details, and adding a quantity of his own results, so that we finally have a basic reference on this aspect, probably the central aspect, of the theory of commutative formal groups.

An n-dimensional (coordinatized) formal group is simply an n-tuple $\mathbf{F} = (F_1, \ldots, F_n)$ of formal power series, subject to a single condition expressing a kind of associativity. Here, $F_i = F_i(\mathbf{x}, \mathbf{y})$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$; and $x_1, \ldots, y_n$ are $2n$ independent indeterminates. For instance, the expansion at the origin of the group law of an n-dimensional complex analytic Lie group gives rise to such series, once a coordinate system is chosen; the standard coordinatization of the one-dimensional multiplicative Lie group $\mathbf{C}^*$, for example, gives the single power series $F(x,y) = x + y + xy$.

The advantage in talking about formal groups rather than local groups is that the single relation of associativity $\mathbf{F}(\mathbf{F}(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{F}(\mathbf{x}, \mathbf{F}(\mathbf{y}, \mathbf{z}))$ makes sense algebraically, in the ring of formal power series $A[[\mathbf{x}, \mathbf{y}, \mathbf{z}]]$, where A is any commutative ring whatever. We need not restrict ourselves to the groundrings $\mathbf{C}$ and $\mathbf{R}$, not even to topological rings, and can now ask the relationship between Lie algebras over the ring A and formal groups over A. We then find that if A is a $\mathbf{Q}$-algebra, i.e. if every positive integer is invertible in A, then the categories of finite-dimensional Lie algebras over A and of