Unitary dilations of Hilbert space operators and related topics by B. Sz.-Nagy, CBMS No. 19. Amer. Math. Soc., Providence, R. I., 1974.

In 1972 V. I. Lomonosov discovered a technique which settled the longstanding problem of whether or not two commutative compact operators have a common invariant subspace. He actually proved more: If A is a compact operator, then A shares a common invariant subspace with every operator that commutes with it. In fact as he asserted, a slight modification of his proof shows that the conclusion holds if A merely commutes with a compact operator. His technique, which utilizes the Schauder fixed point theorem, was immediately seized upon by many people and used to produce even stronger invariant subspace theorems. The paper *A survey of the Lomonosov technique in the theory of invariant subspaces*, by C. Pearcy and Allen L. Shields takes us through Lomonosov's contribution to some of its consequences and also discusses the current interesting state of the invariant subspace problem. A goodly portion of this material has also appeared in the monograph *Invariant subspaces* of H. Radjavi and P. Rosenthal, Springer-Verlag, Berlin, 1973.

There is much of interest in this book. The writing is generally brisk and meets a high standard for mathematical exposition. These essays can contribute a great deal to showing students some of the areas of operator theory that have been and are still the subject of considerable research.

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Model theory, by C. C. Chang and H. J. Keisler, Studies in Logic and the Foundations of Mathematics, Vol. 73, North-Holland, Amsterdam, 1973, xii+550 pp., $26.50.

1. **General remarks.**[1] This, in many ways remarkable, book is the first attempt at a systematic exposition of a young discipline, model theory, written by two of the main contributors to the subject. Naturally, the reviewer felt tempted to seize the opportunity to give a general discussion of the subject itself but unfortunately most of his general remarks had to be eliminated to bring the review down to a size acceptable to the Editors. To appreciate another difficulty of writing this review, consider one of the most striking features of the book, and in fact of model theory itself, namely the immense variety of topics, methods and orientation. One could hardly find two subjects further apart than, e.g. Artin's conjecture on p-adic number fields on the one hand, and the theory of measurable cardinals on the other, both given full expositions in the book. And these are just two examples of the large number of similarly disparate (at least, apparently disparate) matters in

[1] The reviewer would like to express his thanks to Stephen Garland, Victor Harnik, Jan Mycielski, Gonzalo Reyes, H. Jerome Keisler and Allan Swett for their helpful criticism of the original version of this review.

the book. Hence, any reasonably complete description of the contents alone has already to be quite long.

And now we have to say that the book does not contain enough! Naturally, it is impossible to blame the authors for this. Rather, one has to "blame" the explosion in model theory that has taken place especially during the last few years. This has produced not only many new results but also it has changed our general outlook considerably. The book does not quite, and in fact, cannot possibly, do justice to the emergence of these new aspects.

The authors employ an apparently natural principle to limit their material; namely, they consider only the model theory of (finitary) first order logic. Actually, this limitation no longer seems natural, as surely many teachers of model theory realize today. (We think here of generalized quantifiers and infinitary logic in particular.) But even the model theory of first order logic in the strict sense has important aspects that cannot be much more than guessed at on the basis of the book. Perhaps a slight blame is due to the authors who write in the preface that "... this book covers most of first-order model theory". There are qualifications in the next few paragraphs of the preface but they are not quite adequate.

A good reason for the particular choices the authors have made for their material is the existence of books in the literature dealing with subjects omitted from the present book. Keisler's *Model theory for infinitary logic*, North Holland, 1971, is in fact quite close in spirit and notation to the present book and (as Keisler has said to the reviewer) should be regarded as a continuation of the present book. Later we will mention other books as well that supplement the present one in various ways.

In summary, let us then warn the reader that he should not expect an exposition of an elegant uniform single theory but rather the spectacle of several basic methods developed and applied, sometimes in combinations, into widely divergent directions. And let us add that there is much more to practically each of those directions than could possibly be dealt with in the book, but which is indispensable material for those wishing to do research in the respective area. On the other hand, let us also state that it is at least difficult to argue with the judgment of the authors as to what are the fundamental facets that have to be put into a global introduction to model theory. If we add that the authors have done a truly admirable, meticulously careful job in planning the structure and in the actual writing of their work, the reader should have a first general impression of the book.

First order logic is perhaps the central construction of all of mathematical logic. Model theory in particular, at least as presented in this book, and as initially conceived, is simply a general theory of first order logic. However, it is interesting to note that first order logic appears in a quite different light to the "metamathematician" working on the foundations of mathematics than to the model theorist. In foundations, one is interested only in a few special "foundational" structures, such as Cantor's universe of sets, the structure of natural numbers, etc., and one perceives first order logic as a practically universal language in which "everything" one is interested in can be expressed.

