BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

BOOK REVIEWS

Stability of solutions of differential equations in Banach space, by Ju. L. Daleckiı and M. G. Kreĭn, Translations of Mathematical Monographs, Volume 43, American Mathematical Society, Providence, Rhode Island, 1974, vi + 386 pp. $36.40.

If challenged to describe the subject-matter of this book while standing on one foot, one might say: It is the study of the behavior (mostly at infinity) of the solutions of equations of the form $dx/dt=F(t, x)$ living in a Banach space (often a Hilbert space), with F linear in x or nearly so, and of almost all imaginable cognate matters. One might quickly state a few things the book does not do: it does not treat unbounded operators or strongly nonlinear ones; it is not interested in results specific to non-Hilbert phase space; there is little or nothing on solutions belonging to various function spaces; there are no functional-differential equations. If at that point one were to set down the other foot, muttering: But everything else is there, one might be confident of having met the challenge.

The authors have patiently accumulated over many years all the information on their topic that they could lay their hands on or, in many cases, generate themselves. The book grew through several stages of papers and lecture notes, unpublished and published, from 1947 to 1964, had a period of incubation, and emerged in 1970 in its present intricately integrated shape; some finishing touches appear only in this 1974 translation.

Stability theory is of course an almost classical branch of analysis at present and has been very active since Poincaré and Ljapunov. The specific point of view that informs the bulk of this book evolved out of the work of Ljapunov, to whom we owe our main concept of stability and who developed many of the tools still in use. It was, in my view, decisively influenced by two further events: Perron's discussion **[11]** (1930) of the relationship between the existence properties of the inhomogeneous linear equation and the exponential asymptotic behavior of the solutions of the homogeneous equation, and M. G. Kreĭn's observation (1947)—made independently by Bellman (1948)—that methods of functional analysis were available—at first to prune away unsightly and artificial computations, and then to deepen the understanding of the problems and to allow a more powerful attack on them. The possibility of dealing with equations in infinite-dimensional spaces was but a by-product of this insight.

To these two turning-points the authors would feel compelled to add a third: the long-neglected memoir of Bohl **[2]**. The authors record their amazement at the fact that this work, though published in the Crelle Journal in 1913, was for so long overlooked and that so many of its incisive contributions were rediscovered, some quite often (note, e.g., the scornful remarks on p. 147 anent inequalities of the G------l type). The authors themselves seem to have discovered this aspect of Bohl's mathematical oeuvre rather late in the process leading to this book; the discovery seems to have triggered the final effort ending in publication.

Bohl's contribution centers on the concept of the Bohl exponents of a solution (authors' terminology). For a given solution x, the upper Bohl exponent is the infimum of the real numbers ν such that for some $N>0$ we have $\|x(t)\| \leqq Ne^{\nu(t-t_0)}\|x(t_0)\|$ for all $t \geqq t_0$ in the interval of definition; the lower Bohl exponent is defined similarly as a supremum, and with the inequality sign reversed. Bohl substantially demonstrated the "stability" of the Bohl exponents (a suitable semicontinuity) for near-linear equations under perturbations.

It seems to me that this notion of Bohl exponents is the central organizing idea of the book, the clue in this veritable labyrinth of intricate design, leading in and out of pleasant nooks and intimate recesses (e.g., a section on the localization of the spectrum of the monodromy operator of a periodic linear equation in Hilbert space), threading grand vistas (e.g., a chapter on exponential splittings of the solutions of a linear equation), even ultimately guiding the wanderer past arid stretches of blank walls (e.g., the technical work in Hilbert spaces with an indefinite metric). As any visitor of a museum (a better metaphor, were it not for the utterly inappropriate suggestion of the dead-and-done-with), the reader, student, and reference user will spend more time in congenial halls and corners and speed by some important exhibits that do not strike his/her fancy. I confess an unsurprising partiality for Chapters IV and V; others have noted particularly well-done accounts of linear equations in the complex plane (Chapter VI) and asymptotic representation (Chapter VII).

