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GENERALIZED ZETA-FUNCTIONS FOR AXIOM A BASIC SETS

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Let X be a set, $f: X \mapsto X$ a map, $\varphi: X \mapsto \mathbf{C}$ a complex-valued function. We write formally

$$D(\varphi) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} \varphi(f^k \xi)\right]$$

Taking φ constant, i.e. replacing φ by $z \in \mathbf{C}$, we can interpret $1/D(z)$ as a zeta-function proved to be rational for Axiom A diffeomorphisms by Guckenheimer and Manning [6].

Similarly, if (f^t) is a flow on X, we write formally

$$d(A) = \prod_{\gamma} \left[1 - \exp \int_0^{\lambda(\gamma)} A(f^t x_\gamma)\, dt\right]$$

where the product extends over the periodic orbits γ of the flow, $\lambda(\gamma)$ is the prime period of γ and x_γ a point of γ.

In this note we indicate analyticity properties of $A \longrightarrow D(e^A)$ or $A \longrightarrow d(A)$ for diffeomorphisms or flows satisfying Smale's Axiom A, assuming only that A is Hölder continuous. Our results hold in particular for Anosov diffeomorphisms and flows, and when A is C^1. Stronger properties of meromorphy hold under suitable assumptions of real-analyticity and will be published elsewhere by P. Cartier and the author.

Let Λ be a basic set for a C^1 diffeomorphisms or flow satisfying Smale's Axiom A (see [13]). Choosing a Riemann metric d, and $\alpha \in (0, 1)$ we let C^α be the Banach space of real Hölder continuous functions of exponent α, with the norm

$$\|A\|_\alpha = \sup\left\{|A(x)| + \frac{|A(y) - A(x)|}{(d(x, y))^\alpha} : x, y \in \Lambda \text{ and } x \neq y\right\}$$

We denote by $C^\alpha_{\mathbf{C}}$ the corresponding space of complex functions.

1. THEOREM. *Let the Axiom* A *diffeomorphism f restricted to the basic set Λ be topologically mixing. We denote by $P(A)$ the (topological) pressure of a real continuous function A on Λ (see* [8], [14], [4]). *There is a continuous real function R on $C^\alpha_{\mathbf{C}}$ satisfying*

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$$R(A) \geqslant \exp[-P(\operatorname{Re} A)] > 0,$$

$$R(A + c) = e^{-\operatorname{Re} c} R(A) \quad \text{when } c \in \mathbf{C}$$

and such that

(a) *if* $A \in \mathcal{C}^\alpha_{\mathbf{C}}$, *the following power series in* z,

$$D(ze^A) = \exp\left[-\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \exp \sum_{k=0}^{m-1} A(f^k x)\right]$$

converges for $|z| < R(A)$. *The function* $A \longmapsto D(e^A)$ *is analytic in* $\{A \in \mathcal{C}^\alpha_{\mathbf{C}} : R(A) > 1\}$.

(b) *If* $A \in \mathcal{C}^\alpha$, *then* $R(A) > \exp[-P(A)]$, *and* $z \longmapsto D(ze^A)$ *has only one zero in* $\{z : |z| < R(A)\}$. *This zero is simple and located at* $\exp[-P(A)]$.

We shall also write P_f, R_f, D_f instead of P, R, D, to indicate the dependence on f.

We outline the proof of Theorem 1. First suppose that (Λ, f) is a subshift of finite type (see [13]). Then the theorem can be proved by the "transfer matrix" method of statistical mechanics (see [7], [1], [12], [9], [10]). The general case reduces to that one: using a Markov partition for Λ (see [11], [2]) one can, by a combinatorial lemma of Manning [6], write

$$D_f(ze^A) = \prod_{i \in I} [D_{\tau_i}(ze^{A \circ \pi_i})]^{s_i}.$$

In this formula the index set I is finite, $s_i = \pm 1$, the τ_i are shifts acting on spaces Ω_i and the $\pi_i : \Omega_i \longmapsto \Lambda$ are Holder continuous maps such that $\pi_i \tau_i = f \pi_i$. Furthermore there is an index $1 \in I$ such that $s_1 = +1$ and

$$P_f = P_{\tau_1} \circ \pi_1 > P_{\tau_i} \circ \pi_i \quad \text{if } i \neq 1$$