On the other hand, the modeltheorist is interested in all models (more precisely, in the class of all models) of a theory and it turns out that this point of view leads him to view first order logic as having little expressive power. E.g., a characteristic model theoretical theorem, the Löwenheim-Skolem theorem implies that there is no way to make a model uncountable by imposing first order axioms on it. Thus, systematic model theory at its beginnings (around 1950) did not promise to be a logical continuation of the earlier development of mathematical logic. Also, in sharp contrast to other parts of mathematical logic, and to some other parts of mathematics as well, model theory did not start out with the aim of solving philosophically or otherwise intriguing specific problems. Rather, model theory for its existence depended on some basic phenomena discovered earlier plus an essential amount of enthusiastic philosophical faith in the intrinsic interest and in the potential usefulness of a general study of the semantics of first order logic. Alfred Tarski is the one who undoubtedly is responsible, both by his work and by his direct influence on his students (who are key figures of model theory today), for the emergence of model theory as we see it today. A particularly important feature that is connected to Tarski's influence is the decisive set-theoretical orientation of model theory.

To start with, model theory inherited two fundamental results which should in our view both be considered to belong to model theory proper but which had been obtained at times when systematic model theory did not exist yet. One is the Löwenheim-Skolem theorem, the other is Gödel's completeness theorem. The first is a 'purely semantical' one in the sense that it makes reference to the fundamental semantical notion of truth of a sentence in a model, it also involves a structural property of models, namely, the cardinality of the underlying set, but it makes no reference to the syntactical structure of formulas. Now, the adjective 'model theoretical' is sometimes used synonymously with 'semantical' and the use signifies the absence of reference to syntax. From this point of view, the Löwenheim-Skolem theorem belongs to model theory, but the completeness theorem does not. Though we do not agree with this point of view, it is easy to see why it has evolved. Mathematical logic as it became defined in its first era of maturity before around 1950 was primarily concerned with formalized proof procedures and it had a general tendency, and sometimes the explicit goal, of eliminating semantical considerations from the investigation of syntactical structure. For the emancipation of model theory it was important to emphasize the relative independence of the semantical aspect in view of the fact that other parts of mathematical logic had claimed self-sufficiency of the syntactical aspect. But it seems to this reviewer that today model theory is better defined as the investigation of the interplay of semantics and syntax than as a study of semantics alone. In this view, Gödel's completeness theorem is the most typical result of all of model theory. This position should be seen partly as a result of recent shifts of emphasis and interests in model theory. E.g., Keisler (Ann. Math. Logic, 1970) has proved a completeness theorem for the logic with the quantifier "there are uncountably many" that has the same form as Gödel's completeness theorem. It seems hardly possible to exclude this result from model theory proper (of a logic extending first order logic, of course), e.g., in view of the advanced use of the typical methods of model theory in the proof. As another example, in so-called infinitary first order logics on admissible sets syntactic notions that are generalizations of 'recursive' and 'recursively enumerable' are indispensable for formulating the results.

It is quite fortunate that the completeness theorem is treated fully in the present book despite the fact that it might have been tempting to take a shortcut to its 'purely semantical' consequence, the compactness theorem since completeness as it is not used (except once in Chapter 6) again in the book. Nevertheless, in the book there are traces of the view excluding syntax from model theory. E.g., questions of decidability (most typically a syntactical notion) are not considered to belong to model theory (p. 49). Actually, the classical proofs of *un*decidability have nothing to do with semantics, and so, with model theory either. But model theory provides excellent examples of proofs of decidability of theories and such decidability proofs have played an important role in model theory. E.g., A. Robinson's theory of model completeness was developed for giving a model-theoretical approach to the completeness *and* decidability of 'specific' theories such as that of algebraically closed fields or real closed fields. Incidentally, in those proofs the abstract completeness theorem, a not purely semantical consequence of the original formulation: the set of logically valid sentences is recursively enumerable, plays an important role.

As a matter of fact, model theory is full of results that are at least partly syntactical and the book gives an ample sampling of them.

To return to the origins of model theory, the Löwenheim-Skolem theorem and the compactness theorem stand out clearly as the foundations. Much later it turned out that, by a fundamental result of Lindström (Theoria **35** (1969), 1–11), first order logic is "characterized" by these two theorems, a fact that is almost ironic in its fittingness.