A sketchy summary with some comments is now in order.

Chapter I: "Some information from the theory of bounded operators in Banach spaces". Definitions and basic properties (including the statement of the invertibility of bounded linear bijections); spectrum and resolvent (emphasis on analytic methods; spectral projections and their continuous dependence); solutions of linear operator equations including $AX+XB=Y$. Exponential operator function; criteria for stability operators in Hilbert space; power-bounded operators. Hilbert space with indefinite metric induced by a symmetric operator W; W-dissipativity, W-contractions, dichotomic operators, W-unitary operators. Elementary properties of nonlinear operators.

Chapter II: "The linear equation with a constant operator", is intended to familiarize the reader with some of the properties that will be discussed later for general linear equations; it does that (especialty with respect to equations with a variable but precompactly-valued operator function), but has its own special spectral and analytic flavor: use of renormings, dissipativity criteria in Hilbert space, explicit structure of Green's function. Some discussion of second-order equations, especially in Hilbert space. Periodic and almost-periodic equations and solutions.

Chapter III: "The nonstationary linear equation. Bohl exponents", supplies the parts out of which much of the remaining work is assembled. It studies the equation $dx/dt=A(t)x+f(t)$, where A and f are locally Bochner integrable; A is occasionally "integrally bounded", i.e., $\int_t^{t+1} \|A\|$ is bounded. Basic existence, uniqueness, and growth results, also for operator equations. Integral inequalities. Stability. Ljapunov and Bohl exponents (the upper Ljapunov exponent of a solution x is $\limsup_{t\to\infty} t^{-1}\log\|x(t)\|$; it plays a subordinate role to the upper Bohl exponent described above; the upper and lower Bohl exponents of the equation are the supremum and infimum of the upper and lower Bohl exponents of the solutions). Stability (i.e., semicontinuity) of the Bohl exponents of the equation under certain perturbations. THEOREM: *The initial-value problem for given integrally bounded A has a bounded solution for every bounded continuous f and every initial value if and only if the upper Bohl exponent of the equation is negative.* Equations with precompactly-valued operator functions: how little does the behaviour of the equation differ from that of the constant-operator "limit" equation?

Chapter IV: "Exponential splitting of the solutions of the linear equation". This chapter is probably the core of the book (but I may be biased). An exponential splitting for equation $dx/dt=A(t)x$ is given by a finite direct decomposition of the space into subspaces such that: (a) the interval just containing all lower and upper Bohl exponents of the solutions starting from one subspace is disjoint from the intervals for all other subspaces; (b) solutions starting from different subspaces remain uniformly angularly apart. For the interval $[0,\infty[$ and two subspaces this is an exponential dichotomy (as introduced by Massera and Schäffer **[9]** (1960); the authors only deal with exponential dichotomies induced by complemented subspaces). There are two important preliminary sections. One is on "conjugation operators for projections", describing the elegant method developed by Coppel **[4]**, **[5]** that permits one to track the direct decomposition of a space produced by time-varying projections. This is particularly useful in a Hilbert space. The other preliminary section is on kinematical similarity and reducibility. Exponential dichotomies are introduced and representative theorems linking them for integrally bounded A to the existence of bounded solutions of the inhomogeneous equations with bounded continuous f are proved. Exponential splittings are defined and the stability of the intervals of Bohl exponents associated with them are discussed. In a Hilbert space, an equation having an exponential splitting is kinematically similar to one in which the corresponding splitting has invariant subspaces: the latter equation thus breaks into separate equations. Again in Hilbert space, the case of a precompact-valued operator function has a more detailed analysis paralleling the constant-operator case through study of the spectra of the values of the operator function.

