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# HOMOGENEOUS MECHANICAL SYSTEMS WITH SYMMETRY, AND GEOMETRY OF THE COADJOINT ACTION

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In this note we announce further results on the global behaviour of the so called transitive mechanical systems [4], i.e., mechanical systems with symmetry on homogeneous spaces with an invariant riemannian metric. In the first three sections we see that many facts about the global parameters of such systems are reduced to geometrical properties of the coadjoint action (a linear one). The germ of this idea appeared in Arnold [1, especially §6], linked to the Euler equations for Lie groups. Detailed proofs will appear in [5]. Some results here, like Theorems 1, 2, Corollary 7, etc., are implicit in [6] (noted by the referee).

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1. **Parameters determined by the coadjoint.** Our notation is essentially as in [4]. Given a map $f$, denote by $\sigma(f)$, $\Sigma'(f)$ and $\operatorname{Im}(f)$ its critical points, critical values, and image, respectively. Let $M = G/H$ be a homogeneous space. Let $\operatorname{ad}(G) \subset GL(\mathcal{G})$ be the adjoint group of $G$, and $\operatorname{ad}(H)$ its restriction to $H$. The induced action in $\mathcal{G}^*$ is named coadjoint (Souriau [8, §11]). The kinetic energy $K\colon TM \longrightarrow \mathbf{R}$ is the square norm of a $G$-invariant metric on $M$, and the momentum $J\colon TM \longrightarrow \mathcal{G}^*$ of the symmetry is linear injective on fibers. $I = (K, J)\colon TM \longrightarrow \mathbf{R} \times \mathcal{G}^*$ is equivariant under $G$-actions: $I(g_* X_a) = (K(X_a), \operatorname{ad}(g)_* J(X_a))$. If $S^0$ is the unit sphere at the base point $0 = eH$, $J(S^0)$ is an algebraic sphere of the linear subspace $J(T_0M) \subset \mathcal{G}^*$. We have elegant results about critical values in $J(T_0M)$. The first one generalizes Theorem 3 [1]. Their proof is carried out by equivariance and definition of $DJ$. Denote transversality of manifolds by $\pitchfork$. The orbit $\operatorname{ad}(G)_* p$ in Theorems 1, 2 depends only on the Lie algebra, because if $G$, $G'$ are connected Lie groups with the same Lie algebra, then $\operatorname{ad}(G) = \operatorname{ad}(G')$ [2, p. 118]. In some sense $J\colon S^0 \longrightarrow \mathcal{G}^*$ together with the coadjoint group $\operatorname{ad}(G)_*$ contains most of the relevant information about $I$.

THEOREM 1. *Let $p \in J(S^0)$. Then $(1, p) \notin \Sigma'_0(I)$ if and only if $J(S^0) \pitchfork \operatorname{ad}(G)_* p$.*

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THEOREM 2. *Let* $p \in J(T_0M)$. *Then* $p \notin \Sigma_0'(J)$ *if and only if* $J(T_0M) \pitchfork \mathrm{ad}(G)_* p$.

COROLLARY 3. *If* $J(S^0)$, $J(T_0M)$ *in* $\mathcal{G}^*$ *are given, the sets* $\Sigma'(I)$, $\mathrm{Im}(I)$, $\sigma^0(I)$ *depend only on* $\mathcal{G}$, *but not on* $G$.

2. **Properties of critical points.** We know $\sigma^0 = \sigma^0(I)$ gives $\sigma = \sigma(I)$ via the $G$-action and can be obtained as the cone over $\sigma^0 \cap S^0$ in $T_0M$. We get $\sigma^0 \cap S^0$ from Theorem 1, as follows, where $J_0 = J | T_0M$. See [1], [4] for direct computation of $\sigma^0$, or part of it.

COROLLARY 4. *We have* $\sigma^0 \cap S^0 = J_0^{-1}(\Sigma_0'(I) \cap J(S^0))$.