To give a very rough classification of model theory, we note that, in retrospect, model theory seems to have had two main lines of inquiry, the first of which we would like to call generalized algebra, the other descriptive formal logic. Very roughly speaking, the emphasis in generalized algebra is on models as opposed to formulas, whereas in descriptive logic, the situation is the reverse. Of course, no claim is being made about the possibility of a rigid division, but we nevertheless feel that the distinction is quite real. The word "algebra" in the first phrase refers to "modern algebra" conceived as a "theory of structure" and not the earlier meaning of the "formal aspect of mathematics" that (confusingly) would have more to do with the second subdivision!

2. **The plan of the book.** The book has seven chapters. The first five (with the possible exception of parts of §§5.4 and 5.5) form the basic part of

the material, the rest being a glimpse into advanced model theory. After the introductory Chapter 1, the basic part is organized around the following methods of model constructions:

1. Models constructed from constants (Chapter 2). 2. Elementary substructures and elementary chains (§§3.1 and 3.2). 3. Skolem functions and indiscernibles (§3.3). 4. Ultraproducts (Chapter 4). 5. Saturated and special models (Chapter 5).

3. **The first two groups of constructions.** The first fundamental theorem is the Compactness Theorem, Theorem 1.3.22, saying that a set of sentences Σ has a model iff every finite subset of Σ does. It is derived as a consequence of the Gödel completeness theorem that, in turn, is proved by Henkin's 1949 method. Henkin's method constructs the desired model M such that the underlying set (domain) of M is essentially a set of individual constants (formal symbols capable of denoting individual elements of models), hence the title of Chapter 2. Actually, Henkin's method as well as a related method, the method of diagrams (p. 68) can be more appropriately described by saying that the model is constructed from formulas; in the case of Henkin's method, a full 'description' of a model is constructed in the form of a set of sentences. The compactness theorem is the most frequently used theorem of model theory.

The second fundamental result is the (extended) Omitting Types Theorem (Theorem 2.2.15) that we decline to state here. It is an extremely useful result, despite the fact that it concerns countable models only.

The notion of *elementary extension* was introduced by Tarski and Vaught. An extension B of A is an elementary extension of A if every finite sequence of elements in A satisfies the same formulas in B as in A. The third basic result is the downward Löwenheim-Skolem-Tarski theorem that in a special case says that any infinite structure of a countable language has a countable elementary substructure. By an application of the Compactness Theorem, this can be extended to the full Löwenheim-Skolem-Tarski theorem (Theorems 3.1.5 and 3.1.6 together). The fourth fundamental result is Theorem 3.1.13, the Tarski-Vaught elementary chain theorem: the union of a directed system of structures in which every "smaller" one is an elementary substructure of a "greater" one is an elementary extension of each structure in the family.

Let us look at some applications of the fundamental results listed so far. There are, first of all, some rather direct applications to special situations, such as the existence of "nonstandard" models of various theories. A first coherent group of model theoretical results constitutes a descriptive theory of countable models of countable complete theories (and is most typically a part of "generalized algebra" as we understand this term) (§§2.3 and parts of 3.2). The special kinds of countable models that are isolated in the discussion are the prime, countably saturated, countably universal, and countably homogeneous models. The most striking results are Ryll-Nardzewski's characterization of $\aleph_0$-categorical theories and Vaught's theorem that a complete countable theory cannot have exactly two nonisomorphic countable models (but can have exactly 1, 3, 4, etc.).

Under a second heading fall the two-cardinal theorems. The two-cardinal theorem of Vaught says that if a countable theory has a model $(A, U, \cdots)$ with card $A>$card $U \geqq \aleph_0$, then it has a model $(B, V, \cdots)$ with card $B=\aleph_1$, card $V=\aleph_0$. This theorem is a downward Löwenheim-Skolem type theorem but it is much more difficult to prove (the 'substructure' version, requiring that the second model be a submodel of the first, cannot be proved in ZFC, Zermelo-Fraenkel set-theory with the axiom of choice, cf. §7.4). There are two distinct refinements of this theorem, one being the actual (stronger) result proved by Vaught (Corollary 3.2.13), the other the two-cardinal theorem of Keisler (3.2.14). Vaught's proof evolved from his notion of homogeneous models. Keisler's proof is entirely different and relies on the Omitting Types Theorem. There is one feature of Keisler's proof that often arises in other situations and that constitutes the main idea in many proofs. This has been called the *method of expansions.* With this, one exploits (expresses part of) a hypothesis made on a structure by introducing new predicates that, in combination with the old ones, satisfy interesting first order properties. In Keisler's proof, one has a structure A of power α^+ and a distinguished subset U of power α. One introduces a predicate denoting a well ordering of A of order type α^+. The interesting (useful) first order properties will be those that express that the whole structure is not cofinal with any (first order definable) sequence indexed by elements of U (a consequence of the regularity of α^+). These results, besides being interesting themselves, are also very useful (often in ways going beyond the material of the book).