Chapter V: "The equation with a periodic operator function", goes into great detail in discussing several problems on linear periodic equations. The key to all is of course the "monodromy operator", the value after one period of the principal operator solution of $dU/dt=A(t)U$ with initial value

I. In a complex finite-dimensional space, this operator is in the range of the exponential function, and a periodic transformation reduces the equation to one with constant operator (Floquet's theorem). In an infinite-dimensional space this is not always the case; the question remains: When is it? Another question: How can one get information on the equation without attempting this reduction? After some generalities on these questions (including tests for dichotomy in Hilbert space using indefinite metrics), the authors provide some tests for Floquet reducibility in Hilbert space that sharpen the known test $\int \|A\| < \pi$ (integral over one period). Detailed sections are devoted to canonical linear equations and to second-order linear equations (like Hill's equation). A final section deals with analytic dependence of the monodromy operator on a parameter.

Chapter VI: "Linear differential equations in the complex plane", deals, in great and explicit detail, with the equation $dx/dz = A(z)x$, where the meromorphic function A has a pole of order 1 at 0 and no other singularity. The discussion hinges on the existence and nature of pairs of points with integral differences in the spectrum of the principal part of A.

Chapter VII: "Nonlinear equations", deals with perturbations of linear equations, and the stability of their asymptotic behavior (e.g., of the Bohl exponents). A very detailed account is given of the case where the unperturbed equation has constant operator, though a method for generalizing to other cases is described. Stability and instability manifolds are carefully pinned down, and the case where the spectrum of the unperturbed operator meets the imaginary axis is examined with the help of exponential splittings. There is a brief account of the method of averaging.

Chapter VIII: "Asymptotic representation of the solutions of a linear differential equation with a large parameter". Section headings: Approximate decomposition of the equation; Estimate of the error; The equation with a rapidly oscillating coefficient. (At this point, "museum feet" compel me to rely on the catalogue.)

Each chapter is followed by a section deceptively headed "Exercises". Most of these are very nontrivial sketches of results in and out of the literature; some sequences are developments of entire little theories. Elaborate hints are given, but I suppose many exercises will require a reference to the sources. Much in these sections thus functions as a welcome guide to the literature. The "Exercises" in Chapter V are particularly impressive in their range and depth. There are also "Notes" to each chapter, mostly on sources for the text.

There are a few books that have some overlap in subject-matter and viewpoint with this one. We should mention Krasovskiĭ [**8**] (1959), Hartman [**7**] (1964), Coppel [**3**] (1965), Massera and Schäffer [**10**] (1966), Demidovič [**6**] (1967), Barbašin [**1**] (1970). This list does not include work on evolution equations with unbounded operators, nor strongly nonlinear equations, two very active fields today. In several cases, topics treated here are dealt with in greater depth in one or the other of these books; but none quite represents the same mix of topics nor the same organizing principle as the book under review, not do they have quite as far-ranging reports on the literature. The authors claim that they are not specialists in the specific topic of this book—indeed, that it was their "hobby". Instead of disputing this claim, let us wish there were more such "dilettanti".

The style of the book, is readable, with some arid stretches. Very careful attention must be paid by the reader to terminology; the index is very helpful. It would be pleasant to report an absence of detected errors and misprints; unfortunately the book is riddled with them. A cursory reading of several sections detected about one per page. Most are trivial (wrong signs, 0 for ∞, misplaced exponents, etc.), quite a few are more serious (e.g., on p. 114, line 5, replace "the spectrum $\sigma(A)$ lies on the imaginary axis" by "the equation is bistable"[1]); none of those detected is crippling, but their accumulation is most annoying. More to the point, such carelessness in small things makes one wonder about the great ones that one would gladly trust.

Recommendation: a must for the specialist in stability theory (who does not need this review); an important reference book; a source—with very judicious selection—for an inspiring seminar or even a graduate course for enthusiasts.

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Indefinite inner product spaces, by János Bognár, Springer-Verlag, New York, 1974, 223 pp., $19.70.

This is the first book on infinite-dimensional vector spaces with a signed inner product, a subject which frequently goes under the title 'spaces with an

[1].This is a translator's error.