[π_1 defines the symbolic dynamics associated with the Markov partition; therefore $P_f = P_{\tau_1} \circ \pi_1$ (see for instance [4]). If $i \neq 1$, $\pi_i \Omega_i \neq \Lambda$ and therefore the pressure of f restricted to $\pi_i \Omega_i$ is $< P_f$. This gives bounds on f-periodic points in $\pi_i \Omega_i$, and therefore on τ_i-periodic points in Ω_i, implying $P_{\tau_i} \circ \pi_1 > P_{\tau_i} \circ \pi_i$]. The conditions of the theorem are satisfied if we take

$$R_f(A) = \min\left\{ R_{\tau_1}(A \circ \pi_1), \min_{i \neq 1} \exp(-P_{\tau_i}(\operatorname{Re} A \circ \pi_i))\right\}.$$

COROLLARY. *P is a real-analytic function on* $\mathcal{C}^\alpha$; $e^{P(A)}$ *is the radius of convergence of the series*

$$\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in K_m} \exp \sum_{k=0}^{m-1} A(f^k x)$$

where K_m *consists of the f-periodic points of prime period m.*

2. THEOREM. *Let* Λ *be a basic set for an Axiom A flow* (f^t). *We denote by* $P(A)$ *the topological pressure of a real continuous function* A *on* Λ *(see* [5]). *There is a continuous real function* $r \geqslant 0$ *on* $C_{\mathbf{C}}^{\alpha}$ *such that*:

(a) *if* $A \in C_{\mathbf{C}}^{\alpha}$, *the product*

$$d(A-u) = \prod_{\gamma}\left[1 - \exp\int_0^{\lambda(\gamma)}(A(f^t x_\gamma) - u)\,dt\right]$$

is convergent for $\operatorname{Re} u > P(\operatorname{Re} A)$ *and extends to an analytic function of* u *for* $|u - P(\operatorname{Re} A)| < r(A)$. *The function* d *is analytic in* $\{A \in C^{\alpha}: P(\operatorname{Re} A) < r(A)\}$;

(b) *If* $A \in C^{\alpha}$, *then* $r(A) > 0$, *and* $u \longmapsto d(A-u)$ *has only one zero in* $\{u: \operatorname{Re} u > P(A)$ *or* $|u - P(A)| < r(A)\}$. *This zero is simple and located at* $P(A)$.

The proof is based on a technique of counting periodic orbits due to Bowen [3, §5].

COROLLARY. *P is a real-analytic function on* C^{α}; $P(A)$ *is the abscissa of convergence of the Dirichlet series* $\Sigma_\gamma \exp \int_0^{\lambda(\gamma)}(A(f^t x_\gamma) - u)\,dt$.

REMARK. The functions $z \longmapsto D(ze^A)$ of Theorem 1 and $u \longmapsto d(A-u)$ of Theorem 2 do not in general extend to meromorphic functions in the whole complex plane. Counterexamples have been constructed by G. Gallavotti (private communication).

REFERENCES

1. H. Araki, *Gibbs states of a one dimensional quantum lattice*, Comm. Math. Phys. 14 (1969), 120–157. MR 42 #8834.
2. R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. 92 (1970), 725–747. MR 43 #2740.
3. ———, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. 95 (1973), 429–460. MR 49 #4041.
4. ———, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms* (to appear).
5. R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math., 29 (1975), 181–202.
6. A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. 3 (1971), 215–220. MR 44 #5982.
7. D. Ruelle, *Statistical mechanics of a one-dimensional lattice gas*, Comm. Math. Phys. 9 (1968), 267–278. MR 38 #3013.
8. ———, *Statistical mechanics on a compact set with* Z^{ν}*-action satisfying expansiveness and specification*, Bull Amer. Math. Soc. 78 (1972), 988–991; Trans. Amer. Math. Soc. 185 (1973), 237–251. MR 47 #4292.
9. ———, *A measure associated with Axiom A attractors*, Amer. J. Math. (to appear).
10. ———, *Notes on classical statistical mechanics* (to appear).
11. Ja. G. Sinaĭ, *Construction of Markov partitionings*, Funkcional. Anal. i Priložen. 2 (1968), no. 3, 70–80 = Functional Anal. Appl. 2 (1968), 39–52. MR 40 #3591.

12. Ja. G. Sinaĭ, *Gibbsian measures in ergodic theory*, Uspehi Mat. Nauk 27 (1972), no. 4 (166), 21–64 = Russian Math. Surveys 27 (1972), no. 4, 21–70.

13. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817. MR 37 #3598; erratum 139, p. 1593.

14. P. Walters, *A variational principle for the pressure of continuous transformations* (to appear).

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On p. 823 of the September 1975 Bulletin the name of Robert L. Anderson was inadvertently included as a panel member for the AMS-MAA Committee on the Training of Graduate Students to Teach. He should have been listed as a panel member for the AMS Committee on Employment and Educational Policy discussion on "Seeking employment outside academia: Views from some who have recently succeeded".