A third topic is model completeness (pp. 110–115), a notion introduced by A. Robinson, which is very important for applications to algebra. A theory T is model complete if for models A and B of T, if A is a submodel of B, then A is an elementary submodel of B. The primary example is the theory of algebraically closed fields.

A fourth group of results belongs to "descriptive logic". This includes Craig's interpolation theorem, its application to the proof of Beth's theorem, preservation theorems for substructures, homomorphisms, and unions of chains of models.

4. **Indiscernibles.** The notion of order indiscernibles due to Ehrenfeucht and Mostowski (1956) is perhaps the most typical of model-theoretical ideas in its being a mixture of algebraic and set-theoretical elements (if one regards linear orderings as "set-theoretic"). Let A be a structure, X a subset of A, $<$ a linear ordering of X ($<$ is not necessarily a distinguished relation of A). We call $(X, <)$ a set of (order) indiscernibles of A if any two n-tuples of elements of X, ordered increasingly by $<$, have the same first order properties in A. An important but obvious point about this notion is that it "has a finite character" meaning that $(X, <)$ is a set of indiscernibles in A iff every finite subset of X is, "relative to any finite set of formulas" in the

obvious sense. This enables one to use the compactness theorem to construct models with indiscernibles and with or without additional properties. Let us note that there are other notions related to indiscernibles with a similar finite character; these are certain 'patterns' of elements consisting of a set (such as X above) and having some prescribed behavior with respect to formulas. Many such patterns occur in Shelah's work on stability.

The fundamental general theorem on order indiscernibles is Theorem 3.3.11 that concerns so called theories with built-in Skolem functions. For example, a part of Theorem 3.3.11, the stretching theorem tells us how to generate models with arbitrary ordered sets as indiscernibles once one is given with an infinite set of indiscernibles.

The decisive step in proofs using indiscernibles with or without special properties is to create them outright, or, to create their arbitrarily large finite "subpatterns". In the simplest case when one is interested in the mere existence of a model with infinitely many indiscernibles, for theories having infinite models, it turns out that the finite patterns can be created by using Ramsey's famous 1930 combinatorial theorem. Together with the fundamental properties of indiscernibles (Theorem 3.3.11), this construction already gives very good results. One is the Ehrenfeucht-Mostowski result on the existence of models with many automorphisms (3.3.13), another is Morley's theorem on the existence of models realizing 'few' types over 'small' subsets (3.3.14). The latter uses a set of indiscernibles that is well ordered (although the conclusion has nothing to do with such matters). Below we will see how indiscernibles with special properties are constructed and used.

Skipping Chapter 4 for a moment, let us turn to

5. **Saturated models (Chapter 5).** Given a model A and a subset X of A, a set $\Sigma(v)$ of formulas $\phi(v)$ with the single free variable v but with (names for) elements in X might be considered as a specification of the behaviour of an undetermined and hypothetical element v with respect to X. $\Sigma(v)$ is realized in A if there actually is an element in A satisfying each formula in $\Sigma(v)$. $\Sigma(v)$ is a *type* over X in A if (i) it is consistent in the sense that every finite subset of it is realized in A and (ii), it is complete, i.e. maximal among consistent such sets. Structures that are *saturated* to a certain degree are those that realize all their types of a certain kind. In the case of the α-saturated structures, for a cardinal α, these are the types over sets X of power less than α. A structure is (simply) saturated if it is α-saturated in its own power α.

One of the basic theorems of the theory of saturation is the existence theorem of α-saturated models in certain powers. Unfortunately, (simply) saturated structures can be proved to exist in sufficient generality only if one assumes the generalized continuum hypothesis (GCH), or else, the existence of inaccessible cardinals. This is unfortunate because saturated structures have very nice properties. One is their uniqueness, meaning that they are determined up to isomorphism by their first order theory and their cardinality.

A rather more complicated notion (that is, more complicated to use) is the notion of special model which, however, does have most of the nice properties (including uniqueness) and also exists in abundance. Notions of saturation are closely related to homogeneity and universality (treated in §5.1) which are the basic notions for the first systematic theory of the subject given by Morley and Vaught in 1962.