THEOREM 5. *We have a* (*topological*) *fibration* $\sigma^0 \longrightarrow \sigma \longrightarrow M$, *and moreover,* $\sigma$ *is a manifold* (*or stratified set*) *if and only if* $\sigma^0$ *is.*

We warn that $\sigma(I)$, like $M = G/H$, depends globally on the $G$-action. This theorem is proved by taking a local cross section [2, p. 113] of $G/H$. The linear isotropy group $H^*$ [2, p. 115] provides some symmetries, and this effect can be observed in $\mathcal{G}^*$ too. The invariance of $J(S^0)$ below, carries implicitly the condition for existence of a $G$-invariant metric [3, p. 200].

THEOREM 6. *The group* $H^*$ *leaves* $\sigma^0$, $\mathrm{Re}^0$ *and* $S^0$ *invariant.*

COROLLARY 7. *The subgroup* $\mathrm{ad}(H)$ *of* $\mathrm{ad}(G)$ *leaves invariant the sets* $J(T_0M), J(S^0), J(\sigma^0 \cap S^0)$ *in* $\mathcal{G}^*$.

We consider now the effect on the momenta of a homogeneous space $G/H$ under different $G$-invariant metrics. The following proposition comes from the fact that $J$: $TM \longrightarrow \mathcal{G}^*$ does not depend on $K$, except for diffeomorphism [7].

PROPOSITION 8. *The sets* $\mathrm{Im}(J)$, $\Sigma'(J)$ *are the same for any $G$-invariant metric $K$. The image* $J(T_aM) \subset \mathcal{G}^*$ *does depend on the point* $a \in M$, *but not on* $K$. *Up to a fiber diffeomorphism of* $TM$, $\sigma(J)$ *does not depend on* $K$, *either.*

3. **Semialgebraic sets.** Since the action of an algebraic subgroup of $GL(k)$ on an algebraic set of $\mathbf{R}^k$ generates a semialgebraic set, the following refines results in [4, §4].

COROLLARY 9. *If* $\mathrm{ad}(G) \subset GL(\mathcal{G})$ *is algebraic, then* $\mathrm{Im}(J)$ *and* $\mathrm{Im}(I)$ *are semialgebraic sets for any transitive mechanical system with $G$ as its group of symmetries. If, in addition,* $\Sigma_0'(J)$ *is semialgebraic, then* $\Sigma'(J)$ *and* $\Sigma'(I)$ *are semialgebraic, too.*

4. **Amended potential.** To end up, we consider a generalization of the notion of amended potential [7] for transitive mechanical systems.

Let $p \in \mathcal{G}^*$ such that $J_p = J^{-1}(p)$ is a submanifold of $TM$, and $G_p$ be the isotropy group of $p$. Then $\Pi: TM \longrightarrow M$ projects $J_p$ diffeomorphically onto $\Pi(J_p) \subset M$. We define the amended potential $\widetilde{V}_p: \Pi(J_p) \longrightarrow \mathbf{R}$, by $\widetilde{V}_p(a) = K(J_a^{-1}(p))$. If $\dim H = 0$, then $\Pi(J_p) = M$, and $\widetilde{V}_p$ is Smale's amended potential. Most of Proposition 6.5 [7] is true:

THEOREM 10. *Assume $J_p$ is a manifold. Then $\widetilde{V}_p$ has the properties*:

(i) *$\sigma(K|J_p)$ projects by $\Pi$ onto $\sigma(\widetilde{V}_p)$, and $(c, p) \in \Sigma'(I)$ if and only if $c \in \Sigma'(\widetilde{V}_p)$ or $p \in \Sigma'(J)$.*

(ii) *Both $\widetilde{V}_p$ and $\sigma(\widetilde{V}_p)$ are $G_p$ invariant.*

(iii) $\Pi(I_{cp}) = \widetilde{V}_p^{-1}(c)$.

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