As far as the aesthetic appropriateness of the means for the aim is concerned, perhaps the best applications of saturated and special models are to definability theory (§5.3). Here one can exploit very effectively the possibility given in saturated structures of making compactness arguments without moving out of the given structure. Definability theory has evolved from Beth's 1953 theorem and is the most typical part of "descriptive logic". Interest in it has not decreased up to now when it is being done very effectively in infinitary logic. There are many elegant results in the field that also have the advantage of being intrinsically more interesting than most other results of the descriptive theory.

Other applications given in §5.2 of these tools are new proofs of previous preservation theorems as well as improvements of them. A fairly large body of results concerning intersections of models is included here, partly in the exercises. These contain ingenious applications of the method of expansions (see above).

§5.4 contains some of the representative applications of model theory to algebra. Theorem 5.4.4 is Tarski's theorem saying that the theory of real closed fields is complete (i.e., any two real closed fields satisfy the same sentences). The proof given in the book also establishes a result due to Erdös, Gillman and Henriksen (1955) stating that any two real closed fields whose order structures are isomorphic saturated orderings of a successor cardinality are isomorphic. To infer Tarski's theorem from the last statement one needs the GCH (to show the existence of saturated models). Using an argument from axiomatic set theory (due to Gödel), on the basis of the logical form of Tarski's theorem, one sees however that once Tarski's theorem is established using the GCH, it follows that it is a theorem of ZFC. This curious argument is used repeatedly in the same chapter, and in a more essential way, in Chapter 6. As the authors remark, Tarski's theorem itself could be established quite similarly by the use of special models without a detour via GCH but at the expense of more work in the main part of the proof. This remark applies to the rest of Chapter 5 too but apparently not to Chapter 6. There are several different treatments in the literature, mostly quoted in the book, of the material of Chapter 5.

The fundamental theorem of Ax and Kochen, and independently, of Ershov, is somewhat analogous to Tarski's theorem but is more complex. It states that a so-called Henselian valued field with cross section having a residue class field of characteristic 0 has a first order theory that is determined by the first order theory of the residue class field and that of the value group (Theorem 5.4.12). The proof (besides having a very similar outline to that of Tarski's theorem) is full of algebraic details, many of them only quoted. The famous application is an 'almost affirmative' solution to Artin's conjecture, viz. the result that for each positive integer d there exists a finite set ψ of primes such that for every prime $p \notin \psi$, every polynomial with more than d^2 variables of degree d over the field Q_p of p-adic numbers having zero constant term has a nontrivial zero in Q_p (5.4.19).

The last section is on the first order theories of Boolean algebras and it is a nice example of the use of model theory for a complete analysis of a concrete situation. It should be compared to Chapter 1 where the elementary "method" of the elimination of quantifiers is used for direct analyses of similar situations. The results of this section are used in Chapter 6. In the exercises, the student is asked to perform such analyses, sometimes with different methods.

6. **Ultraproducts and generalizations (Chapters 4 and 6).** Ultraproducts stand rather apart from the concerns of the parts of model theory related above. The basis of interest in the operation of ultraproduct is the fact that it is an 'algebraic' operation that preserves elementary properties, a fundamental discovery of Łós (1954). One defines the ultraproduct of a family $\langle A_i : i \in I \rangle$ of structures A_i, modulo an ultrafilter D of the Boolean algebra of all subsets of I to be the quotient of the full Cartesian product of the A_i modulo the equivalence defined as the equality of elements of the Cartesian product for 'almost all' (in the sense of D) indices $i \in I$.

From Łós' theorem (Theorem 4.1.9), the main interest in ultraproducts is almost immediate and it lies in the possibility of characterizing model-theoretical notions (such as 'elementary class') purely algebraically (i.e., by eliminating reference to formulas). This actually can be done to a remarkable extent, not readily foreseeable from Łós' theorem. Another possibility is to exploit the special properties of models constructed as ultraproducts. It is probably fair to say that ultraproducts applied for the latter purpose are less interesting, at least from the point of view of the usual concerns of model theory. However, a very important and very special kind of application of ultraproducts belonging to the category in question is made for models of set theory. It is characteristic of the spirit of the book that the applications of ultraproducts to the study of measurable cardinals are presented in the 'basic' part of the book. The results presented in 4.2 constitute one of the two breakthroughs of set theory achieved around 1960, and together with later developments it constitutes a fundamental contribution to our metamathematical knowledge. The Hanf-Tarski result of the 'largeness' of measurable cardinals (Theorem 4.2.14) can be (and was first) proved without ultraproducts but the stronger result, 4.2.23, seems to depend on ultraproducts more essentially. This latter result also includes the statement that measurable cardinals (if they exist) are much greater than the first so-called weakly compact cardinal whose largeness was originally shown by Hanf. Theorem 4.2.23 is based on a truly great theorem, 4.2.21, of the kind that is both powerful as far as consequences are concerned and striking as it is that states that for a measurable cardinal α the model R_α of all sets of rank $<\alpha$ with the ε-relation is isomorphic to the ultraproduct of the models R_β ($\beta<\alpha$) modulo a normal ultrafilter.

The power of ultraproducts is also demonstrated by the proof of Scott's theorem to the effect that the axiom of constructibility implies that there is no measurable cardinal, a result that has important refinements discussed in Chapter 7. The subject of 'large cardinals' (a phrase referring to various conditions on cardinals ensuring their being 'large', i.e. inaccessible and more) receives a remarkably complete coverage in the book despite the fact that it does not belong to model theory in the strict sense. In §4.2 and in the related exercises the reader learns about measurable, weakly compact, and various 'indescribable' cardinals, with many details of interesting facts.

Let us return now to what we described as the first source of interest in ultraproducts. After initial results (presented in 4.1 and 4.3), the definitive results are given in 6.1 and 6.4. The reasons for this procedure of putting this material into the 'advanced' part is the considerable technical sophistication of the proofs. The main result here is the "isomorphism theorem" (Theorem 6.1.15): two structures satisfy the same sentences iff they have isomorphic ultrapowers. This theorem has an interesting history. Keisler proved it first using the GCH in 1961 and for the purposes of the proof he (later) isolated the notion of a 'good' ultrafilter. One (but not the only) problem left was proving the existence of good ultrafilters without the GCH. This was settled in the positive by Kunen (1973) (Theorem 6.1.4). Considerable further work by Shelah was needed to prove the isomorphism theorem without GCH. The various kinds of ultrafilters (regular, good, etc.) are covered to a remarkable extent partly in the exercises. These present some interesting unsolved problems, heavily set-theoretical in nature.

Without going into details about the contents of 6.4 and 6.5, let us note that it is possible to formulate the results in 6.4 in a way (that the reviewer learned from A. Joyal) that brings them neatly in line with the idea of algebraic characterization of first order "operations". 6.5 is on iterated ultraproducts whose main applications are in set-theory.

Under this heading, there remains to discuss §§6.2 and 6.3 (also belonging to 'advanced' model theory) which are quite different from the previous ones and in fact describe quite curious matters. The material here originates from some three sources. One is Keisler's theorem that characterizes up to logical equivalence those sentences that are preserved under reduced products (similar to ultraproducts but with an arbitrary proper filter in place of an ultrafilter) as the so-called Horn sentences (prenex sentences with matrices that are conjunctions of implications whose antecedent is a conjunction of atomic formulas and whose succedent is a single atomic formula). Actually, this theorem was proved by Keisler using the GCH through the use of saturated models, and the use of GCH was later eliminated by Galvin. Galvin's proof is based on the same principle as the elimination of GCH from the proof of Tarski's theorem mentioned above but it is much more complex. This brings us to the other source, upon which also Galvin's work is ultimately based. This is the 'early' but in itself quite comprehensive and complete work of Feferman and Vaught (1959) on 'generalized products'. This work is not treated in full in the book but it should be mentioned as one of the most influential papers of model theory, both completing a long line of investigations and giving rise to new ones. The gist of the Feferman-Vaught paper is a very elementary method, the elimination of quantifiers (Chapter 1), lifted to a higher level and giving results of a very general nature on the dependence of the first order theories of results of many algebraic operations ('products') on those of the 'factors'. The work of Weinstein and Galvin ('autonomous sets of formulas'), although in spirit similar to the Feferman-Vaught work, was a new twist and it applied to the more special situation of direct products and reduced products only; this work is reproduced to a large extent in 6.3. Besides the elimination of GCH, the most striking result emerging is Galvin's theorem (amazingly, not stated in these words in the book!) stating that every sentence is logically equivalent to a Boolean combination of Horn sentences (6.3.18 and 6.2.5′ jointly). The reviewer does not know of a proof of this purely syntactic result that does not go along something like the long and tortuous way traversed in the book. (A third source is the analysis of elementary theories of Boolean algebras, §5.4, which gives rise to Ershov's theorem, 6.3.20.)

7. **More of 'advanced' model theory (Chapter 7).** Chapter 7 contains varied material and is the result of an effort to give a view of further topics of 'advanced' model theory. §§7.1 and 7.2 start developing themes that are at the heart of model theory as commonly understood, whereas §§7.3 and 7.4 are, to some extent, in the vein of applications of model-theoretic ideas to set theory. Compared to the first two sections, the last two are understandably more complete since what comes after those contributions to set theory can no longer be considered model theory. But the first two sections just touch the tips of (two) icebergs in the middle of model theory.

§7.1 is on Morley's work on categoricity. A theory T is called categorical in power α if any two models of power α of T are isomorphic. Łós conjectured in 1954 that if a countable theory is categorical in one uncountable power, it is in any other. Morley proved this conjecture and in the process, he introduced many important tools and concepts, like the property of a theory being ω-stable. The treatment in the book is the result of a gradual simplification of Morley's proof, but it is still quite involved. The basic tools in 7.1 are Keisler's two-cardinal theorem (3.2.14) which, through the "method of expansions" allows one to conclude e.g., that if a countable theory has a nonsaturated model in an uncountable power then it has one in power ω_1 (this is the key to the proof that if the theory is categorical in $\aleph_1$, it is in all higher powers). Another tool is Morley's theorem (3.3.14) on models realizing 'few types' which is used to establish that a theory categorical in an uncountable power is ω-stable. The main use of ω-stability is to develop a relative version of the theory of atomic (prime) models, also no longer confined to countable models as before. Using the latter theory, an ingenious idea of Baldwin and Lachlan gives a direct way to construct arbitrarily large nonsaturated models, given that there is one in power $\aleph_1$, which establishes the other 'direction' of Morley's theorem.

Indiscernibles that figured prominently in Morley's work (e.g. in the last part of the proof) are almost eliminated in the above proof (except for the implicit use through Morley's theorem). Also, the concept of rank of transcendence is eliminated. There is some indication of this latter notion as an addendum to the proof of the categoricity theorem.

For the reader who is interested in related subjects, G. Sacks' book *Saturated model theory* (Benjamin, 1972) is warmly recommended. This work develops the general theory of ω-stability and uses ranks of transcendence heavily; it contains more on categorical theories and interesting connections to differential fields. It should be pointed out further that "stability theory" (dealing with notions related to ω-stability) has been vastly developed mainly by Shelah but also by others. It has turned out that this theory has relevance to questions of a more general nature than categoricity, viz. the determination of the cardinal number of nonisomorphic models in a given power and the possibility of a structure theory for models of a given theory.

§7.2 contains three, both historically and theoretically, fundamental theorems. The first, Theorem 7.2.2, is Morley's theorem on omitting types, which determines the cardinal ("Hanf number") α such that for any countable theory T and any type $\Sigma(v)$, in order to conclude that there are arbitrarily large models of T omitting Σ, it is enough to know that T has a model in each power less than α that omits Σ. α is $\beth_{\omega_1}$, the $\omega_{1\mathrm{st}}$ iterated power of $\aleph_0$. Morley's proof of this theorem was probably suggested by his earlier work on categoricity since in both cases there is a similar use of indiscernibles that, through the "stretching theorem", give a way of "generating" arbitrarily large models. Compared to the Ehrenfeucht-Mostowski work, the basic 'pattern' now, however, is more difficult to construct because we have to make sure that the models generated by the indiscernibles will actually omit the type Σ. The tool is a transfinite generalization of Ramsey's theorem, the Erdös-Rado partition theorem.

The true context of Morley's theorem is infinitary logic as explained in Exercises 7.2.13.

The second theorem is another two-cardinal theorem of Vaught, given here with a proof found by Morley, again using indiscernibles in a similar way. The last of the three theorems is a two-cardinal theorem of Chang that says if a countable theory T has a model $(A, U, \cdots)$ such that card $A >$ card $U \geqq \aleph_0$ and α is a regular cardinal such that $2^\alpha = \alpha^+$, then T has a model $(B, V, \cdots)$ such that card $B = \alpha^+$ and card $V = \alpha$. The proof of this theorem uses a surprising and ingenious trick of 'coding' ('expansion') and a relativized version of saturation.

The 'iceberg', the tip of which is §7.2, consists of extensions of model theory to infinitary logic and generalized quantifiers. Vaught's and Chang's two-cardinal theorems were directly applied by Fuhrken (Fund. Math. **54,** 291–302 and *The* *theory of models*, North-Holland, 1965, 121–131; for an exposition, see also Bell and Slomson, *Models and ultraproducts*, North-Holland) to generalized quantifiers. For the same purposes Fuhrken also used the important notion of α-like orderings mentioned in Exercises 7.3.42–7.3.49 of the book. After Fuhrken, several people including Keisler and Shelah made important contributions to the subject.

The book mentions (without proofs, of course) some interesting developments that took place in axiomatic set-theory in relation with two-cardinal theorems.

The final two sections, 7.3 and 7.4, of the book, return to subjects of a more metamathematical interest. The main results presented here are statements of the effect of the existence of large cardinals on the structure of the class of constructible sets, culminating in Theorem 7.4.7, a result of successive approximations by Gaifman, Rowbottom and Silver. But even this purely set-theoretic result is derived more or less directly from general model-theoretical theorems; thus here we have genuine applications of model theory to set theory.

The notion of a Ramsey cardinal is introduced; Ramsey cardinals are 'large' cardinals 'between' weakly compact and measurable cardinals. The two main results of §7.3 are Rowbottom's theorem 7.3.16 and Silver's theorem 7.3.18; both are downward Löwenheim-Skolem results and assert the existence of special kinds of elementary submodels of models whose power is a Ramsey cardinal. Silver's theorem is a much more elaborate result than Rowbottom's earlier result. There are numerous other results, partly on Jonsson models (models not having proper elementary submodels of their own power), partly on "Chang's conjecture" that is the problem of downward two-cardinal theorems holding in the strong "substructure" version. In 7.4 it is shown among others that "Chang's conjecture" does not hold in the constructible universe for any nontrivial combination of the cardinalities involved.

8. **Concluding remarks.** As we said above, the book is written with extreme care in all respects, reflecting the styles of both of the authors in their other writings. Great care was taken to ensure that each chapter and each section be a balanced unit, with its due share of 'basic' work and 'deep' insights. Many hundreds of exercises, ranging "from extremely easy to impossibly difficult", complete the material presented in the text. In their other writings the authors have been instrumental in developing the flexible and elegant modes of notation presently widely used in model theory; they will be even more widely popularized through the present book (and can be appreciated best by trying to read model theory *not* written with this notation). There are very complete historical remarks.

As is obvious, it must have been very difficult to decide on the material to be taken up in the book. At any rate, as a result of the authors' choices, one feels that the rather more essentially set-theoretical parts of model theory, such as ultraproducts and models of large cardinality received a better treatment than the rest. One feels this not so much because of the relative amount of space devoted to the various subjects but rather because these 'set-theoretical' results are more final and self-sufficient in character than the others, some of which sometimes appear to be somewhat technical and not as much justified in themselves.

Many things (Keisler's two-cardinal theorem, Morley's "Hanf-number" theorem on omitting types) could have been put in their true contexts only in extensions of first order model theory (generalized quantifiers, infinitary logic).

A particular matter that should have received more attention in the book is Fraissé-Ehrenfeucht games (and some generalizations). These are treated only in exercises. These games are important, particularly through the work of Lindström who applied them to give a theory of preservation theorems ("regular relations") (Theoria **32** (1966), 171–185), and to his celebrated work on characterizing first order logic.

It should be added to the discussion of transcendence rank that the rank of a formula defined in a not necessarily ω_1-saturated model is simply taken to mean rank in any (cf. Lemma 7.1.20) ω_1-saturated elementary extension. Then the first sentence of the proof of 7.1.23 can be deleted, and it should be because as it stands it is incorrect. Furthermore, the proof of 7.1.23 uses the fact that α is regular (and Victor Harnik tells us that the theorem is false without this assumption).

The proof of 7.2.2 is written up in a somewhat awkward way, and in fact, the induction hypothesis (4) is not stated correctly. In the proof of 7.3.7, the definition of the structure A was omitted (but can be guessed). On p. 480 the numerical code (1) should be shifted to the next displayed formula.

In the review, the reviewer could not bring himself to suppressing the use of the word "structure" in favor of the word "model" as it is done in the book.

In conclusion, let us say that in this book model theory has received a thoroughly worthy exposition that will no doubt help establish the deserved status of model theory as an original, rich, useful and mature branch of mathematics.

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Mathematical theory of dislocations and fracture, by R. W. Lardner, University of Toronto Press, Toronto, 1974, xi+363 pp.

The mathematical theory of elasticity has a rich and varied history. It is concerned with the mathematical study of the response of elastic bodies to the action of forces. There is no doubt that the linear theory is one of the more successful theories of mathematical physics. A beautiful account of this theory is found in Gurtin (1970).

The first attempt to set the elasticity of bodies on a scientific foundation was undertaken by Galileo and is described in his Discourses, published in